Sovereign Debt and Domestic Economic Fragility

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Abstract

Recent sovereign default episodes have been associated with substantial output costs. In this paper, we construct a parsimonious finite horizon model which captures two key market imperfections: (i) the government’s inability to commit to repay debt; (ii) liquidity constraints in the domestic financial sector. We use the model to answer two sets of questions. First, we characterize the optimal sovereign default decision. The optimal haircut on debt is decreasing in the productivity shock and the exposure of domestic banks to the debt, and increasing in the proportion of the debt held by foreign agents. Second, we characterize the optimal sovereign debt issuance decision. The government can effectively “purchase commitment” by engineering a high exposure of the domestic banking system to defaults, but this has the negative side-effect of limiting international risk diversification benefits. If government debt is tradable between domestic and foreign holders, the government may have to constrict domestic activity and push up interest rates in order to ensure that domestic banks are willing to hold sufficient quantities of the debt. The methodology used enables tractable analysis by effectively reducing the number of state variables, and it can be easily generalized.

Keywords: Sovereign Default, Capital Flows, Emerging Markets, Optimal Government Policy

JEL Classifications: E61, F34, F36, G15

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1 Introduction

For economies which rely on external debt financing, it is of paramount importance to determine optimal government policies regarding both debt issuance and sovereign default. From a theoretical perspective, policy recommendations are highly contingent on the nature of imperfections in domestic and international capital markets. This paper proceeds as follows. To begin with, we construct a parsimonious model which captures the key stylized facts of recent default episodes. Then, we use this model to answer two questions. First, what is the optimal government default decision ex post as a function of economic conditions and the debt structure? Second, what is the optimal government debt issuance decision ex ante – in terms of not only the volume of debt, but also the nature and distribution of debt across different categories of domestic and international debtholders?

Which stylized facts of default episodes should our model attempt to match? Most of the evidence indicates that sovereign default affects the domestic economy, specifically through the domestic financial sector. Sturzenegger and Zettelmeyer (2005) report that both domestic and foreign creditors to the government suffer losses on their holdings of government debt in the event of default. De Paoli, Hoggarth and Saporta (2006) record that sovereign default is often associated with substantial output costs for the domestic economy, especially when the default episode is mired in concurrent banking and/or currency crises.

Sturzenegger and Zettelmeyer (2006) elaborate upon these findings in a detailed survey of defaults and debt restructurings. In the run up to the Russian debt crisis of 1998, domestic banks had increased their exposure to government debt, so that in the first quarter of 1998 income from government securities amounted to 30 percent of total bank income. The default by the government on domestically held public debt was roughly equal to the economy’s aggregate banking capital. In the aftermath of the default, there were runs on some banks. Interbank transactions ground to a halt and the payments system became non-functional. Real GDP fell by 5.3 percent that year, before rebounding in subsequent years. In the Argentinian debt crisis of 2001-2005, 60 percent of the defaulted debt was held by domestic residents. Forced pesification of dollar-denominated assets and liabilities of the financial sector transferred resources from the banks to the government. The banking system became insolvent. Output fell by 3.4 percent in 1999. After the default, it fell by 4.4 percent in 2001 and 10.9 percent in 2002. Clearly, not all the output costs in these default episodes arose from the default decision – in both cases, the default decision was influenced by a prior negative shock to the domestic economy. Nevertheless, the decision by the government to default on its debt contributed to a worsening of the initial crisis, in particular through a disruption to the financial system. The economy rebounded in the years following the default.

In this paper, we construct a finite horizon model which captures two key market imperfections: (i) the government’s inability to commit to repay debt; (ii) liquidity constraints in the domestic financial sector (due to a lack of insurance against shocks to bank capital). The government may issue cash or debt to domestic agents, and may influence the distribution of debt across domestic
and foreign holders. In the event of a sovereign default, the government is constrained to default equally on both domestic and foreign holders of debt. Default on foreign lenders improves the asset position of the country, but default on domestic lenders (the banks) reduces aggregate banking capital, and thereby generates a domestic economic contraction that is increasing in the scale of the default. The model has two periods, and we solve it backward.

First, we characterize the optimal sovereign default decision. The optimal haircut on debt (the proportion of debt that is not repaid) is a function of inherited debt variables and the current productivity shock. The higher is the volume of debt held by domestic banks, the more the domestic economy is exposed to the adverse consequences of default. The optimal haircut is therefore correspondingly lower. The higher the proportion of sovereign debt claims held by foreigners relative to that held by domestic agents, the lower the optimal volume of debt repayments by the government. Notice that the optimal sovereign default decision depends not on the total debt stock of the government, but instead on the ratio of foreign to domestic debtholders. We believe that this comparative static prediction is useful to help us understand the difference between advanced and emerging economies. Finally, our model predicts counter-cyclical default risk: the optimal haircut on debt is larger, the lower is the domestic productivity shock. In the low productivity state, the default decision then reduces output further, amplifying the effect of the productivity shock. This amplification effect is higher for economies with larger external to domestic debt ratios, because the optimal haircut is larger.

In our model, the optimal haircut decision is expressed as a tractable function of inherited debt variables and parameters related to the domestic production function. The utility function is not needed to determine the optimal haircut. The tractability of the model yields the price of government debt, and hence the total external debt level, as tractable functions of government policy and production side variables. If sovereign debt is tradable between domestic and foreign holders, the functional specifications mentioned remain unaffected, but in this case utility function parameters do place a restriction on the feasible set of government policies, and hence on the feasible set of debt levels.

Second, we characterize the optimal sovereign debt issuance decision. The government can influence three key variables through its debt issuance decision: the total volume of government claims (cash and debt) issued, the proportion of debt (as opposed to cash) in domestic banks’ portfolios, and the ratio of debt held by foreigners relative to that held by domestic agents. The government recognizes that its actions affect the domestic economic cost of default in future periods, and it can effectively “purchase commitment” by engineering a high exposure of the domestic banking system to defaults. On the one hand, such exposure is indeed valuable ex post, in enforcing repayment after positive productivity shocks. On the other hand, domestic exposure increases the cost of default after adverse productivity shocks, when debt repudiation is actually beneficial for international risk diversification. The optimal exposure decision balances these countervailing effects on welfare. Given this decision, the optimal ratio of foreign to domestic held debt is determined by balancing another trade-off: a higher ratio may (over some range) enable higher borrowing from abroad, but
also reduces future output because of a concomitant increase in future average haircuts. If government debt is tradable between domestic and foreign holders, the government must also satisfy an equal marginal valuation restriction: it must ensure that both domestic banks and foreign lenders are willing to hold some, but not all of the debt. In other words, it must offer a sufficiently high domestic interest rate on its debt, perhaps by pushing down aggregate domestic credit to firms.

Emerging market economies with higher growth rates may wish to borrow more from abroad than advanced economies. According to our model, countries that wish to issue more external debt must typically increase both the exposure of the domestic economy to debt (as opposed to safe assets such as cash), and the proportion of debt that is held by foreign agents. These render the economy more vulnerable to default, and increase the frequency of default. Such countries may also need to contract the level of domestic activity in order to push up interest rates and persuade domestic banks to hold government debt.

To analyze the optimal government policy problem in a tractable manner, we draw upon methodological insights from the literature on optimal fiscal policy with non-contingent debt – in particular, Werning’s (2003) analysis of the setup in Aiyagari, Marcet, Sargent and Seppälä (2002). Using these techniques, we are able to reduce the number of state variables (from the three government policy variables mentioned above) and split the problem into intertemporal and intratemporal subproblems. The first subproblem concerns the optimal external debt level, while the second focuses on the optimal configuration of government policies to achieve the chosen level of external debt. The approach described in this paper can be easily applied more generally to explore what happens if we endow the government with more policy levers.

How should this two period model be extended to an infinite horizon framework? The most natural extension would be to maintain the assumption that sovereign default affects the domestic economy in a similar manner, but does not lead to reduced access to international capital markets. Then according to the model, optimal default in a particular period would result in lower output in the same period, but higher consumption in current and future periods. Default on external debt would also lead to an enhanced ability to borrow new resources from abroad because it would reduce the total pre-existing external debt stock without affecting the feasible set of external debt levels. This extension is conducted formally in Basu (2009). Clearly, this approach diverges from the bulk of the traditional literature on sovereign default. Following Eaton and Gersovitz (1981), this literature has assumed that default leads to higher consumption in the period of default, but lower welfare in the future due to lost capital market access. Bulow and Rogoff (1989) showed that cut-off from capital markets is not sufficient to sustain debt if governments have access to savings technologies of appropriate contingency. This has spurred a series of insightful papers expanding the scope of reputation. Cole and Kehoe (1998), Eaton (1996) and Kletzer and Wright (2000) construct models where the default decision adversely affects the economy’s future consumption possibilities. Amador (2003) derives changes in the set of future feasible allocations via political economy considerations. The empirical evidence suggests that countries do experience some reduction in capital market access following defaults, but this reduction is temporary (Gelos,
Sahay and Sandleris 2004, Sturzenegger and Zettelmeyer 2006). Therefore, we believe that our paper is complementary to the previous literature. Our model’s exclusive focus on domestic output costs is obviously a simplification, but we believe that such costs are an important ingredient of a more general model of sovereign borrowing.

