Monetary Policy and Heterogeneous Expectations

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Abstract

This paper studies the implications for monetary policy of heterogeneous expectations in a New Keynesian model. The assumption of rational expectations is replaced with parsimonious forecasting models where agents select between predictors that are underparameterized. In a Misspecification Equilibrium agents only select the best-performing statistical models. We demonstrate that, even when monetary policy rules satisfy the Taylor principle by adjusting nominal interest rates more than one for one with inflation, there may exist equilibria with Intrinsic Heterogeneity. Under certain conditions, there may exist multiple misspecification equilibria. We show that these findings have important implications for business cycle dynamics and for the design of monetary policy.

JEL Classifications: G12; G14; D82; D83

Key Words: Heterogeneous expectations, monetary policy, multiple equilibria, adaptive learning.

1 Introduction

The monetary policy literature has, for the most part, been developed within the rational expectations paradigm. One consequence to the assumption of rational expectations is that all agents in the economy hold homogeneous expectations formed with respect to the economy’s true probability distribution. Recent studies of the implications for monetary policy of private agents following adaptive learning rules typically also maintain the assumption of homogeneity in expectation formation. There is, however, considerable empirical evidence that agents – households, firms, economists... – hold heterogeneous beliefs.

This paper focuses on the implications for monetary policy of heterogeneous expectations. To do this we adopt the Misspecification Equilibrium framework of Branch and Evans (2006, 2007, 2009), itself an extension of Brock and Hommes (1997) to
stochastic environments. Here we apply this approach to the New Keynesian model by assuming that agents select between forecasting models that differ by their explanatory variables. In the standard New Keynesian model of Woodford (2003) inflation and the output gap are driven by two exogenous processes corresponding to “demand” and “price” shocks. In a determinate model, the unique rational expectations equilibrium depends linearly on both demand and price shocks. We instead assume that agents favor parsimony in their forecasting models and, as a consequence, adopt underparameterized models: agents choose models that forecast the state based on demand shocks or price shocks, but not both. In a Misspecification Equilibrium, agents only select the best performing statistical models. In this sense, although agents hold misspecified beliefs, they will be optimal within the class of underparameterized models. A Misspecification Equilibrium is “in the spirit” of rational expectations equilibria by imposing important cross-equation restrictions while incorporating reasonable assumptions about the manner in which agents form their expectations.

Our approach is motivated by our earlier findings that even though all agents optimally select their underparameterized forecasting models, in equilibrium they may be distributed across all models (Branch and Evans (2006b)) or there may exist multiple equilibria (Branch and Evans (2007), Branch and Evans (2006a)). Heterogeneity in beliefs and dispersion in forecasts is an empirical regularity (Branch (2004), Mankiw, Reis, and Wolfers (2003), Kurz and Motoles (2007)). Thus, it is an important issue whether, and to what degree, monetary policy can affect the diversity of beliefs. We assume that monetary policy can be described by a policy rule that adjusts interest rates in response to the current state of the economy. We find that, depending on the stochastic properties of the exogenous disturbances, there may exist multiple misspecification equilibria or equilibria exhibiting Intrinsic Heterogeneity.

These theoretical findings have several unique implications for business cycle dynamics and monetary policy. We illustrate that “bad luck”, in the form of more volatile and persistent price shocks, can turn an economy with a unique low volatility equilibrium into one with multiple equilibria. Real-time learning and dynamic predictor selection in such an economy will produce endogenous regime-switching volatility of the kind documented by Sims and Zha (2006), among others. However, if policy becomes increasingly aggressive in responding to inflation, as the Federal Reserve appears to have done during the mid 1980’s, the economy will then coordinate on a unique equilibrium.

We also demonstrate the implications of heterogeneous expectations for a policy-maker seeking to implement optimal discretionary policy. When monetary policy is set by a nominal interest rate rule that responds optimally to private-sector expectations, there exists a unique misspecification equilibrium. Interest rate rules of this form have been known to perform well when agents are homogeneous and use least-squares to update their forecasting models. This finding indicates that expectations-based policy rules perform well even when agents have heterogeneous expectations.
This paper proceeds as follows. In section 2 we develop the New Keynesian model with diverse beliefs while section 3.1 present the main analytic results. Section 3.3 presents numerical examples of the number and nature of misspecification equilibria while section 4 focuses on the business cycle and policy implications of heterogeneity. Section 5 concludes.

2 Model

2.1 A New Keynesian Model with Diverse Beliefs

We develop our results within the framework of a standard “New Keynesian” model along the lines of Woodford (2003), where aggregate output and inflation are given by the following equations

\[x_t = \hat{E}_t x_{t+1} - \varsigma \left( i_t - \hat{E}_t \pi_{t+1} \right) + g_t\]  \hfill (1)

\[\pi_t = \beta \hat{E}_t \pi_{t+1} + \kappa x_t + u_t\]  \hfill (2)

where we assume \(g_t = \rho g_{t-1} + \varepsilon_t\), \(u_t = \phi u_{t-1} + \nu_t\), \(0 < \rho, \phi < 1\). Here \(x_t\) is the aggregate output gap, \(\pi_t\) is the inflation rate, and \(\hat{E}_t\) is a convex combination of expectations operators to be specified below. Note, that under rational expectations \(\hat{E} = E\).

Equations (1)-(2) constitute a New Keynesian model where conditional expectations have been replaced by a convex combination of boundedly rational expectations operators. The model shares the same reduced-form as the homogeneous expectations version of the New Keynesian model with either rational expectations or adaptive learning formulations along the lines of Bullard and Mitra (2002) and Evans and Honkapohja (2003). Branch and McGough (2009) derive these reduced-form equations from linear approximations to the optimal decision rules of a Yeoman-farmer economy extended to include two types of agents, \(\text{ex ante}\) identical except with respect to the manner in which they form expectations. The first equation (1) is an IS relation that describes the demand side of the economy. In an economy with homogeneous agents, rational or boundedly rational, it is a linear approximation to a representative agent’s Euler equation. With agents heterogeneous in their expectation formation process, it is found by aggregating Euler equations across agents. The parameter \(\varsigma\) is the real interest elasticity of output. Equation (2) describes the aggregate supply relation. It is found by averaging each firms’ pricing decisions, and as in the representative agent economy, \(\lambda\) is the output elasticity of inflation.