The prediction of countercyclical default risk is found in some other recent papers. In an infinite horizon model, Arellano (2008) shows that default on non-contingent debt is more likely after adverse endowment shocks, because it is more painful for risk-averse consumers to repay at this point. Our optimal default predictions do not depend on risk aversion, and are related to the production side of the economy instead: the optimal haircut is higher after adverse productivity (rather than endowment) shocks. However, our papers share the feature of non-contingent debt. In contemporaneous work to ours, Mendoza and Yue (2008) also derive the prediction of countercyclical default risk. In their thought-provoking framework, sovereign default also leads to a domestic output cost, although through a mechanism which is quite different to that in our paper. Default leads to a cut-off of market access for domestic firms, which causes an efficiency loss because foreign inputs are replaced by imperfect domestic substitutes. Our focus is instead to build a framework where the government is also allowed to manipulate the output cost of default via its debt issuance decision. We also wish to relate the default decision to the structure of debt chosen.

In our paper, sovereign default generates a domestic output cost via a contractionary balance sheet mechanism within the domestic banking sector. As in Woodford (1990), private agents (here, banks) save between periods using government assets, in this case cash and debt. By assumption, banks cannot receive transfers from the government or from consumers except via repayment of government debt. This means that a haircut on debt results in a reduction of aggregate banking capital, and thereby lower domestic credit and production. The stark liquidity constraint is an extreme assumption. For example, the Argentinian government stepped in to attempt a bailout of the banking system after the default decision and pesification of 2001-2002 had rendered it insolvent. Nevertheless, such bailouts and insurance mechanisms are rarely sufficient to insulate the domestic economy entirely. To the extent that the insurance is imperfect, the mechanism in this paper will be active. What is more, the logic of the paper suggests that in order to be able to sustain more debt, the government would want to commit in advance to an institutional setup which provides poor insurance of the domestic production sector.

In more recent work than ours, Gennaioli, Martin and Rossi (2009) also build a model with balance sheet effects following default. They focus on a different question, in particular the effect of private capital inflows on the incentives of the government to repay debt. The authors show that the existence of private capital inflows amplifies the domestic economic costs of default, and thereby reduce the government’s incentives to repudiate debt. In our model, we abstract from private capital inflows entirely, in order to focus on a tractable characterization of the optimal default decision, and the optimal use of various policy levers by the government.

Other papers have emphasized different ways in which government actions may affect the domestic economy. Broner and Ventura (2006) construct a model where insurance contracts are
written both between domestic agents and between domestic and foreign agents. The government must decide to either enforce all contracts or none. It opts to enforce contracts if the benefits of domestic risk-sharing dominate the costs of making payments to foreigners. Rappoport (2005) elegantly analyzes optimal fiscal policy in a similar framework, where domestic and foreign agents both use government debt instead of insurance contracts for risk-sharing purposes. The government chooses the return on its debt to solve the optimal trade-off between domestic and international risk diversification. The closest part of her paper to ours is her consideration of the effect on government incentives of forcing domestic agents to hold more government bonds ex ante for risk sharing purposes. Our paper focuses on how the government can use various policy variables ex ante (including without having to force agents to hold more bonds than they wish), in a framework with a balance sheet mechanism instead. We believe that this choice of mechanism better matches the stylized facts from the sovereign debt literature, which of course is not Rappoport’s focus. Finally, the exposure mechanism of domestic banks in our paper is related in spirit to Tirole’s (2003) work. In our paper, exposure of domestic banks helps to align the interests of the government and foreign creditors more closely. Tirole considers a different economic environment, where the relationship between domestic firms and foreign financiers more generally affects the government’s incentives to pursue sound policies. He concludes that the promotion of “safer” forms of finance may be insufficient to achieve the required match of interests between stakeholders and the government.

Throughout the bulk of this paper, we assume that the government defaults equally on domestic and foreign debtholders. The appendix considers possible justifications for this setup. Equal haircuts may result from an inability on the part of the government to distinguish between holders of the debt in the period of repayment, or from a legal obligation to repay all debtholders within an asset class equally. Alternatively, even if the government may in principle choose to make different repayments to different categories of debtholders, the existence of secondary markets for debt may constrain its ability to do so. Broner, Martin and Ventura (2006) explore the effects of introducing secondary markets on the ability of the government to issue debt in a finite horizon model. They examine a number of different scenarios, including a version where the government has short-term commitment within periods. It can credibly announce the haircut decision a short interval prior to the execution of haircuts, and secondary markets are open during this interval. Then, the appendix confirms that equal haircuts on foreign and domestic debt are an equilibrium outcome. More generally, the mechanism in our paper operates under the weaker condition that haircuts on domestic and foreign debtholders are positively related.

The remainder of the paper is structured as follows. Section 2 summarizes the model. There are two specifications of interest. In the first specification, government debt is not tradable between domestic and foreign agents in the period of issue. In the second specification, it is tradable. Section 3 solves some benchmark cases of the model (including the first best case). Section 4 summarizes the construction of the program, the theoretical results and numerical simulations for the first specification. Section 5 does the same for the second. Section 6 considers policy implications arising from the model. Section 7 concludes.
2 Model

2.1 Preferences and Technology

The model has two periods, \( t = 1 \) and 2. There are five categories of actors in our framework: consumers, firms, banks, the government and foreign creditors. There is a continuum of consumers and firms, both of measure 1, and a continuum of banks. There is also a continuum of foreign creditors.

Preferences Each consumer is identical, with preferences over consumption streams \( \{c_1, c_2\} \) given by the expression
\[
u(c_1) + \beta \mathbb{E}u(c_2).
\]
\( \beta \in (0, 1) \) is the discount factor and the period utility function is continuously differentiable and strictly increasing: \( u'(c) > 0 \).

The government is benevolent and maximizes the utility of the representative consumer. Firms, banks and foreign creditors are risk neutral and maximize expected profits.

Technology In the first period, each consumer receives an endowment \( y_1 \). In addition, it is possible for the economy as a whole to borrow resources \( z \) from foreign creditors. There is no domestic storable good between periods. Accordingly, the resource constraint for the economy in this period is written:
\[
c_1 \leq y_1 + z.
\]

At the beginning of the second period, each consumer receives an endowment \( y_2 \). Then the economy has access to a production technology, operated by firms. Specifically, the economy can invest \( x \) units of its endowment income in the production sector, which produces \( F(x, \tilde{R}) \) units of output:
\[
F(x, \tilde{R}) = x + \tilde{R} f(x).
\]
\( \tilde{R} \in \mathcal{R} \) is a stochastic productivity variable. Its value is realized at the beginning of the second period. We assume \( \tilde{R} \geq 0 \), with highest and lowest values \( \hat{R} \) and \( \bar{R} \) respectively. The production function \( f(x) \) is strictly increasing and strictly concave up to an input level \( \bar{x} \), and is flat for input levels beyond this:
\[
\begin{align*}
f'(x) &\geq 0, f''(x) < 0 \quad \forall x \in [0, \bar{x}] \\
f(x) &= f(\bar{x}) \quad \forall x \geq \bar{x}.
\end{align*}
\]
\( f(x) \) is twice differentiable. We impose \( \lim_{x \to 0} f'(x) = \infty \), \( f'(\bar{x}) = 0 \) and in addition the condition \( \lim_{x \to 0} xf'(x) = 0 \). The output of the production sector cannot be reinvested in the same sector.

At the end of the second period, the economy makes repayments \( v \) to foreign creditors. The
resource constraint is derived:
\[ c_2 \leq y_2 + \tilde{R}f(x) - v. \]

Foreign creditors maximize profits from their lending to the domestic economy, and they have access to an international riskless asset which yields the interest rate \( r \) between periods. This imposes the following rational expectations restriction across periods:
\[ z = \frac{1}{1 + r} \mathbb{E}v. \]

### 2.2 Market Structure

Figure 1 illustrates the order of events and actions in periods \( t = 1 \) and \( 2 \). This subsection uses the timeline to describe the market structure we impose in our framework.

**Figure 1: Model Timeline**

<table>
<thead>
<tr>
<th>Period 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Endowment ( y_1 ) realized.</td>
</tr>
<tr>
<td>• Government issues debt ( A_d, B_d, B_f ) and transfers proceeds ( T_1 ) to consumers.</td>
</tr>
<tr>
<td>Consumers consume ( c_1 ) goods and save ( s_1 ) in banks.</td>
</tr>
<tr>
<td>Banks invest in government debt ( A_d, B_d ).</td>
</tr>
<tr>
<td>Foreigners purchase government debt ( B_f ).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Period 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Productivity shock ( \tilde{R} ) realized.</td>
</tr>
<tr>
<td>• Government imposes lump sum taxes ( T_2 ) and applies haircut ( h ) on debt ( B_d, B_f ).</td>
</tr>
<tr>
<td>• Banks lend ( x ) to firms.</td>
</tr>
<tr>
<td>Firms borrow and produce ( F(x, \tilde{R}) = x + \tilde{R}f(x) ).</td>
</tr>
<tr>
<td>• Consumers consume ( c_2 ) goods.</td>
</tr>
</tbody>
</table>

**Consumers** Each consumer solves the following maximization problem:

\[
\max_{\{c_1,s_1,c_2\}} u(c_1) + \beta \mathbb{E}u(c_2) \quad (1)
\]

subject to

\[
c_1 \leq y_1 + T_1 - s_1 \quad (2)
\]
\[
c_2 \leq y_2 + T_2 + \Pi_B + \Pi_F + S(s_1, \tilde{R}) \quad (3)
\]
\[
c_{1,2} \geq 0. \quad (4)
\]

In the first period, each consumer decides on its consumption and savings \( \{c_1, s_1\} \), taking transfers from the government \( T_1 \) and \( T_2 \) as given. Savings are deposited in the banks, and yield a gross investment return of \( S(s_1, \tilde{R}) \) in the subsequent period. Each consumer owns an equal share in
all the banks and all the firms that exist in the second period. $\Pi_B$ and $\Pi_F$ denote bank and firm profits respectively.