Although equations (1)-(2) share the same reduced-form as a representative agent economy, the key distinction here is that the diversity of beliefs leads the equilibrium processes for inflation and output to depend on the distribution of agents’ expectations. This paper takes as given that these equations govern the economy, and is interested in the interaction of monetary policy and the equilibrium distribution of agents across (underparameterized) forecasting models.
2.2 Monetary Policy Rules

A fully specified model prescribes a nominal interest rate rule followed by the central bank. Below we specify the precise form of the rule assumed to be followed by the monetary authority. Throughout this paper we assume that monetary policy is set according to the nominal interest rate rule

\[ i_t = \chi_\pi \hat{E}_t \pi_t + \chi_x \hat{E}_t x_t \]  

(3)

According to this rule, policymakers respond to contemporaneous private sector expectations. Basing policy on observable variables is easily implementable and has been suggested by, among others, McCallum (1999). The precise form of the policy rule is inconsequential for the qualitative theoretical results presented in this paper. For example, we looked at rules where policymakers set interest rates in response to their own forecasts of contemporaneous inflation and output gap

\[ i_t = \chi_\pi \tilde{E}_t \pi_t + \chi_x \tilde{E}_t x_t \]

where it may be the case that \( \tilde{E} \neq \hat{E} \). We also looked at rules where they respond to contemporaneous state variables

\[ i_t = \chi_\pi \pi_t + \chi_x x_t \]

which has a form close to the celebrated Taylor-rule, or based on expectations of future economic variables

\[ i_t = \chi_\pi \hat{E}_t \pi_{t+1} + \chi_x \hat{E}_t x_{t+1} \]

With the latter rule, we examined cases where policymakers have rational expectations, their own boundedly rational forecasts, or respond to private-sector expectations. In each instance, the qualitative results are similar. In section 4.2, we also consider optimal discretionary policy. We focus on contemporaneous expectations-based policy rules of the form (3) because of the central role it places on private-sector expectations. This paper primary purpose is to study the joint determination of monetary policy and the distribution of agents’ beliefs and so policy rules of the form (3) deliver the sharpest results.

In each of the policy rules considered, nominal interest rates respond to aggregate state variables, thus it is possible to represent the state of the economy in terms of an aggregate state vector \( y = (x, \pi)' \).

2.3 Misspecification Equilibrium

In a minimum state variable rational expectations equilibrium, the law of motion for the state vector \( y \) depends linearly on the exogenous processes for \( q_t, u_t \). In this paper, we assume that agents favor parsimony in their forecasting model and instead select
the best performing statistical models from the set of underparameterized forecasting models. Since there are only two exogenous variables, we assume that private sector agents underparameterize by selecting from two possible perceived laws of motion:

\[
P LM_1 : y_t = b^1 g_t + \eta_t \\
P LM_2 : y_t = b^2 u_t + \eta_t
\]

where \( \eta_t \) is a perceived exogenous white noise shock. This implies expectations of the form,

\[
E_1^t y_t = b^1 g_t, \quad E_1^t y_{t+1} = b^1 \rho g_t \\
E_2^t y_t = b^2 u_t, \quad E_2^t y_{t+1} = b^2 \phi u_t
\]

Let \( n \) denote the fraction of agents who use \( P LM_1 \). Then for any state variable \( w \)

\[ \hat{E} w = nE^1w + (1 - n)E^2w. \]

Substituting in the diverse beliefs along with the policy rule (3), the system can be written in the form

\[ y_t = A\hat{E}_t y_{t+1} + B\hat{E}_t y_t + Dz_t \]

where \( z = (g, u)' \) and the expressions for \( A, B, D \) are provided in the Appendix. Plugging in private-sector expectations produces a reduced-form actual law of motion

\[ y_t = \xi_1(n) g_t + \xi_2(n) u_t \quad (4) \]

where

\[
\xi_1(n) = n (\rho A + B) b^1 + De_1 \\
\xi_2(n) = (1 - n) (\phi A + B) b^2 + De_2
\]

and \( e_i, i = 1, 2 \) is a unit row vector with the ith component equal to one and zeros elsewhere. In the sequel, we will suppress the dependence of \( \xi_j \) on \( n \).

Although private-sector agents are assumed to underparameterize their forecasting models, that is they hold restricted perceptions, we require that they forecast in a statistically optimal manner; we require that the forecast model parameters are optimal linear projections. It follows then that the beliefs \( b^j, j = 1, 2 \) satisfy the following least-squares orthogonality conditions,

\[
E g_t \left( \xi_1 g_t + \xi_2 u_t - b^1 g_t \right) = 0 \quad (5) \\
E u_t \left( \xi_1 g_t + \xi_2 u_t - b^2 u_t \right) = 0 \quad (6)
\]

It follows that

\[
b^1 = \xi_1 + \xi_2 \bar{r} \\
b^2 = \xi_2 + \xi_1 \bar{r}
\]
where \( r = E_g u_t/E_g^2 \), \( \hat{r} = E_g u_t/Eu^2 \). Least squares orthogonality conditions like (5) or (6) appear frequently in the macroeconomics literature. For example, Sargent (1999) and Sargent (2008) define a self-confirming equilibrium with respect to a very similar condition. Sargent (2008) and Evans and Honkapohja (2001) show that many learning models will converge to a set of parameters that satisfy orthogonality conditions like (5)-(6) rather than to their rational expectations values. The key feature of beliefs that satisfy orthogonality conditions like (5) or (6) are that within the context of their forecasting model, agents are unable to detect their model misspecification.

Plugging into the expectational difference equation (4), it is possible to solve for the reduced-form coefficients:

\[
\begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix} = \Delta^{-1} \begin{bmatrix}
De_1' \\
De_2'
\end{bmatrix}
\]  \( \text{(7)} \)

where

\[
\Delta = \begin{bmatrix}
I - n (\rho A + B) & -n (\rho A + B) r \\
-(1-n) (\phi A + B) & I - (1-n) (\phi A + B)
\end{bmatrix}
\]

We are now ready to define a Restricted Perceptions Equilibrium.

**Definition.** Given exogenous processes for \( g_t, u_t \) and given the proportion \( n \) of agents using forecast model \( j = 1 \), a Restricted Perceptions Equilibrium is a stochastic process \( \{y_t\} \) of the form (4), where the coefficients satisfy (7).

**Remark.** A unique RPE exists provided \( \Delta^{-1} \) exists.

A general existence result is not available, however the following result holds for very weakly correlated demand and price exogenous shocks.

**Proposition 1** Let \( r, \hat{r} \to 0 \). For sufficiently small values of \( \rho, \phi \), there exists a unique RPE.

Because the model is self-referential, \( b^j \), hence \( \xi_j \), are objects pinned down by the equilibrium. Similarly, we do not want to treat \( n \) as a free parameter and we now make it endogeneous. Thus, even though agents are assumed to forecast with misspecified models, there are still important equilibrium restrictions analogous to the restrictions obtained under fully rational expectations.

In order to determine \( n \) endogenously, we need a metric for forecast success. Following Branch and Evans (2007), we assume that agents seek to minimize their forecast mean square error (MSE). Thus, we assume that agents rank their forecasting model according to

\[
EU^j = -E \left( y_{t+j} - E_{t+j}^j y_{t+j} \right)' W \left( y_{t+j} - E_{t+j}^j y_{t+j} \right)
\]

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where $W$ is a weighting matrix that is assumed to be equal to the identity matrix. This is a natural metric since the prediction problem confronting agents, according to equations (1), (2), is to form one step ahead forecasts of the output gap and inflation.

Plugging in for private-sector expectations, in an RPE, and the actual law of motion (4), leads to the MSE expressions

$$EU^1 = -E \left( (\xi_2 \phi u_t - \xi_2 \rho g_t) + \xi_1 \epsilon_{t+1} + \xi_2 \nu_{t+1} \right)^{\prime} W \left( (\xi_2 \phi u_t - \xi_2 \rho g_t) + \xi_1 \epsilon_{t+1} + \xi_2 \nu_{t+1} \right)$$

$$EU^2 = -E \left( (\xi_1 \rho g_t - \xi_1 \phi \tilde{r} u_t) + \xi_1 \epsilon_{t+1} + \xi_2 \nu_{t+1} \right)^{\prime} W \left( (\xi_1 \rho g_t - \xi_1 \phi \tilde{r} u_t) + \xi_1 \epsilon_{t+1} + \xi_2 \nu_{t+1} \right)$$

The endogenous value for $n$ depends on the relative forecast performance. Thus, define $F(n) : [0, 1] \to \mathbb{R}$ as $F(n) = EU^1 - EU^2$. We can write the expression for relative forecast performance as

$$F(n) = Eg_t^2 \left[ \rho^2 (\xi_1 \xi - r^2 \xi_2 \xi_2) + 2 \rho \phi (r \xi_2 \xi_2 - \tilde{t} \xi_1 \xi) \tilde{r} + \phi^2 (\tilde{r}^2 \xi_1 \xi_1 - \xi_2 \xi_2) Q \right]$$

where $Q = Eu_t^2 / Eg_t^2$. The relative forecasting performance depends on the distribution of beliefs because $\xi_1, \xi_2$ depend on $n$.

As in our earlier papers, we follow Brock and Hommes (1997) in assuming a multinomial logit (MNL) approach to predictor selection. The MNL approach has a long history in discrete decision making and is a natural way of introducing randomness in forecasting into a monetary model. Young (2004) argues that randomness in forecasting provides robustness against model uncertainty and flexibility in economic environments with feedback. We assume that agents select their predictor from a discrete set of forecasting models, they are uncertain about which model forecasts best, and so the MNL map is natural in this framework:

$$n = \frac{\exp (\alpha EU^1)}{\exp (\alpha EU^1) + \exp (\alpha EU^2)}$$

which can be re-written as

$$n = \frac{1}{2} \left[ \tanh \{\alpha F(n)\} + 1 \right] \equiv T_\alpha(n) \quad (8)$$

We remark that $T : [0, 1] \to [0, 1]$ is a continuous and well-defined function provided that an RPE exists. The parameter $\alpha$ is called the ‘intensity of choice’. The neoclassical limit arises in the limit $\alpha \to \infty$ as in this case all agents select the best-performing statistical model. In our theoretical results below, we focus on the case of large $\alpha$ as we are interested in whether monetary policy can affect the diversity of beliefs when agents behave in an optimal manner subject to the restriction that they only forecast with parsimonious models.

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1All qualitative results are robust to alternative assumptions about how much relative weight agents place on inflation or output forecast errors. We note, however, that to remain consistent with the micro-foundations of the model the matrix $W$ should be diagonal (c.f. Woodford (2003)), and so the identity matrix assumption is a natural choice.
It is important to emphasize that the feedback effects in this model: the RPE parameters, the distribution of agents, and the stochastic process for the state are all jointly determined in an equilibrium. We are now ready to define our equilibrium concept.

**Definition.** A Misspecification Equilibrium \( n^* \) is a fixed point of the map \( T : n^* = T(n^*) \).

### 3 Results

#### 3.1 Analytic Results

Since the MNL is a monotonic function of the relative predictor fitness measure, it should be clear from (8) that the number and nature of Misspecification Equilibria depends on the properties of \( F(n) \). Using the arguments of Branch and Evans (2007), the following proposition characterizes the possible equilibria.

**Proposition 2** Let \( N_{\alpha}^* = \{n^*|n^* = T_{\alpha}(n^*)\} \) denote the set of Misspecification Equilibria. In the case of large \( \alpha \), \( N^* \) has one of the following properties:

1. If \( F(0) < 0 \) and \( F(1) < 0 \) (Condition P0) then \( n^* = 0 \in N^* \).
2. If \( F(0) > 0 \) and \( F(1) > 0 \) (Condition P1) then \( n^* = 1 \in N^* \).
3. If \( F(0) < 0 \) and \( F(1) > 0 \) (Condition PM) then \( n^* \in \{0, \hat{n}, 1\} \subset N^* \), where \( \hat{n} \in (0, 1) \) is such that \( F(\hat{n}) = 0 \).
4. If \( F(0) > 0 \) and \( F(1) < 0 \) (Condition P) then \( n^* = \hat{n} \in N^* \), where \( \hat{n} \in (0, 1) \) is such that \( F(\hat{n}) = 0 \).

In general, we do not know whether \( F \) is monotonic, so we can not rule out the existence of additional equilibrium besides those listed in Proposition 2. When Condition P0 or Condition P1 hold then either \( n^* = 0 \) or \( n^* = 1 \) is a Misspecification Equilibrium because in those instances agents would always want to select the demand disturbance or supply disturbance model when all other agents do as well. On the other hand, if Condition P holds then there is an incentive for an individual agent to deviate from a consensus model and instead Intrinsic Heterogeneity arises. Under Condition PM it follows that both \( n^* = 0 \) and \( n^* = 1 \) are Misspecification Equilibria. Thus, Condition PM is a sufficient condition for the existence of multiple equilibria. Under Condition PM there is also an interior equilibrium with agents distributed across both the demand and supply shock forecasting models. However, because \( F(n) \) is a continuous function, Condition PM implies that whenever \( F(n) \) is monotonic, we have \( T'(F(\hat{n})) > 1 \) and hence \( \hat{n} \) is unstable. Under Condition P, whenever \( F(n) \) is monotonic then it crosses through zero from above, and we showed in Branch...
and Evans (2005) that a Misspecification Equilibrium with Intrinsic Heterogeneity is stable. Proposition 2 does not state the circumstances under which these conditions will arise. The model is multivariate and \( F(0) \) and \( F(1) \) depend in a complicated way on the parameters of the model. However, it is possible to demonstrate that conditions exist under which each of the various cases might arise.