Consumers, firms and banks cannot borrow from or save abroad.

**Bank Deposit Contracts** Banks compete for savings of the consumers in period 1. They can offer contracts $\chi$ to consumers of the form:

$$\chi : s_1 \rightarrow S(s_1, \tilde{R}).$$

The contract takes $s_1$ from consumers in period 1 and returns $S(s_1, \tilde{R})$ to consumers in period 2. No other transfers between consumers and banks are allowed. Consumers observe the set of contracts available $\{\chi(s_1)\}$ and choose the contract that maximizes their expected utility. In equilibrium, the banks will make zero profits ($\Pi_B = 0$) and they will invest in assets so as to maximize consumer utility. Since there is no storable good between periods, the only means by which the banking system can transfer resources between periods is via the purchase of government-issued securities. The set of available assets is described next.

**Government Debt** In the first period, the government can issue two types of securities, cash $A_d$ and debt $B$. Of the total debt issuance, $B_d$ is purchased by domestic banks and $B_f$ is purchased by foreign creditors. There is no government expenditure in this model. The government may transfer to the consumers any resources raised from debt issuance:

$$T_1 \leq p_A A_d + p_B B_d + q B_f,$$

where positive quantities are used to denote debt. $p_A$ is the price of cash in terms of output. $p_B$ and $q$ are the prices of debt held by domestic banks and foreign creditors respectively. If debt is not tradable between domestic and foreign agents in the period of issue, these prices may differ. If debt is tradable in the period of issue, then:

$$p_B = q.$$

In the second period, the government observes the productivity shock and then decides on its repayments to debtholders. In our model, the government cannot default on cash, and it must default on all lenders by an equal haircut $h$. The haircut is the proportion of the face value of debt that is not repaid. The government imposes lump sum transfers on consumers in order to make repayments on its debt:

$$-T_2 \geq A_d + (1 - h) [B_d + B_f].$$

The key feature of the government is that it cannot commit in period 1 to the level of the haircut $h$ in period 2.
Bank Holdings of Government Debt Cash and debt are issued by the government in the first period. Banks choose their holdings of these categories of securities in order to maximize their profits, taking the prices \( p_A, p_B, q \) as given.

Foreign Creditors The rational expectations restriction imposed in the previous subsection may now be rewritten in terms of the debt variables:

\[
\max_{B_f} \left\{ \frac{1}{1 + r} \mathbb{E}(1 - h)B_f - qB_f \right\}
\]

\[\implies q = \frac{1}{1 + r} \mathbb{E}(1 - h).\]  

(7)

This equation determines the price of foreign-held debt.

Loans Market in Period 2 Since banks enter the second period with holdings of government-issued cash and debt, the government’s haircut decision affects the banks’ asset position. Let

\[ X = A_d + (1 - h)B_d \]

denote the resources in the banking system after the default decision. We assume that the government cannot transfer resources from consumers to banks except through repayment of cash and debt, and that banks have no other means of raising funds from consumers in period 2 (i.e. the banks cannot purchase insurance against shocks to bank capital).

Banks can choose to either hold these resources \( X \) until the end of the second period, or to lend them to firms in a competitive market for loanable funds. In the latter case, firms use the loaned funds as inputs in production and repay the banks with interest before the end of the period. The supply of loanable funds by banks takes the shape illustrated in Figure 2. At the end of the second period, banks transfer the promised units of output \( S(s_1, \tilde{R}) \) to consumers.

Firms take the loan rate for funds \( \rho \) as given and choose to borrow \( x \) units of input in order to maximize profits:

\[
\max_x \left\{ x + \tilde{R}f(x) - \rho x \right\}
\]

\[\implies 1 + \tilde{R}f'(x) = \rho.\]  

(8)

The resulting demand curve is shown in Figure 2. By inspection, the equilibrium loan rate is given by

\[ \rho = 1 + \tilde{R}f'(X). \]
The key constraint that summarizes the market imperfection on the production side of the economy is

\[ x \leq A_d + (1 - h)B_d. \]

In the final period, inputs into the domestic production sector are less than or equal to the total value of repaid cash and government debt. Effectively, we have the following structure. In period 1, consumer savings are invested in cash and government bonds. In period 2, inputs into production are constrained by the gross return on these investments.

### 2.3 Equilibrium Definition

We use the following equilibrium definition in this paper.

**Definition 1** A Rational Expectations Equilibrium for this economy comprises sequences for allocation rules \( \{c_1, s_1, c_2, \{\chi\}, x\} \), prices \( \{p_A, p_B, q, \rho\} \) and policies \( \{A_d, B_d, B_f, h, T_1, T_2\} \) that satisfy:

(a) Consumers choose \( \{c_1, s_1, c_2\} \) to maximize utility (1) subject to the budget constraints (2), (3) and the nonnegativity constraints on consumption (4), taking prices, bank contract offers, government policies and the endowment as given.

(b) Banks offer contract schedules \( \chi : s_1 \rightarrow S(s_1, \tilde{R}) \) in period 1 to maximize expected profits, taking prices and government policies in period 2 as given.

Banks choose lending quantity \( x \) in period 2 to maximize profits, taking the loan rate \( \rho \) as given.

(c) Firms choose borrowing level \( x \) to maximize profits (8), taking the loan rate \( \rho \) as given.

(d) Government chooses \( \{h, T_2\} \) in period 2 to satisfy the government budget constraint (6) in that period, taking \( \{A_d, B_d, B_f\} \) and the shock \( \tilde{R} \) as given.
Government chooses \( \{A_d, B_d, B_f, T_1\} \) in period 1 to satisfy the government budget constraint (5) in that period, taking the price functions \( \{p_A (A_d, B_d, B_f), p_B (A_d, B_d, B_f), q (A_d, B_d, B_f, \rho(x))\} \) and government policies in period 2, \( h\left(A_d, B_d, B_f, \bar{R}\right) \) and \( T_2 \left(A_d, B_d, B_f, \bar{R}\right) \), as given.

(e) All markets clear for the economy. In particular, the markets for cash, debt, goods and loans clear.

(f) Bond prices for foreign-held debt follow rational expectations: \( q (A_d, B_d, B_f) = \frac{1}{1 + \rho} \mathbb{E} \{1 - h\} \), taking the government policy \( h\left(A_d, B_d, B_f, \bar{R}\right) \) in period 2 as given.

Now we turn to the optimal policy problem for the government. In the second period, the government observes the shock to productivity \( \bar{R} \) and then makes a haircut decision. The government lacks commitment: it cannot credibly commit in period 1 to the haircut it will impose in period 2.

**Definition 2** The **Government Problem** is to maximize utility (1) over time consistent rational expectations equilibria. In particular, we must satisfy not only the equilibrium conditions above but also the additional optimization decisions:

(g) Government chooses \( \{h, T_2\} \) in period 2 to maximize \( u(c_2) \) given \( \{A_d, B_d, B_f\} \) and the shock \( \bar{R} \).

Government chooses \( \{A_d, B_d, B_f, T_1\} \) in period 1 to maximize \( u(c_1) + \beta \mathbb{E} u(c_2) \), taking the price functions \( \{p_A (A_d, B_d, B_f), p_B (A_d, B_d, B_f), q (A_d, B_d, B_f, \rho(x))\} \) and government policies in period 2, \( h\left(A_d, B_d, B_f, \bar{R}\right) \) and \( T_2 \left(A_d, B_d, B_f, \bar{R}\right) \), as given.

In this paper, we consider two different scenarios. In the first specification, government debt is not tradable between domestic banks and foreign creditors in the period of issue. In the second specification, debt is tradable in the period of issue. In the latter case, we impose the additional restriction:

\[
p_B = q.
\]

2.4 **Discussion of Model Assumptions**

The two period model summarized above is as parsimonious as possible, yet captures two key market imperfections: (i) the government’s inability to commit to repay debt; (ii) liquidity constraints in the domestic financial sector (due to a lack of insurance against shocks to bank capital). In period 1, the government may issue cash and debt. The former is held by domestic agents, while the latter may be held by domestic banks or foreign creditors. In period 2, the government repays all cash, and is constrained to default equally on both domestic and foreign holders of debt. Default on foreign creditors improves the asset position of the country, and improves consumption. However, default on domestic banks reduces aggregate banking capital, and thereby generates a domestic economic contraction that is increasing in the scale of the default.
Our model imposes a sharp liquidity constraint on the domestic production sector. After a default event, we assume that the government cannot transfer resources from consumers to banks except through repayment of cash and defaultable debt, and that banks have no other means of raising funds from consumers in period 2. This feature of the model is intended to reflect as simply as possible a stylized fact from the empirical literature (for example, the survey of debt restructurings and defaults in Sturzenegger and Zettelmeyer 2006): namely, that governments are typically unable to fully insulate the domestic banks and productive sector in the event of default. Of course, some imperfect level of insurance is possible in reality, for example via attempted bank bailouts. Even though our model captures the extreme case where no insurance of bank capital is possible, all the qualitative features of the model mechanism (and later, of the results) remain valid as long as governments cannot provide full insurance for the banking system in the aftermath of default.

For simplicity, all domestic securities are issued by the government in our model. It can issue both cash and debt. Cash can only be held by domestic agents. Debt in our model can be held by both domestic and foreign lenders. We do not consider types of government-issued securities that can only be held by foreign creditors. In our model, the government will always fully default on the entirety of such debt. In section B of the appendix, this result is proved formally.

For clarity of exposition, the model environment dictates that the government cannot concurrently purchase assets issued by foreign institutions and issue its own debt. This assumption rules out scenarios where the government both saves abroad and issues debt to domestic and foreign lenders. Section C of the appendix considers an environment where this assumption is relaxed. The feasible set of debt levels is unchanged from the model considered in the main text of this paper.

Why does the government default equally on both domestic and foreign debtholders? Section D of the appendix considers possible justifications for equal haircuts for domestic and foreign debtholders. One possible justification is that the government cannot observe who holds its debt. Alternatively, even if the government may in principle choose to make different repayments to different categories of debtholders, the existence of secondary markets for debt may constrain its ability to do so. Broner, Martin and Ventura (2006) show that in a setup where the government lacks commitment to repay its debt, it would like to treat domestic and foreign lenders differently. In the appendix we consider a model related to one of the setups that they consider. We assume the government has “short-term (within-period) commitment.” In period 2, the government can credibly announce the haircut decision a short interval prior to the execution of haircuts, and secondary markets are open between the announcement and execution of the haircuts. On these secondary markets, domestic and foreign lenders may trade the debt with each other. For this sequence of actions, we show that it is an equilibrium for the government to choose the same haircut for domestic and foreign lenders. More generally, the mechanism (and results) in our paper are qualitatively valid under the weaker condition that haircuts on domestic and foreign debtholders are positively related.
Our model has a finite horizon, and has by construction ruled out the possibility of sanctions by foreign lenders in the event of default. This helps to emphasize that the results in this paper regarding the nature and feasibility of sovereign debt do not rely upon reputation effects.

2.5 Nontradable versus Tradable Government Debt

Throughout this paper, we consider two different specifications regarding the tradability of debt in the period of issue. Our motivation for proceeding in this fashion is to clarify the mechanisms operating in our model. Section 4 solves the model for the nontradable debt case. The focus of this section is on the trade-offs that are relevant for determining the optimal exposure of the domestic banks to debt (as opposed to cash), and the optimal ratio of foreign to domestically held debt. The decision of the government regarding the total combined volume of domestic cash and debt is degenerate for the nontradable debt case. Moreover, we show that the feasible set of debt values depends only on production side parameters, not the utility function. Then, we analyze the tradable debt case in section 5. For this section, we have to impose another condition on the feasible set – namely, that both domestic and foreign lenders must find it optimal to hold some, but not all, of the debt stock at the margin. This reduces the size of the feasible set of debt values. In particular, unlike in the nontradable debt case, the discount factor and risk aversion of the representative consumer are now relevant for the characterization of this set. The government’s decision regarding the total value of cash and debt is no longer degenerate. Analysis of the tradable debt case provides us with an understanding of the overall model when all the relevant mechanisms are combined.

3 Benchmark Cases

In this section, we solve two benchmark cases of the model: the first best case and the case where the government cannot default on its debt. Our model can then be compared to these benchmarks. Proofs are contained in the appendix.

3.1 First Best Case

Suppose that the government can both (i) contractually commit in period 1 to the haircut schedule in period 2 (full commitment), and (ii) save abroad and issue debt at the same time. Then the first best is achieved.

The full commitment case is a major difference from the model with lack of commitment studied in the subsequent sections. The requirement that the country also be able to save and borrow at the same time allows the sovereign to make its debt repayment in period 2 fully contingent, so that it may actually make net repayments in high productivity states and receive net transfers from abroad in low productivity states. This configuration is not possible if the government can either save or borrow, but cannot do both contemporaneously.
Proposition 1 (First Best Case) Assume that $y_2$ is sufficiently high. The optimal consumption schedule $(c_1, c_2)$ is the same whether debt is tradable or not. It has the properties:

1. Production by domestic firms is equal to $\bar{x} + \bar{R}f(\bar{x})$ when the productivity shock is $\bar{R}$.

   The optimal allocation solves:
   $$\max_{B_f(h)} \left\{ u \left( y_1 + \frac{B_f}{1 + r} \mathbb{E} (1 - h) \right) + \beta \mathbb{E} u \left( y_2 - (1 - h) B_f + \bar{R}f(\bar{x}) \right) \right\}$$

2. Consumption $c_1$ and borrowing in period 1 are chosen to satisfy the representative consumer’s Euler equation.

3. Consumption $c_2$ is equalized across states of nature $\bar{R}$ in period 2 (by appropriate selection of haircuts in period 2).

At the first best, the total output of domestic firms is at the maximum level in every state of nature $\bar{R}$ in period 2. The output of this sector does vary due to the fluctuation in the productivity shock value. The government fully insures the consumption of domestic consumers against the productivity shock, via state contingent transfers to and from foreigners. The corollary is that repayments to foreigners in period 2 vary across different states of nature.

To achieve the allocation described, the government can issue $A_d \geq \bar{x}$. $B_f$ solves the expression above. $B_d$ is set arbitrarily in the nontradable debt case. In the tradable debt case, $B_d$ is equal to desired debtholdings by domestic banks at the optimal allocation.

3.2 Nondefaultable Debt

Now consider the case where the government is not able to default at all on its debt, whether to foreign or domestic agents. In effect, $h = 0$ for all values of the productivity shock $\bar{R}$. The following proposition applies for this case.

Proposition 2 (Nondefaultable Debt) Assume that $y_2$ is sufficiently high. The optimal consumption schedule $(c_1, c_2)$ is the same whether debt is tradable or not. It has the properties:

1. Production by domestic firms is equal to $\bar{x} + \bar{R}f(\bar{x})$ when the productivity shock is $\bar{R}$.

   The optimal allocation solves:
   $$\max_{B_f \in (-\infty, y_2]} \left\{ u \left( y_1 + \frac{B_f}{1 + r} \right) + \beta \mathbb{E} u \left( y_2 - B_f + \bar{R}f(\bar{x}) \right) \right\}$$

2. Consumption $c_1$ and borrowing in period 1 are chosen to satisfy the representative consumer’s Euler equation for noncontingent and nondefaultable debt.

3. Consumption $c_2$ is increasing in the value of the productivity shock $\bar{R}$. 
Again, the total output of domestic firms is at the maximum level in every state of nature \( \tilde{R} \) in period 2. The maximum output level varies with \( \tilde{R} \). However, in this case the government is not able to fully insure domestic consumers against the productivity shock, because the repayments to foreign creditors are not state contingent in the final period. Therefore, consumption in period 2 is increasing in the value of the productivity shock.

The government can choose \( A_d \geq \tilde{x} \), with \( B_f \) as given above. \( B_d \) is set arbitrarily in the nontradable debt case. For tradable debt, it is equal to desired debtholdings by domestic banks at the optimal allocation.

4 Nontradable Debt

In this section, we characterize and solve the government problem for the case where government debt is not tradable between domestic and foreign lenders in the period of issue. The crucial element of the analysis is the reduction of the number of state variables to just one variable, the total level of real resources raised from abroad in period 1. On the theoretical front, the resulting program can be broken up into two parts: an intratemporal component, which calculates the optimal combinations of debt and exposure for any given level of borrowing from abroad; and an intertemporal component, which determines the optimal level of borrowing in period 1. Both of these subproblems are analyzed. On the numerical side, the reduction of the number of state variables renders the model more tractable for simulations.

Subsection 4.1 characterizes the Ramsey problem for the government in the rational expectations economy. In subsection 4.2, we derive the optimal government default decision in period 2. Subsection 4.3 derives the optimal debt issuance decision in period 1, following a number of steps. First, we characterize the feasible values of external debt, and describe the combinations of government policies that may be used to support each level of debt. Then, we solve for the optimal government policy for each level of external debt, and finally, we derive the optimal total level of external debt as a function of economic variables in period 1. In the final subsection, we also tie together some of the predictions for optimal debt issuance policy.

The proofs of the results in this and subsequent sections are contained in section A of the appendix.

For the remainder of the paper, we make a variable transformation that enables us to visualize more clearly the exposure of the domestic economy to government debt. We may rewrite any combination of government securities issuance \( (A_d, B_d, B_f) \) as a combination \( C = (\alpha, D, B_f) \) such that:

\[
D = A_d + B_d
\]

where

\[
A_d = (1 - \alpha) D
\]

\[
B_d = \alpha D.
\]
$D$ is a measure of the total face value of government-issued cash and debt held by banks at the beginning of period 2. $\alpha$ is the fraction of government debt (as opposed to cash) in total bank assets.

### 4.1 Construction of Government Program

We apply Definitions 1 and 2 to derive the program for the government problem. In period 1:

\[
U_1 = \max_{c_1, \alpha, D, B_f} \left\{ u(c_1) + \beta \mathbb{E} U_2 \left( \alpha, D, B_f, \tilde{R} \right) \right\}
\]

subject to

\[
c_1 = y_1 + q B_f
\]

\[
c_1 \geq 0
\]

\[
q = \frac{1}{1 + r} \mathbb{E} \left\{ 1 - h \left( \alpha, D, B_f, \tilde{R} \right) \right\}
\]

\[
B_f < 0 \Rightarrow \alpha = 0,
\]

where the expression for the period utility in period 2 is given by

\[
U_2 \left( \alpha, D, B_f, \tilde{R} \right) = \max_{c_2, h} u(c_2)
\]

subject to

\[
c_2 = y_2 - (1 - h) B_f + \tilde{R} f \left( \left[ (1 - \alpha) + (1 - h) \alpha \right] D \right)
\]

\[
c_2 \geq 0
\]

\[
y_2 \geq (1 - \alpha) D + (1 - h) \left[ \alpha D + B_f \right]
\]

\[
0 \leq h \leq 1.
\]

In period 1, the government may borrow or save abroad. Each combination $C = (\alpha, D, B_f)$ corresponds to a default schedule across states $h \left( \alpha, D, B_f, \tilde{R} \right)$ in the next period, and hence to the bond price function $q = Q(\alpha, D, B_f)$. This function is calculated using rational expectations over the default schedule in period 2, and it is taken as given by the government in period 1.