**Corollary 3** Conditions \( P_0, P_1, PM \) and \( P \) can each be satisfied for appropriate choices of structural parameters.

By focusing on the special case of weakly correlated demand and supply shocks, more precise results are available. In particular, the following result provides conditions under which either multiple Misspecification Equilibria or Intrinsic Heterogeneity arise. First, though, we define the following expressions:

\[
B_0 = \frac{\rho^2(1 - \phi^2)(1 + \kappa^2)[(1 - \phi)(1 - \beta\phi) + \kappa \zeta(\chi - \phi) + \zeta \chi(1 - \beta\phi)]^2}{\phi^2(1 - \rho^2) \left[ \zeta^2(\chi - \phi)^2 + (\zeta \chi + 1 - \phi)^2 \right]}
\]

\[
B_1 = \frac{\rho^2(1 - \phi^2)}{\phi^2(1 - \rho^2)} \left[ (1 - \beta\rho)^2(1 - \rho) + \chi \zeta(1 - \beta\rho) + \kappa \zeta(\chi - \rho) \right]^2
\]

**Proposition 4** For \( r, \tilde{r} \) sufficiently small we have

1. Condition \( P_0 \) holds if
   \[
   \frac{\sigma^2_\nu}{\sigma^2_\varepsilon} > \max(B_0, B_1)
   \]

2. Condition \( P_1 \) holds if
   \[
   \frac{\sigma^2_\nu}{\sigma^2_\varepsilon} < \min(B_0, B_1)
   \]

3. Condition \( PM \) holds if
   \[
   B_0 < \frac{\sigma^2_\nu}{\sigma^2_\varepsilon} < B_1
   \]

4. Condition \( P \) holds if
   \[
   B_1 < \frac{\sigma^2_\nu}{\sigma^2_\varepsilon} < B_0
   \]

This result illustrates that either multiple equilibria or Intrinsic Heterogeneity may arise under many different parameterizations of the model. Since \( B_0, B_1 \) do not depend on the variances of the exogeneous white noise shocks, we are free to choose the relative ratio of these shocks to satisfy either of these conditions.

**Proposition 5** Let \( r, \tilde{r} \) be sufficiently small and assume \( \alpha \) is large.
1. For $\chi_\pi$ and/or $\chi_x$ sufficiently large, depending on $\frac{\sigma^2_\nu}{\sigma^2_\varepsilon}$, either Condition $P$ holds, so that there exists a Misspecification Equilibrium with Intrinsic Heterogeneity, or Condition $P0$ holds, so that there exists a Misspecification Equilibrium with $n^* = 0$.

2. For $\frac{\sigma^2_\nu}{\sigma^2_\varepsilon}$ sufficiently large, there exists a Misspecification Equilibrium at $n^* = 0$.

3. For $\frac{\sigma^2_\nu}{\sigma^2_\varepsilon}$ sufficiently small, there exists a Misspecification Equilibrium at $n^* = 1$.

**Remark.** In case 1 of Proposition 5 Intrinsic Heterogeneity arises when $\frac{\sigma^2_\nu}{\sigma^2_\varepsilon}$ is less than a threshold value and $n^* = 0$ arises when $\frac{\sigma^2_\nu}{\sigma^2_\varepsilon}$ is greater than the threshold. For further details, see the Appendix.

### 3.2 Intuition

A simple example helps illustrate the intuition for why different equilibria may arise. The form of the actual law of motion (4) illustrates that the exogenous disturbances $g_t, u_t$ have direct and indirect effects via the feedback from expectations onto the state. The strength of the indirect effects are controlled by the parameters $\varsigma, \kappa, \beta, \chi_\pi, \chi_x$. The number and nature of Misspecification Equilibria depend on a balancing of these direct and indirect effects. To illustrate how multiple equilibria or intrinsic heterogeneity might arise, take the starkest parameterization of the model: suppose that demand and supply shocks are uncorrelated (i.e. $r = \tilde{r} = 0$) and the indirect effects are weak, i.e. $\varsigma = 0, \kappa = 0, \beta = 0$. Under these assumptions, the model (1)-(2) become

$$x_t = E_t x_{t+1} + g_t$$
$$\pi_t = u_t$$

Suppose at first that $n = 1$. The RPE parameters are

$$\xi_1 = \left(\frac{1}{1 - \rho}, 0\right)'$$
$$\xi_2 = (0, 1)'$$

Agents within this RPE will have mean-square forecast errors $\frac{1}{1 - \rho^2} \sigma^2_\nu$ for inflation and $\frac{1}{(1 - \rho)^2} \sigma^2_\varepsilon$ for the output gap. Now consider a zero-mass agent deciding whether to deviate from the consensus and select the supply-shock model instead. Their beliefs, which satisfy their restricted perceptions orthogonality condition, are $b^2 = (0, 1)'$. Their forecast errors are thus $\sigma^2_\nu$ for inflation and $\frac{1}{(1 - \rho)(1 - \phi)} \sigma^2_\varepsilon$ for the output gap. Whether $n = 1$ can be a Misspecification Equilibrium depends on whether the zero-mass agent has an incentive to deviate. That is, $n = 1$ will be an equilibrium whenever

$$\frac{\sigma^2_\nu}{\sigma^2_\varepsilon} < \frac{\rho^2 (1 - \phi^2)}{\phi^2 (1 - \phi^2)(1 - \rho)^2} = B_1$$
Now suppose that the economy is at \( n = 0 \). The RPE parameters are

\[
\begin{align*}
\xi_1 &= (1, 0)' \\
\xi_2 &= (0, 1)'
\end{align*}
\]

Agents using the supply shock model will have forecast errors for inflation and output gap of \( \sigma_{\nu}^2, \frac{1}{1-\rho^2} \sigma_\xi^2 \), respectively. An agent deciding to deviate from the consensus and forecast based on the demand shock model will have beliefs \( b^1 = (1, 0)' \) and forecast errors for inflation and output gap of \( \frac{1}{1-\rho^2} \sigma_{\nu}^2, \sigma_\xi^2 \). Thus, \( n = 0 \) will be an equilibrium whenever

\[
\frac{\sigma_{\nu}^2}{\sigma_\xi^2} > \frac{\rho^2(1-\phi^2)}{\phi^2(1-\rho^2)} = B_0
\]

Since \( B_1 > B_0 \) we have condition \( P0, P1 \) or \( PM \), depending on the size of \( \sigma_{\nu}^2/\sigma_\xi^2 \). However, stable Intrinsic Heterogeneity, i.e. Condition \( P \), cannot arise. However, Intrinsic Heterogeneity can arise if there is negative feedback, e.g. from policy.