Expression (9) states that government repayments of cash and debt in period 2 must be less than or equal to the consumer endowment in that period. For the remainder of this paper, we assume that $y_2$ is large enough so that this constraint is never binding. This approach is valid as long as the repayments to foreigners at the optimum have a finite upper bound irrespective of the productivity shock $\tilde{R}$. The condition we imposed on the production function in section 2.1, $\lim_{x \to 0} x f'(x) = 0$, is sufficient to ensure that this holds (for more details, see the proof of Proposition 4 below).

An important observation to make from the program above is that the haircut decision in the
last period can be analytically derived. Simply apply the first order condition with respect to
the haircut for interior values of $h$, and apply the boundary condition as required for values of $h$
that are not interior. In the next subsection, we examine the expression for the haircut. For the
purposes of the analysis in this subsection, it suffices to note that the expression for the haircut
may be written:

$$ h = H \left( \alpha, D, B_f, \bar{R} \right). $$

In turn, this means that we can also derive the expression for the bond price schedule:

$$ Q(\alpha, D, B_f) = \frac{1}{1 + r} \mathbb{E} \left\{ 1 - H \left( \alpha, D, B_f, \bar{R} \right) \right\}. $$

Observe also that consumption in period 1 depends on the combination $(\alpha, D, B_f)$ only to
the extent that it affects the total real resources raised by the government from foreign creditors
$z = qB_f$. Therefore, we can rewrite the problem as one in which the government chooses how much
to raise from abroad $z$, and then decides the optimal combination $(\alpha, D, B_f)$ that achieves this level
of borrowing. The optimal combination $(\alpha, D, B_f)$ is decided before the state of nature in period
2 is realized, therefore we may rewrite the government problem as follows:

$$ V_1 = \max_{c_1, z} \mathbb{E} \left\{ u(c_1) + \beta V_2 (z) \right\} $$

subject to

$$ c_1 = y_1 + z $$
$$ c_1 \geq 0 $$
$$ z \in G, $$

where we define $V_2 (z)$ as follows:

$$ V_2 (z) = \max_{c_2, \alpha, D, B_f} \mathbb{E} \left\{ u(c_2) \right\} $$

subject to

$$ c_2 = y_2 - (1 - h)B_f + \bar{R}f \left( (1 - \alpha) + (1 - h)\alpha \right) D $$
$$ h = H \left( \alpha, D, B_f, \bar{R} \right) $$
$$ z = Q(\alpha, D, B_f) \cdot B_f $$
$$ z < 0 \Rightarrow \alpha = 0, $$

for some set $G$. Our notation suppresses the dependence of $h$ on $\left( \alpha, D, B_f, \bar{R} \right)$ in the consumption
equation. Note that the combination $C = (\alpha, D, B_f)$ is still chosen before the productivity shock
in period 2 is realized.

This formulation separates the problem into two subproblems. The intertemporal component of
the problem concerns how much to borrow in the initial period, \( z \), in order to smooth consumption between periods. The intratemporal component takes the default decision \( h(\alpha, D, B_f, \bar{R}) \) in the final period as given, and uses this information in order to calculate the optimal combination \((\alpha, D, B_f)\) for the chosen \( z \) value. Section A of the appendix explains the generation of the set of feasible debt values \( G \). For sufficiently high \( y_2 \), the set can be characterized using only the relation \( Q \).

4.2 Optimal Government Default Decision

In this subsection, we characterize the optimal haircut in period 2 as a function of inherited debt variables and the productivity shock. The main result of this subsection is Proposition 3, which is illustrated in Figure 3.

**Proposition 3 (Optimal Haircut Decision)** For \( B_f > 0 \), the optimal haircut decision \( h = H(\alpha, D, B_f, \bar{R}) \) satisfies the following formulation:

\[
h = \max \{0, \min \{1, \theta\}\}
\]

where \( \theta \) satisfies

\[
\frac{B_f}{R\alpha D} = f'(\{1 - \alpha\} + (1 - \theta)\alpha D).
\]

(10)

For \( B_f = 0 \), the optimal haircut is zero if \( D \leq \bar{x} \). For \( D > \bar{x} \), any haircut such that total post-default bank capital remains weakly above \( \bar{x} \) is optimal.

![Figure 3: Optimal Haircut Decision](image)

The haircut is selected by the government to maximize consumption in period 2. On the one hand, a higher haircut benefits consumption by reducing repayments abroad. On the other hand, if the haircut reduces total post-default banking capital below \( \bar{x} \), then it also leads to a contraction.
in domestic credit and output. This output cost is increasing and convex in the size of the haircut, and directly reduces consumption.

First, assume that the optimal haircut is interior. Equation (10) is the appropriate formula for this case. How is this illustrated in Figure 3? Each ratio of foreign to domestically held debt \( \frac{B_f}{D} \) and productivity shock \( \tilde{R} \) corresponds to a horizontal line in the diagram. The intersection of this line and the marginal product function \( f'(x) \) is the optimal haircut. Comparative static results follow immediately. For any given ratio of foreign to domestically held debt and productivity shock, the optimal haircut is lower if the proportion of debt (as opposed to cash) in domestic banks’ portfolios, \( \alpha \), is higher. This is because in this case, the domestic economy is more exposed to the adverse consequences of default. For any given levels of cash and debt in domestic banks’ portfolios, the haircut is larger if the ratio of foreign to domestically held debt, \( \frac{B_f}{D} \), is higher. Therefore, the relevant variable for default risk is not the total volume of debt, but the ratio of foreign to domestically held debt. If the government’s total debt stock is large but most of it is held by domestic agents, then the default risk is low. This prediction is useful to help us understand the difference between advanced and emerging economies. A lower productivity shock \( \tilde{R} \) corresponds to a higher horizontal line on the diagram. The model yields countercyclical default risk: the optimal haircut is high when the realized productivity shock is low.

Next, consider cases when the optimal haircut is not interior. If the level of cash held by domestic banks is equal to \( (1 - \alpha^I)D^I \), then \( h = 1 \) will be binding after the lowest productivity shock. If the total debt level is equal to \( D^{II} \), then \( h = 0 \) will be binding in the highest productivity state.

Finally, notice that the optimal haircut decision in our model is a tractable function, related both to inherited debt variables and parameters related to the domestic production function. Utility function parameters are not relevant for the optimal haircut.

### 4.3 Optimal Government Debt Issuance Decision

The analysis below follows a number of steps. First, we focus on feasible allocations. We characterize the feasible values of external debt \( z \), and then we present some results regarding the government policy combinations \( (\alpha, D, B_f) \) that may be feasibly used to support each level of debt \( z \). Next, we turn to optimal allocations. We solve for the optimal government policy \( (\alpha, D, B_f) \) for each level of external debt \( z \), and finally we derive the optimal total level of external debt \( z \) as a function of the endowment \( y_1 \) in period 1. In the final subsection, we tie together some of the model’s predictions for optimal debt issuance policy.

#### 4.3.1 Feasible Levels of External Debt \( z \)

The tractability of the haircut function means that we can also derive the price of foreign held debt \( q \), and the total external debt level \( z \), as tractable functions of government policy choices \( (\alpha, D, B_f) \). When sovereign debt is not tradable between domestic and foreign creditors in the period of issue,
there is no further restriction on the set of feasible combinations \((\alpha, D, B_f)\) of government policies. Therefore, the set of feasible external debt values depends purely on production side variables.

**Proposition 4 (Feasibility of External Debt)** It is feasible for the sovereign to issue external debt in period 1. The feasible set of debt is \([0, z_{\text{max}}]\), where the maximum feasible level of external debt solves:

\[
    z_{\text{max}} = \max_{\gamma} \frac{\gamma}{1 + r} \sum_{R \in \mathcal{R}} (f')^{-1}\left[\frac{\gamma}{R}\right] \Pr(\tilde{R}).
\]

The assumption on the production function, \(\lim_{x \to 0} x f'(x) = 0\), ensures that \(z_{\text{max}}\) is finite.

Let us now turn to some special cases to illustrate how this result is related to the optimal haircut function derived in the previous section.

**Case 1: \(\alpha = 0\)** Domestic banks only hold cash, and all debt is held by foreigners. Consumption in period 2 is given by

\[
    c_2 = y_2 - (1 - h)B_f + \tilde{R}f(D).
\]

The optimal haircut is \(h = 1\) for all realizations of the productivity shock \(\tilde{R}\). The price of debt in period 1 is given by rational expectations:

\[
    q = \frac{1}{1 + r} \mathbb{E}\left\{1 - h \left(\alpha, D, B_f, \tilde{R}\right)\right\} = 0,
\]

which immediately yields the result that \(z = 0\). No debt can be sustained without exposing domestic banks to the adverse consequences of default.

**Case 2: \(\alpha = 1, D = \bar{x}, B_f > 0\)** Cash does not exist in the economy, so domestic banks must invest solely in government debt. Consumption in period 2 is given by

\[
    c_2 = y_2 - (1 - h)B_f + \tilde{R}f((1 - h)\bar{x}).
\]

The optimal haircut is \(h \in (0, 1)\) for all realizations of the productivity shock \(\tilde{R}\). The price of debt in period 1 is given by rational expectations:

\[
    q = \frac{1}{1 + r} \mathbb{E}\left\{1 - h \left(\alpha, D, B_f, \tilde{R}\right)\right\} > 0.
\]

For \(B_f > 0\), this means that \(z > 0\). The sovereign can raise resources from abroad in period 1 using such a combination of government debt issuance policies.

Therefore, the specific functional form of the optimal default decision derived in the previous section is important for the feasibility of external debt. In this section, we show that it is also crucial for the government’s optimal debt issuance decision in period 1.
Notice that contrary to the bulk of the literature on sovereign debt, we have assumed that default by the sovereign does not lead to any sanctions by foreign creditors. Nevertheless, debt can be sustained in period 1 because default hurts the domestic production sector in period 2. This makes the government more reluctant to initiate default in period 2, and thereby supports a non-zero price for debt held by foreigners.

### 4.3.2 Feasible Combinations $(\alpha, D, B_f)$ for each debt level $z$

In this subsection and the next, we focus on the intratemporal dimension of the problem. In other words, we take the level of $z$ as given and find the sets of feasible and optimal combinations $(\alpha, D, B_f)$ that raise this level of resources from abroad in period 1. Let us begin by describing the most important features of the set of feasible combinations.

When the government chooses to save rather than borrow, i.e. $z < 0$, the government is constrained to issue only cash.

**Proposition 5 (Saving)** For $z < 0$, the government chooses: (i) $\alpha = 0$; (ii) $D = D$; (iii) $B_f = (1 + r)z$.

Now let us focus on the case where $z > 0$. If domestic and foreign debtholders are not allowed to trade the debt in the period of issue, all combinations $(\alpha, D, B_f)$ can feasibly be achieved by the government. Nevertheless, without loss of generality we state a simple Lemma that allows us to focus on a restricted subset of $D$ values.

**Lemma 1** For any combination $C = (\alpha, D, B_f)$ such that $D > \bar{x}$, there exists some other combination $C' = (\alpha', D', B'_f)$ where $D' = \bar{x}$, such that $C'$ raises the same revenues as $C$ in period 1 and is equivalent to $C$ in terms of repayments abroad, output and hence consumption for all values of the productivity shock $\tilde{R}$ in period 2.

**Corollary 1** We can restrict our attention to combinations $C = (\alpha, D, B_f)$ such that $D \in [0, \bar{x}]$.

For intuition, consider the case where the government issues no cash at all, i.e. $\alpha = 1$. In this case, if the quantity of total domestically held debt exceeds $\bar{x}$ in magnitude, the government can default on the portion $(D - \bar{x})$ for every realization of the productivity shock $\tilde{R}$ in period 2 without any adverse output effect. Indeed, it will exercise this option. Any issuance of debt in excess of the output-maximizing value $\overline{D}$ merely increases the haircut on debt for every shock realization $\tilde{R}$, and therefore depresses the price of foreign held debt in period 1. The same total revenues may be raised from abroad by issuing the output-maximizing level of domestic debt and a lower volume of foreign debt. The appendix formalizes this intuition and shows that an amended argument can be applied for any possible configuration $(\alpha, D, B_f)$. Therefore, we can restrict our attention to the set $\{(\alpha, D, B_f) : D \in [0, \bar{x}]\}$.

The sovereign can borrow resources from abroad in period 1 only if it can be relied upon to make some level of repayments in period 2. Clearly, the government can only persuade foreigners to hold
debt in period 1 if domestic banks are also holding some of the debt, so that the domestic production sector is exposed to the adverse consequences of default. This ensures that the government does not default fully on its debt in period 2. What level of domestic exposure $\alpha$ is needed in order to raise any given level of debt $z$ from abroad? The answer to this question is characterized in the next proposition.

**Proposition 6 (Minimum Domestic Exposure)**  Fix $D = D'$. For any level of borrowing in the set $[0, \max(D')]$ to be achieved, it is required that the level of domestic exposure is sufficiently high, i.e., $\alpha \in [\alpha(z), 1]$. The necessary exposure level has the following properties:

1. $\alpha(0) = 0$.
2. $\alpha(z)$ is weakly increasing in $z$.
3. $\alpha(\max(D')) = 1$.

Increasing the exposure of the domestic economy to the adverse effects of default reduces the optimal haircut on debt in period 2, by raising the costs of default relative to the benefits. This in turn sustains more external debt in the first period. In other words, although the government suffers from a problem of lack of commitment, it can effectively “purchase commitment” by increasing the vulnerability of the domestic economy to a default episode. This vulnerability is what sustains foreign debt issuance, and is a necessary side-effect of the sovereign’s lack of commitment. The shape of the function $\alpha(z)$ depends on the production function.

We characterize the minimum domestic exposure function $\alpha(z)$ numerically, using the following production function:

$$f(x) = \begin{cases} x^\theta - \delta \bar{x} & \text{for } x \leq \bar{x} \\ \bar{x}^\theta - \delta \bar{x} & \text{for } x > \bar{x}, \end{cases}$$

where $\bar{x}$ is set to the value that maximizes $f(x)$, i.e., $\bar{x} = \left(\frac{\theta}{\delta}\right)^{\frac{1}{\theta - 1}}$. This production function satisfies the assumptions in subsection 2.1. In addition, it satisfies the property that $\lim_{x \to 0} f'(x) = 0$.

Parameter values are selected as follows. The riskless rate of return is set to $r = 0.05$ and the production function parameters are $\theta = \delta = 0.5$. The implied $\bar{x}$ is therefore equal to unity. In each period, there are ten possible values of the productivity shock, which occur with equal probability. Possible values of the shock are located between $R = 8$ and $\bar{R} = 12$, with equal intervals between the possible shock realizations. Figure 4 plots the production function for different values of the shock. With the above parametrization, the upper bound of the set $G$ is $\max = 1.1857$. This is the highest value of external debt the economy can support. Figure 5 plots the function $\alpha(z)$ for this production function, setting $D = \bar{x}$. It satisfies the properties described in the above proposition.
For any given level of external debt $z$, there exists a set of combinations $(\alpha, D, B_f)$ that achieve this level of borrowing. Within this set, suppose that we choose a lower level of domestic exposure $\alpha$. What does this imply for the choice of $B_f$?

**Proposition 7 (Exposure and Foreign Debt Issuance)** Consider a combination $C = (\alpha, D, B_f)$ which raises debt $z$ such that $\alpha > \underline{\alpha}(z)$. We assume that given $\alpha$ and $D$, the level of $B_f$ chosen is the minimum necessary to support external debt level $z$. Then at the margin, it is feasible to raise the same level of debt $z$ by reducing $\alpha$ and increasing $B_f$. This new combination $C' = (\alpha', D, B'_f)$ has the following properties:

1. $\mathbb{E} \left[ c_2 (\alpha', D, B'_f) \right] < \mathbb{E} [c_2 (\alpha, D, B_f)]$.
2. $\frac{d}{dR} \left\{ c_2 (\alpha', D, B'_f, \bar{R}) - c_2 (\alpha, D, B_f, \bar{R}) \right\} < 0$.
3. It is possible that $c_2 (\alpha', D, B'_f, \bar{R}) > c_2 (\alpha, D, B_f, \bar{R})$ for the worst productivity shocks $\bar{R}$.

Our starting point is a combination $(\alpha, D, B_f)$ where given $\alpha$ and $D$, the level of $B_f$ chosen is the minimum necessary to support external debt level $z$. We could examine comparative statics from other starting points for $B_f$, but the one chosen maximizes consumption in every state of nature $\bar{R}$ in period 2. Therefore, other starting points cannot be part of the optimal government debt issuance decision (which is formally characterized in the next subsection).

If we wish to sustain the same level of external borrowing $z$ as at this starting point, it is only possible to reduce the level of domestic exposure $\alpha$ if there is a concomitant increase in $B_f$. Such a deviation reduces average consumption in period 2. However, due to the lower domestic exposure, consumption is depressed less in states of nature which feature a worse productivity shock. Moreover, if the upper bound for the haircut is binding for the lowest productivity shock

Figure 5: Debt Level $z$ and Minimum Domestic Exposure Level $\underline{\alpha}(z)$
realizations, consumption necessarily increases in these states of nature in period 2. Even for the cases where the upper bound for the haircut is not binding, there exist permissible production functions $f(x)$ such that the combination of deviations actually increases consumption for the worst productivity shock realizations. Therefore, even if average consumption falls, consumption levels in the worst states of nature are more insulated and may increase.

4.3.3 Optimal Combinations ($\alpha, D, B_f$) for each debt level $z$

Having described the set of feasible combinations ($\alpha, D, B_f$) for each debt level $z$, let us now turn to the first optimal policy question: what is the optimal government policy combination ($\alpha, D, B_f$) for each external debt level $z$? Then in the next subsection, we use the results derived here as an input into the amended Euler condition for our model. This condition is used to finally determine the optimal level of debt $z$ chosen by the government.

Proposition 8 (Optimal Total Domestic Issuance) *It is an optimum to set $D = \bar{x}$.*

If debt is not tradable between domestic and foreign holders in the period of issue, then the government can issue securities to domestic banks in period 1 without having to worry that banks will sell the debt on to foreign agents. Therefore the government can choose $B_d = \alpha D$ and $B_f$ independently. What is the optimal choice of $D$? When the government makes its ex ante securities issuance decision in period 1, it anticipates that it will harm domestic production ex post by defaulting in some states of nature in period 2. However, for any productivity shock $\tilde{R}$ in period 2, the government does not want domestic production to be any lower than the level implied by the chosen haircut. Therefore, it optimally sets the total level of cash and domestically held debt ex ante such that the zero bound for the haircut is never binding ex post. Applying Corollary 1, the government sets $D = \bar{x}$.