To see this we extend the special case just analyzed to allow for \( \varsigma, \chi_x > 0 \). One can then show that \( B_1 < B_0 \) when

\[
(1 - \rho + \varsigma \chi_x)^2(1 - \phi + \varsigma \chi_x)^2 > \varsigma \phi^2,
\]

which is satisfied for \( \varsigma \chi_x \) sufficiently large. Intrinsinc Heterogeneity arises in this case because of the negative impact on output of a strong policy response to expected output.

This example nicely illustrates the equilibria that may arise in a New Keynesian model with misspecified beliefs. More plausible specifications for the indirect effects, and in particular the policy effects, lead to a rich set of equilibrium results. The following section illustrates the role that policy plays in determining the number and nature of Misspecification Equilibria.

### 3.3 Numerical Illustrations

The previous subsection demonstrates the number and nature of Misspecification Equilibria in a New Keynesian model. To provide further results, this subsection presents numerical examples. We calibrate the model’s parameters following Woodford (2003) as follows: \( \varsigma = 1/157, \kappa = .024, \beta = .99 \). The weight in the forecast fitness measure \( W \) is set to the identity matrix. Policy is assumed to satisfy the “Taylor principle” which states that nominal interest rates should be raised by more than one for one with inflation, in the case of a contemporaneous expectations rule the response coefficient is on expected inflation. In particular, we set \( \chi_\pi = 1.5, \chi_x = .125 \), which coincide with Taylor’s (1993) original policy rule recommendation. This subsection demonstrates that the model’s equilibria will vary depending on the properties of the exogenous stochastic processes for \( g_t, u_t \).
First, we demonstrate the possibility for multiple equilibria by setting $\rho = \phi = 0.5, \sigma^2 = 0.9, \sigma^2 = 0.25, \sigma_{\varepsilon\nu} = 0$. Figure 1 plots the T-map and relative predictor fitness measure $F(n)$ for various values of $n$.

Figure 1: Multiple equilibria.

The top panel of Figure 1 plots the T-map while the bottom figure plots the relative profit fitness measure. Because the function $F$ is monotonic, so is the T-map. Moreover, the T-map crosses the 45-degree line three times, at $n = 0, n = 1$ and at a point $\hat{n}$ where $F(\hat{n}) = 0$. Clearly, the interior equilibrium has the property that $T'(\hat{n}) > 1$ and so this equilibrium would not be attainable under a reasonable
learning rule. However, the equilibria at \( n = 0 \) and \( n = 1 \) are stable, and so under this parameterization, with policy satisfying the Taylor principle multiple equilibria exist.

The results of the previous section, however, suggest that with alternative choices for the exogenous process governing demand and supply shocks that an equilibrium exhibiting Intrinsic Heterogeneity, where agents are distributed across all forecasting models, may exist. To illustrate this, we adopt the same parameter values as in Figure 1, except now we set \( \phi = .45, \sigma^2_\epsilon = .35, \sigma^2_\nu = 0.2 \). Figure 2 plots the T-map and relative predictor fitness measure. The bottom panel shows that, under this particular parameterization, that the function \( F \) is no longer monotonic, and in particular it satisfies Condition P where \( F(0) > 0, F(1) < 0 \). Under Condition P, equilibria where agents are all massed onto a particular forecasting model do not exist. The top panel illustrates that this is the case for these particular demand and price shock processes. Now the T-map is (monotonically) negatively sloped with a single interior fixed point. Hence, there is the possibility for diverse beliefs even though agents only select the best-performing forecasting models.

Figures 1 and 2 show that when policy satisfies the Taylor principle it is possible to have either multiple equilibria or Intrinsic Heterogeneity. It remains to be seen the role policy plays in the equilibrium properties of the model. To address this issue we turn to bifurcation diagrams. We adopt Woodford’s calibration, make assumptions about the exogenous processes and the policy rule’s response coefficient to output \( (\chi_x) \), and then vary the inflation response coefficient \( \chi_\pi \) between zero and one. For each value of \( \chi_\pi \in [0, 2] \), we calculate the fixed points \( n^* = T(n^*) \), the corresponding unconditional variances of inflation and the output gap, and plot the results.

The top panel of Figure 3 demonstrates that multiple equilibria exist for all values of the inflation response coefficient below two. As in Figure 1 there are multiple equilibria at \( \chi_\pi = 1.5 \). Although Figure 3 shows the existence of multiple equilibria for all values of \( \chi_\pi \) this does not imply that policy does not have an effect on the dynamic properties of the model. First, the top panel shows that the basins of attraction for the \( n = 0 \) or \( n = 1 \) equilibria vary with the policymaker’s inflation response. If there were a real-time learning dynamic (see below) where agents select their forecasting model in real-time based on recursive estimates of the relative fitness of each model, then the size of the basin of attraction to equilibrium \( n = 1 \) can be viewed as the vertical distance between the \( n = 1 \) equilibrium and the unstable interior equilibrium. Clearly, at \( \chi_\pi \approx 0.5 \) the size of this basin is maximized. As \( \chi_\pi \) increases further above one, the basin of attraction for the \( n = 1 \) equilibrium shrinks further and further. We know from Proposition 5 that for sufficiently large \( \chi_\pi \) there is an equilibrium at \( n = 0 \). So this basin of attraction for \( n = 1 \) continues to shrink until the equilibrium disappears altogether.

The inflation response coefficient also affects the equilibrium variances for the economy. The bottom two panels plot the unconditional variances of output gap and inflation, respectively. In the middle panel, the equilibrium variance corresponding to
Figure 2: Intrinsic Heterogeneity.

\[ T(n) \]

\[ F(n) \]

\( n = 0 \) starts off around 30, significantly above the variances for the \( n = 1 \) and interior equilibrium, then approaches the values of the other equilibrium, before increasing sharply for \( \chi \pi \) above one. In the bottom panel, the smaller two variances correspond to the \( n = 1 \) and \( n = \hat{n} \) equilibria, while the higher line corresponds to the “supply shock” equilibrium \( n = 0 \). These two panels, demonstrate that the \( n = 0 \) equilibrium has the usual trade-off between stabilizing inflation and output volatility. Moreover, it demonstrates that the equilibrium stochastic properties for the economy differ by equilibrium. Essentially, there is a low volatility \( n = 1 \), “demand shock”, equilibrium
Figure 3: Bifurcation diagram: multiple equilibria.

$\rho = 0.5, \mu = 0.5, \alpha = 10000, \kappa = 0.024, \gamma_g = 1.5, \gamma_u = 1.5, \sigma_g = 0.9, \sigma_u = 0.25, \sigma_{gu} = 0$

and a high volatility supply shock equilibrium. These results have important economic implications that we elaborate on further below.

Figure 4 demonstrates that some parameterizations can lead to a unique equilibrium with Intrinsic Heterogeneity. Here we adopt the same parameterization as above except now we set $\kappa = 0.3, \sigma^2_\varepsilon = 0.45$. Now for low values of $\chi_\pi$ we see as in Figure 3 the existence of multiple equilibria: high volatility price shock equilibrium and low volatility demand shock equilibrium. However, for moderate inflation responses, including some that satisfy the Taylor principle, there exists a unique low
volatility demand shock equilibrium. Eventually, for values of $\chi_\pi > 1.4$ the $n = 1$ equilibrium bifurcates and Intrinsic Heterogeneity becomes the unique equilibrium. Thus, under this parameterization, the standard Taylor rule coefficients would lead to diverse beliefs in equilibrium. Eventually, though, as Proposition 5 shows, the price shock equilibrium emerges as the unique outcome.

Figure 4: Bifurcation diagram: intrinsic heterogeneity.

So far we have been focusing on the neoclassical case of $\alpha = +\infty$. This case is appealing theoretically because all agents select only the best performing models. However, with finite $\alpha$ heterogeneity becomes a more pervasive feature of the econ-
There are very good reasons to expect that agents will have finite intensities of choice. Young (2004) argues that a finite $\alpha$, which is equivalent to some randomness in predictor selection, is analogous to mixed strategies in game theory: with some uncertainty about the appropriate model agents can benefit by not being dogmatic in the choice of predictor. To see the impact that heterogeneity can have, we compare large $\alpha$ and small $\alpha$ cases.

We first note that it is possible to have situations where there are multiple equilibria with large $\alpha$ and a unique equilibrium, with diverse beliefs, for small values of $\alpha$. To demonstrate this, adopt the Woodford calibration and the parameter settings in Figure 1, and set $\phi = 0.45, \sigma_v^2 = 0.075$. Figure 5 plots the bifurcation diagram for various values of $\chi_\pi$ for both large and small $\alpha$. The left-most plots illustrate that with large $\alpha$ there can be multiple equilibria. The right plots show that when $\alpha$ takes values less than three ($\alpha = 3$ shown) that there exists a unique equilibrium with Intrinsic Heterogeneity.

Finite values of $\alpha$ can also qualitatively affect a model with a unique equilibrium under large $\alpha$. Figure 6 parameterizes the model as in Figure 1 with the exception of $\phi = 0.45, \sigma_v^2 = 0.075$. Then we imagine a “bad luck” structural change where the persistence and variance of the price shock $u_t$ increases via the new parameter values $\phi = 0.5$ and $\sigma_v^2 = 0.4$. For each of these two scenarios, Figure 7 exhibits the equilibrium properties for various values of the inflation response coefficient $\chi_\pi$. The left-most panels of Figure 7 arise under the case of low price shock variance while the right-most panels occur under the bad luck scenario. When price

4 Business Cycle and Policy Implications

Figures 4 and 3 have important business cycle and policy implications. This section demonstrates two implications: a bad luck story with regime-switching output and inflation variances and the implications of heterogeneity for optimal monetary policy.

4.1 Bad luck

One widely cited empirical finding is that inflation and output volatility in the U.S., especially during the 1970’s, follows a regime-switching process alternating between periods of high and low volatility (see Sims and Zha (2006)). Stock and Watson (2003) present evidence that high inflation and output volatility during the 1970’s coincided with a series of “bad luck” price shocks, e.g. high oil prices. This subsection investigates whether bad luck, in the form of greater persistence and volatility of price shocks, might lead to multiple equilibria and, under real-time learning and dynamic predictor selection, endogenous volatility.

We consider the following experiment. We parameterize the model as in Figure 1 with the exception of $\phi = 0.2, \sigma_v^2 = 0.1$. Then we imagine a “bad luck” structural change where the persistence and variance of the price shock $u_t$ increases via the new parameter values $\phi = 0.5$ and $\sigma_v^2 = 0.4$. For each of these two scenarios, Figure 7 exhibits the equilibrium properties for various values of the inflation response coefficient $\chi_\pi$. The left-most panels of Figure 7 arise under the case of low price shock variance while the right-most panels occur under the bad luck scenario. When price
shocks have low persistence and volatility, there is a unique equilibrium at \( n = 1 \) for all values of \( \chi_\pi \). If, on the other hand, there is an episode of bad luck then there are multiple equilibria for values of \( \chi_\pi < 1.5 \). The bad luck economy, therefore, could be in either a low volatility demand shock equilibrium or a high volatility price shock equilibrium.

Figure 7 also shows that if policy is sufficiently aggressive in response to inflation, i.e. \( \chi_\pi > 1.5 \), the the economy with more volatile price shocks will possess a unique equilibrium. This figure, therefore, captures two empirical features of the U.S. econ-
omy. First, bad luck can de-stabilize the economy in the sense that before the increase in the volatility of price shocks there was a unique equilibrium and afterwards there are multiple equilibria. Second, a monetary policy that responds strongly to expected inflation is stabilizing by coordinating the economy on a unique equilibrium. This second feature of Figure 7, the stabilizing effect of anti-inflationary policy, has often been argued as a key component of the lower volatility during the mid-1980’s-1990’s. The model here explains these phenomena by highlighting the role that monetary policy and the stochastic properties of price shocks play as bifurcation parameters.

The existence of multiple equilibria following a bad luck episode suggests that under a real-time learning and dynamic predictor selection formulation of the model, regime-switching output and inflation volatility might arise endogenously. We examine this possibility by turning to a real-time learning environment. In a Misspecification Equilibrium, agents’ expectations satisfy the least-squares orthogonality conditions, where the expectation is taken with respect to population moments. We now assume that rather than knowing these population moments, and hence the equilibrium values for \( b^1_t, b^2_t \) and the relative predictor fitness measure \( F \), private-sector agents infer their values in real-time from historical data. In particular, we assume that agents use recursive least-squares to generate parameter estimates \( b^1_t, b^2_t \) by regressing the state \( y_t \) on demand or price shocks, respectively. In deciding on which forecasting model to adopt, they must also estimate the unconditional mean-square forecast errors recursively and select that model which delivers the lowest estimated squared forecast error.