Proposition 9 (Optimal Domestic Exposure) *The optimal domestic exposure level $\alpha$ for any given external debt level $z$ depends on the risk aversion of the representative consumer.*

This result follows directly from Propositions 6 and 7. Each external debt level $z$ can be supported by a range of government policy combinations ($\alpha, \bar{x}, B_f$). Within this set, for any given $\alpha$, the level of $B_f$ chosen is the minimum necessary to support a given external debt level $z$. Suppose that the domestic exposure level $\alpha$ is higher than $\alpha(z)$. From this point, the only way to reduce domestic exposure $\alpha$, while still supporting the same level of external debt $z$, is by increasing foreign debt issuance $B_f$. At the margin, such a combination of deviations reduces average consumption in period 2, but the effect is dampened for consumption after the worst productivity shocks. Indeed, for the worst productivity shock realizations, consumption may even increase. For this case, such a combination of deviations is actually analogous to purchasing a contract (at a premium) which offers insurance against the period 2 productivity shock $\tilde{R}$. Clearly, this “insurance purchase” is optimal if the representative consumer is sufficiently risk averse.
The minimum domestic exposure level $\alpha(z)$ depends only on the domestic production function. The optimal domestic exposure level also depends on the utility function.

We illustrate Proposition 9 numerically using the same parametrization of the production function that was used to show the minimum domestic exposure result of Proposition 6. We use the following functional form for the utility function:

$$u(c) = \frac{-e^{-\psi c}}{\psi}$$

The value of the endowment income in period 2 is set to $y_2 = 9$ and the discount factor is $\beta = 0.8$. Figure 6 plots the optimal domestic exposure level $\alpha$ as a function of the debt level $z$, for $\psi = 10$, 60 and 100. As the coefficient of absolute risk aversion $\psi$ increases, the optimal domestic exposure level for any given external debt level weakly decreases. For $\psi = 10$, the optimal exposure is $\alpha = 1$ for all debt levels. For $\psi = 100$, the optimal domestic exposure level is very close to the minimum feasible exposure level $\alpha(z)$.

For the special case of a risk neutral representative consumer, $u(c) = c$. From the numerical simulations, it is clear that for this case, the optimal domestic exposure is $\alpha = 1$ for all debt levels $z$.

![Figure 6: Optimal Domestic Exposure Level](image)

What is the optimal foreign debt issuance level $B_f$ for any given external debt level $z$? Proposition 7 indicates that the answer to this question depends on both the particular value of external debt $z$, and the optimal choice of the domestic exposure level $\alpha$. The optimal choice of the domestic exposure level $\alpha$ is already described above. For simplicity and to avoid repetition, let us now focus instead on the relation between the optimal foreign debt issuance $B_f$ and the external debt level $z$ when the optimal domestic exposure level $\alpha$ itself follows an uninteresting path.

**Proposition 10 (Optimal Foreign Debt Issuance)** Suppose that $u(c)$ and $f(x)$ are specified
such that the optimal level of domestic exposure is $\alpha = 1$ for all levels of external debt. Then at the optimum, for $z \in [0, z_{\max}]$:

1. $B_f$ is increasing in $z$.

2. The interest rate on government debt is increasing in $z$ and $B_f$.

As argued above, for any given level of external debt $z$, the level of $B_f$ chosen at the optimum is the minimum necessary to support that debt level. Therefore, as the desired external debt level $z$ increases, it is also necessary for the government to increase the total volume of foreign debt issuance $B_f$. What is the effect of this increase in $B_f$ for the price of, and interest rate on, foreign held government debt? An increase in foreign debt issuance $B_f$ raises the ratio of foreign to domestically held debt, which in this case is given by $\frac{B_f}{x}$. Since this is associated with a higher average haircut in period 2, it is also associated with a lower bond price and a higher interest rate in period 1.

Notice that it is not possible to make the level of external debt $z = qB_f$ arbitrarily large via increases in foreign debt issuance $B_f$. Although such increases do expand the face value of foreign held debt, after some point any further increases always cause a more-than-proportionate decrease in the bond price $q$. This feature is ensured by the condition imposed on the production function in section 2.1, $\lim_{x \to 0} xf'(x) = 0$. Therefore, $z_{\max}$ is finite-valued.

![Figure 7: Optimal Foreign Debt Issuance and Interest Rates](image)

We implement the model numerically using the functional forms for utility and production functions described above. We set the coefficient of absolute risk aversion $\psi$ equal to 10. As mentioned above, the upper bound of the set $G$ is $z_{\max} = 1.1857$. Panel a of Figure 7 captures the evolution of foreign debt issuance $B_f$ as a function of the total level of resources raised from abroad $z$. As the level of external debt grows, the magnitude of $B_f$ required increases at a faster rate. This is because as foreign debt issuance increases, the bond price falls—so to achieve a given increase in external debt, more and more foreign debt issuance is necessary. The fall in the bond
price is reflected in the increasing interest rate schedule shown in panel b. The blue line shows the promised interest rate on foreign held debt, while the green line plots the rate in the absence of any default risk.

4.3.4 Optimal Level of External Debt $z$

In the previous subsection, we have derived results regarding the optimal government policy combination $(\alpha, D, B_f)$ for each level of external debt $z$. In other words, we solved the intratemporal subproblem. In this subsection, we use this information to answer the central intertemporal problem: what is the optimal total level of external debt $z$ in period 1?

The government’s problem may be written as follows.

$$\max_z \mathbb{E} \left\{ u(y_1 + z) + \beta u \left( c_2 \left( z, \tilde{R} \right) \right) \right\}$$

subject to

$$y_1 + z \geq 0$$

$$c_2 \left( z, \tilde{R} \right) \geq 0$$

$$z \in G.$$  

We define $c_2 \left( z, \tilde{R} \right)$ to be the optimal schedule of consumption across states of nature $\tilde{R}$ in period 2, for any given external debt level $z$ chosen in period 1. To be precise: for any given debt level $z$, we choose the optimal combination $C^* = \left( \alpha^*, D^*, B_f^* \right)$ that achieves this debt level; the schedule of consumption (across period 2 states of nature) that corresponds to this combination is denoted $c_2 \left( z, \tilde{R} \right)$. For the main text, we simplify matters by assuming that $c_2 \left( z, \tilde{R} \right)$ is continuous in $z$; the appendix discusses the more general case.

We focus on the choice of external debt level $z$ in the government’s intertemporal subproblem, and suppress the dependence of $c_2$ on $\left( z, \tilde{R} \right)$. The Euler equation for $z$ in the interior of $G$ is written:

$$u' \left( y_1 + z \right) = \beta \mathbb{E} \left\{ u' \left( c_2 \right) \cdot \left| \frac{dc_2}{dz} \right| \right\}, \quad (11)$$

and the equality is replaced with an inequality $\geq$ for $z$ at the upper boundary of $G$.

For $z < 0$, Proposition 5 applies and the Euler equation reduces to the standard formula for nondefaultable noncontingent assets, with $\left| \frac{dc_2}{dz} \right| = 1 + r$.

In the range of debt $z \geq 0$, $\frac{dc_2}{dz}$ depends on the evolution of the optimal combination $C^* = \left( \alpha^*, D^*, B_f^* \right)$ as $z$ changes. At the optimum, the total domestic holdings of cash and debt satisfy $D^* = \bar{x}$. To make it as easy as possible to understand the government’s intertemporal problem, let us focus on the simple case where $u(c)$ and $f(x)$ are defined such that the optimal level of domestic exposure is $\alpha = 1$ for all levels of external debt.
Proposition 11 (Intertemporal Problem) Suppose that $u(c)$ and $f(x)$ are specified such that the optimal level of domestic exposure is $\alpha = 1$ for all levels of external debt. Then for $z \in [0, z_{\text{max}}]$:

1. $\frac{dc}{dz} < 0$ for all values of the productivity shock $\tilde{R}$.
2. $\left| E \left[ \frac{dc}{dz} \right] \right| > 1 + r$.
3. $\frac{dc}{dz}$ is increasing in the value of the productivity shock $\tilde{R}$.

Now let us interpret the three parts of the proposition, one by one. The sign of the derivative in the first result indicates that an increase in external debt $z$ in period 1 necessarily involves a reduction in consumption for every value of the productivity shock $\tilde{R}$ in period 2. This is easy to explain. In our environment, an increase in $z$ necessitates an increase in foreign debt issuance $B_f$, and hence in the ratio $\frac{B_f}{x}$. Therefore, in every state of nature in period 2, the optimal haircut increases and the domestic consumption level falls.

The second result concerns the domestic consumption cost of issuing external debt. In an environment with non-contingent debt and full commitment, an extra unit of external debt in period 1 is associated with an additional repayment abroad in period 2 of $1 + r$ units of consumption goods. In our model, the sovereign cannot commit to repay debt in period 2. Therefore, it must “purchase commitment” to repay by exposing the domestic economy to an output cost in the event of default. The true cost of external debt issuance therefore takes into account not only the repayments abroad in period 2, but also the domestic output costs (due to optimal default decisions) suffered by the economy in period 2. The second result contained in the proposition above states that the true marginal cost of external debt issuance exceeds the gross riskless return $1 + r$. So in expected terms, foreign lenders receive the riskless rate on their sovereign debtholdings, but the domestic economy pays a cost that exceeds the repayments made to foreigners. In other words, the cost of “purchasing commitment” is a series of domestic deadweight losses across states of nature in the final period.

To illustrate this, consider the special case when the domestic representative consumer has an infinite intertemporal elasticity of substitution and $\beta(1 + r) < 1$. Does the representative consumer borrow up to the maximum borrowing limit? For the case with non-contingent debt and full commitment to repay, the trivial answer is yes. For the case with non-contingent debt but no commitment, the maximum borrowing limit may not be binding – because the expected period 2 consumption cost of borrowing exceeds $1 + r$.