Under real-time learning, the economy is generated by an actual law of motion with time-varying parameters:

\[
y_t = \xi_1(b^1_{t-1}, n_{t-1})y_t + \xi_2(b^2_{t-1}, n_{t-1})u_t
\]

where \( b^1_t, b^2_t \) are updated by recursive least squares

\[
\begin{align*}
\hat{b}^1_t &= b^1_{t-1} + \eta_t R^{-1}_{1t} g_t (y_t - b^1_{t-1} y_t) \\
\hat{b}^2_t &= b^2_{t-1} + \eta_t R^{-1}_{2t} u_t (y_t - b^2_{t-1} u_t) \\
R_{1t} &= R_{1t-1} + \eta_t (g_t^2 - R_{1t-1}) \\
R_{2t} &= R_{2t-1} + \eta_t (u_t^2 - R_{2t-1})
\end{align*}
\]

\( \eta_t \) is a deterministic gain sequence such that \( \sum_{t=0}^{\infty} \eta_t = +\infty \). Under recursive least squares \( \eta_t = t^{-1} \) is a decreasing gain and it is possible in many settings, including ours, to show that parameter estimates converge (with probability 1) to their equilibrium values. For the particular experiment under consideration, the \( n = 0, n = 1 \) restricted perceptions equilibria are locally stable under a decreasing gain learning rule. Alternatively, \( \eta_t = \eta \) is a constant gain version that assumes agents discount past data. A constant gain learning rule is desirable in environments where agents may be concerned about structural change. After a bad luck structural change the
economy may be in one of multiple equilibria, so an agent learning about the parameters of their forecasting rule(s) would want to use a constant gain algorithm to remain robust to the possibility of switching between equilibria.

In order to select a particular, recursively updated forecasting model, agents also estimate the unconditional mean square forecast error for each model,

$$EU^1_t = EU^1_{t-1} + \delta_t \left[ (y_t - b^1_{t-1} \rho g_t)^2 - EU^1_{t-1} \right]$$

$$EU^2_t = EU^2_{t-1} + \delta_t \left[ (y_t - b^2_{t-1} \phi u_t)^2 - EU^2_{t-1} \right]$$

where $\delta_t$ is a gain sequence. Following Branch and Evans (2007) we allow $\delta_t \neq \eta_t$ so that agents may be more or less concerned with structural change in predictor fitness than in structural change in model parameters.

To illustrate the real-time learning and dynamic predictor selection dynamics we turn to numerical simulations. We set $\eta_t = 0.03$, $\delta_t = 0.05$, and adopt the parameter values in Figure 7. We first initialize the model by simulating for 5000 periods, with a decreasing gain, allowing the parameters to converge to their $n = 1$ equilibrium values. Then we simulate the model for 8000 periods. The first 4000 periods the economy will be in a good luck scenario with $\phi = 0.2$, $\sigma^2_\nu = 0.1$. Then at period 4001 the economy experiences a bad luck structural change with $\phi = 0.5$, $\sigma^2_\nu = 0.4$. Figure 8 plots a typical simulation.

The top panel of Figure 8 plots the predictor selection in real-time. Prior to period 4000 there is a unique equilibrium, and so even under real-time learning agents coordinate on the demand shock equilibrium. Following period 4000 there are multiple equilibria, and the dynamics of predictor selection switch between the demand and supply shock equilibria as shocks to the system, mitigated through agents’ learning process, switch the economy between basins of attraction.

The bottom two panels of Figure 8 plots a 20 quarter moving average of inflation and output volatility. Prior to period 4000 inflation and output variance are stable and near their low volatility $n = 1$ equilibrium values. Following period 4000, as the economy switches between demand and supply shock equilibria, output and inflation variances fluctuate endogenously between high and low states.

The results in Figure 8 suggest one (possible) interpretation of the high volatility periods in the U.S. as identified by Sims and Zha (2006). The bad luck supply shocks of the 1970’s have been argued, by Stock and Watson (2003) among many others, can account for the high volatility of the 1970’s and the subsequent period of moderation following 1984. However, the standard New Keynesian model can not account for regime-switching volatilities unless the supply shock itself is assumed to follow a regime-switching process. Figure 8 demonstrates that a standard New Keynesian model adapted to take into a very reasonable model of expectation formation, can lead to multiple equilibria when supply shocks switch to a new higher variance process. The greater volatility of supply shocks has the additional indirect effect of leading to the existence of multiple equilibria. Under real-time learning, as agents
learn both about the model parameters and select their models, the economy switches between high and low volatility equilibria. Thus, a bad luck explanation for endogenous volatility. It is important to note that this endogenous volatility arises regardless of whether policy satisfies the Taylor principle. Figure 7 shows that in the bad luck scenario, multiple equilibria exist for both $\chi_\pi$ below and above one.

Figure 7 also demonstrates that the bad luck episode can lead to policy hysteresis. To see this point, imagine that the economy is in a good luck period, policy is passive with $\chi_\pi < 1$ (as found by Clarida, Gali, and Gertler (1999)) and the economy is in the $n = 1$ demand shock equilibrium. Now suppose that a bad luck episode occurs, which bifurcates the model with the existence of multiple equilibria. If policymakers increase $\chi_\pi$ gradually in order to satisfy the Taylor principle, eventually the $n = 1$ equilibrium disappears and the economy will coordinate on the $n = 0$ equilibrium, which has higher output volatility. If after stabilizing inflation policymaker’s then wanted to return to their previous policy by setting lower values for $\chi_\pi$ the economy would remain on the $n = 0$ equilibrium. Hence, there are policy hysteresis effects.

4.2 Optimal Discretionary Policy

Under a contemporaneous policy rule, there are many possible equilibrium outcomes. A natural question then is what if policymakers were to conduct optimal discretionary policy? To address this question, suppose that policymaker’s seek to minimize a quadratic loss function,

$$\sum_{t \geq 0} \beta^t (\omega x_t^2 + \pi_t^2)$$

subject to the aggregate supply curve (2). Under discretion, it is possible to find a nominal interest rate rule that implements the optimal discretionary policy and responds only to the fundamental shocks. This fundamentals based rule was shown by Evans and Honkapohja (2003) to lead to an unstable (and indeterminate) rational expectations equilibrium. Under underparameterized beliefs it is possible to show that the restricted perceptions equilibrium is also unstable when policy is formed via a fundamentals-based rule. Instead, Evans and Honkapohja (2003) propose a rule that responds to private sector expectations and the fundamental exogenous shocks. When monetary policy responds directly to private-sector expectations, under least-squares learning the temporary equilibrium outcomes will converge to a unique rational expectations equilibrium. The good economic properties of an expectations-based rule have been established under the assumption of homogeneous expectations. This subsection considers the properties of expectations-based rules under heterogeneous expectations.

It is possible to show that a policy rule of the form

$$i_t = \kappa^{-1} \hat{E}_{t+1} x_\pi + \left(1 + \frac{\beta \kappa \zeta^{-1}}{\omega + \kappa^2}\right) \hat{E}_{t+1} \pi_t + \frac{\kappa \zeta^{-1} u_t}{\omega + \kappa^2} + \kappa^{-1} g_t$$
will implement optimal discretionary policy under misspecified beliefs. Furthermore, taking the same parameterization as in Figure 1, Figure 9 plots the T-map. The left-most plots of Figure 9 plot the case of the EH-rule that implements optimal discretionary policy. This figure makes clear that the optimal policy rule coordinates the economy on the $n = 0$ equilibrium. This result is expected because optimal policy perfectly offsets demand shocks, leaving the economy only to depend on price shocks. Because the economy is driven entirely by price shocks there is only one possible equilibrium outcome: $n = 0$. Moreover, this equilibrium coincides with the optimal discretionary rational expectations equilibrium.

The right-most plots consider the same parameter values, the same expectations based rule, but where the policymakers do not directly respond to demand shocks $g_t$. In this case, there is a unique equilibrium where agents coordinate on the demand shock model.

5 Conclusion

Most of the literature on monetary policy adheres to the assumption of rational expectations, in which agents have homogeneous expectations. Recent studies of the implications for monetary policy of private agents following adaptive learning rules have for the most part maintained the assumption of homogeneous expectations. However, we know from surveys that heterogeneous beliefs are a salient feature of the data. In our framework, heterogeneous expectations arise naturally because agents adopt one of several competing forecasting models, with the distribution of agents across predictors reflecting the relative success of the alternative forecast rules. In some cases heterogeneity exists even in the limit when agents select only the best predictor. This can arise in particular when monetary policy responds very aggressively to expected output or inflation.

Our framework has turned up new phenomena. Multiple equilibria is possible even when there is a unique rational expectations equilibrium that is stable under least-squares learning. In particular, bad luck in the form of greater persistence and volatility of price shocks, such as occurred in the 1970s, could push the economy from a unique equilibrium to a situation with multiple equilibria. Under real-time learning and dynamic predictor selection the economy would exhibit endogenous volatility. Increasing the policy response to inflation in the Taylor rule, as arguably occurred in the mid 1980s, can eliminate the endogenous volatility.

We also examined optimal discretionary monetary policy, looking in particular at an expectations based-rule known to have good properties under least-squares learning. We found that this rule, designed to respond explicitly to private-sector expectations, continues to perform well in our set-up in the presence of heterogenous expectations. It would be of interest to examine the generality of this finding and to extend the investigation to rules with history dependence of the type discussed in the literature on optimal policy with commitment.
Appendix

Proof of proposition 4. The reduced-form matrix expressions for $A, B, D$ are

$$A = \begin{pmatrix} 1 & \varsigma \\ \kappa & \beta + \kappa \varsigma \end{pmatrix}$$

$$B = \begin{pmatrix} -\chi_x \varsigma & -\chi \pi \varsigma \\ -\chi_x \kappa \varsigma & -\chi \pi \kappa \varsigma \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix}.$$  

Since $b^1 = \xi_1$ and $b^2 = \xi_2$ when $r, \tilde{r} \to 0$ it follows that $\xi_1(0) = (1, \kappa)'$,

$$\xi_2(0) = \left( \begin{pmatrix} \varsigma (\phi - \chi_x) \\ 1 + \varsigma (\chi_x + \kappa \chi_x) - \phi ((1 + \beta + \varsigma (\beta \chi_x + \kappa)) + \beta \phi) \end{pmatrix} \right).$$

Similarly, $\xi_2(1) = (0, 1)'$ and

$$\xi_1(1) = \left( \begin{pmatrix} 1 - \beta \rho \\ 1 + \beta \rho + \varsigma (\chi_x + \kappa \chi_x) - \phi ((1 + \beta + \varsigma (\beta \chi_x)) \right).$$

Evaluating $F(0)$ and $F(1)$ at the above values for $\xi_1, \xi_2$, it can be seen that

$$F(0) < 0 \iff \frac{\sigma_2^2}{\sigma_0^2} > B_0$$

$$F(1) < 0 \iff \frac{\sigma_2^2}{\sigma_0^2} > B_1,$$

from which the result follows.

Proof of proposition 5. For result 1, it is straightforward to see that $\lim_{\chi_x \to \infty} B_1 = \lim_{\chi_x \to \infty} B_1 = 0$. Moreover, using l’Hôpital’s rule

$$\lim_{\chi_x \to \infty} B_0 = \frac{\rho^2 (1 - \phi^2)(1 + \kappa^2)\kappa^2}{\phi^2 (1 - \rho^2)},$$

and

$$\lim_{\chi_x \to \infty} B_0 = \frac{\rho^2 (1 - \phi^2)(1 + \kappa^2)(1 - \beta \phi)^2}{\phi^2 (1 - \rho^2)}.$$  

The result follows. Result 2 follows since $P0$ must hold for $\frac{\sigma_0^2}{\sigma_2^2}$ sufficiently large. Similarly Result 3 follows since $P1$ must hold for $\frac{\sigma_0^2}{\sigma_2^2}$ sufficiently small.
References


Figure 6: Misspecification Equilibria for large and small $\alpha$. 

For Large $\alpha$, we observe a steady increase in variance as $\alpha$ increases. The variance starts at low values and gradually rises to higher values as $\alpha$ approaches its maximum.

For Small $\alpha$, we see a more gradual increase in variance. The variance begins at a slightly higher level than in the Large $\alpha$ case and continues to rise more modestly as $\alpha$ increases.

The graphs illustrate the impact of different values of $\alpha$ on variance, showing how variance changes with respect to $\alpha$ for both Large and Small $\alpha$ scenarios.
Figure 7: Bifurcation diagram: bad luck.

$\mu = 0.2, \sigma_u = 0.1$

$\mu = 0.5, \sigma_u = 0.4$
Figure 8: Bad luck and endogenous volatility.
Figure 9: Optimal discretionary policy and expectations-based rules.