Now we return to the general case. Does the economy pay the same consumption cost in every state of nature in period 2? The third result of the proposition establishes that it does not. Even though a higher external debt level harms domestic consumption for all values of the productivity shock $\tilde{R}$ in period 2, the reduction in consumption is largest in the best states of nature and less severe in the worse states. In other words, the ability to default on nominally non-contingent debt means that the actual domestic cost of debt issuance varies across states of nature in a direction
that we would expect from fully contingent debt. However, there is no reason why the magnitude of the variations in the domestic consumption should be equal to the variations for the first best case. For the full commitment problem with perfectly contingent debt, the repayments made in a particular state of nature would depend on the productivity shock realization \( \tilde{R} \) and the maximum value of output \( f(\bar{x}) \). In contrast, the domestic consumption cost in our model also depends on other factors. These include the marginal product schedule \( f'(x) \), as well as government policy variables from the previous period: the levels of domestic exposure \( \alpha \), total domestic holdings of securities \( D \), and the level of foreign debt issuance \( B_f \).

Figure 8 plots the optimal external debt level \( z \) as a function of \( y_1 \), the endowment in period 1. For the numerical simulation, we use the same parametrization and functional forms for utility and production functions as those chosen for Figure 7. For this specification, the government borrows less when the endowment in the initial period is higher.

Finally, let us conclude by tying together some of the model’s predictions for optimal debt issuance policy, as they relate to actual economies. On a time series dimension, a country which experiences a transitory endowment shock may wish to borrow more resources from abroad \( z \) in that period. On a cross-sectional dimension, we can compare emerging market economies and advanced economies. The fact that emerging market economies have higher average growth rates can be captured in the model by assuming that both sets of countries have identical productivity shock distributions in period 2, but the emerging market economies have a lower endowment in period 1. Then, the emerging market economies may wish to borrow more from abroad \( z \) than advanced economies in period 1. Notice that it is not the total government debt level that is predicted to be different between these two categories of economies, it is the component of government debt that is held by foreigners.

What happens to the optimal structure of government debt issuance when an economy decides
to borrow more from abroad? When one category of economies borrow more from abroad than other, how do their optimal debt structures reflect that? According to the model, countries that issue a higher level of external debt must typically increase the exposure $\alpha$ of the domestic banking system to its debt (as opposed to safe assets such as cash). This helps to increase the price of foreign held debt $q$. For a given exposure of the domestic banks to government debt, a higher optimal external debt level necessitates a greater volume of foreign debt issuance $B_f$. By contrast, when government debt cannot be traded between domestic and foreign agents in the period of issue, the optimal total volume of cash and domestically held debt $D$ is not affected by the level of external debt chosen.

What do these optimal policy predictions imply for realized economic outcomes? A higher level of exposure $\alpha$ of domestic banks to sovereign debt does limit the incentive of the government to default. However, it also means that when the government does default in period 2, liquidity constraints in the domestic financial sector are more binding and the domestic production sector is more severely affected. For a given level of exposure of the banking system, a greater level of foreign debt issuance increases the ratio of foreign to domestic held debt $\frac{B_f}{D}$. At the optimum, this means higher haircuts on debt for economies with a higher level of external debt. Of course, the government optimally defaults more after low productivity shock realizations $\hat{R}$. The default decision amplifies the negative effect of an adverse productivity shock on the domestic output level, because the higher haircut exacerbates the liquidity constraint in the financial sector. Therefore, for the same underlying distribution of productivity shocks ex post, countries with higher external debt levels would tend to observe a higher average haircut, a lower average output level, and greater volatility in realized output. These predictions would apply for the comparison between emerging market economies and advanced economies. They would also be relevant for the analysis of a single country, as a warning of the side-effects of accumulating a higher external debt level in the aftermath of a low endowment shock.

5 Tradable Debt

When sovereign debt is tradable between domestic and foreign agents in the period of issue, the price of debt in domestic markets and the price at which foreigners trade the debt amongst themselves must be equalized:

$$p_B = q.$$ 

For the tradable debt case, it is not possible for the government to directly determine the proportions of cash and debt in the portfolio of the domestic banking system. If domestic banks’ private marginal valuation of debt is lower than the market price, they do not simply hold all the government debt that is issued domestically; instead, they sell any additional debtholdings to foreign agents. Conversely, if the private marginal valuation of domestic banks exceeds the market price, then the government cannot prevent the banks from purchasing additional debt from foreign participants.
The restriction on prices shown above amounts essentially to a restriction on private marginal valuations of debt: in any equilibrium allocation, the valuation of the marginal unit of debt by domestic and foreign bondholders must be equal. Compared to the nontradable debt specification, the condition above constitutes an additional restriction on both the set of feasible debt levels $z$ and the government policy combinations $C = (\alpha, D, B_f)$ that may achieve those debt levels. This additional restriction takes into account the preferences of domestic agents – therefore, parameters related to the utility function now become relevant for the feasible set of debt levels and government policy combinations (by contrast, they were not relevant for the feasible set in the nontradable debt case).

Subsection 5.1 constructs the program for the government’s Ramsey problem. In subsection 5.2, we prove that the optimal default decision of the government is identical to that in the nontradable debt specification. Therefore, the rest of the analysis of the tradable debt case concentrates instead on the optimal government debt issuance problem. In constructing the program and solving the model, we apply the same techniques to solve the model as in the nontradable debt case, and we can again reduce the number of state variables. However, the final number of state variables is two, instead of the one state variable $z$ in the nontradable debt case. The result is that although analysis of the tradable debt case is still facilitated by our methodology, it is not as tractable as the nontradable debt model. This motivates the following approach. Subsection 5.3.1 provides some analytical results on optimal debt issuance policy for a special case of the model: the linear utility case. Clearly, the choice of the debt level $z$ in this case is trivial, so we do not explore this. We focus instead on understanding how the optimal combination $(\alpha, D, B_f)$ that raises any given level of external debt $z$ differs from the nontradable debt case. Finally, subsection 5.3.2 returns to the general case with a concave utility function. The general case is explored via numerical simulations.

Throughout this section, our aim is show how the tradability of debt between domestic and foreign agents changes the feasible and optimal government debt issuance decisions from the nontradable debt case.

### 5.1 Construction of Program

As in the nontradable debt case, we apply definitions 1 and 2 to derive the program for the government problem. However, for the tradable debt case, we add the additional restriction:

$$p_B = q.$$  

The government problem is written as follows. In period 1:

$$U_1 = \max_{c_1, \alpha, D, B_f} \left\{ u(c_1) + \beta \mathbb{E} U_2 \left( \alpha, D, B_f, \tilde{R} \right) \right\}$$

subject to

$$c_1 = y_1 + qB_f$$
\( c_1 \geq 0 \)

\[
q = \frac{1}{1+r} E \left\{ 1 - h \left( \alpha, D, B_f, \bar{R} \right) \right\} \quad \text{for } B_f > 0
\]

\[
\geq \quad \text{"} \quad \text{for } B_f = 0
\]

\[
q u'(c_1) = \beta E \left\{ u'(c_2) \cdot (1 - h) \cdot \left[ 1 + \bar{R} f' \left( (1 - \alpha) + (1 - h) \alpha \right) D \right] \right\} \quad \text{for } \alpha D > 0
\]

\[
\geq \quad \text{"} \quad \text{for } \alpha D = 0
\]

\( B_f < 0 \Rightarrow \alpha = 0 \)

where the expression for the period utility in period 2 is given by

\[
U_2 \left( \alpha, D, B_f, \bar{R} \right) = \max_{c_2, h} u(c_2)
\]

subject to

\[
c_2 = y_2 - (1 - h)B_f + \bar{R} f' \left( (1 - \alpha) + (1 - h) \alpha \right) D
\]

\[
c_2 \geq 0
\]

\[
y_2 \geq (1 - \alpha)D + (1 - h) [\alpha D + B_f]
\]

(12)

\[
0 \leq h \leq 1.
\]

We again assume that \( y_2 \) is large enough so that the constraint (12) is never binding.

Since there is a continuum of domestic banks, all the returns of the banking sector, both from portfolio investments in period 1 and from firm lending in period 2, accrue in the end to their customers, i.e. domestic consumers. These consumers deposit their savings in the banks in period 1, and competition induces the banks to select the portfolio composition (the proportions of cash and debt) that satisfies the Euler conditions of the representative consumer with respect to holdings of each domestic security.

There are two relevant Euler conditions, one for cash and one for debt. Each condition relates the willingness to hold a security to its price. When government debt is not tradable between domestic and foreign debtholders in the period of issue, the prices of cash \( p_A \) and domestically held debt \( p_B \) do not appear in any other equation. Therefore the Euler equations are redundant constraints for the government problem, and are ignored in the analysis. In particular, it is possible to solve the government problem purely in terms of allocations and then derive the prices \( p_A \) and \( p_B \) as residuals of the exercise. When debt is tradable between domestic and foreign agents in the period of issue, the price of cash \( p_A \) can again be derived as a residual of the government problem, but the price of debt \( p_B \) cannot. The tradability of government debt means that domestic and foreign agents must face the same price of debt: \( p_B = q \). This price \( q \) shows up in the rational
expectations condition, and cannot be ignored when characterizing the feasible and optimal sets of external debt \( z \) and government policies \((\alpha, D, B_f)\). Therefore, the Euler equation for defaultable debt is not redundant.

We continue using the \((\alpha, D, B_f)\) notation to characterize and solve the problem. Of course, the government cannot select \(\alpha, D\) and \(B_f\) arbitrarily. The government issues a quantity \(B\) of debt, which is divided between \(B_d = \alpha D\) and \(B_f\) according to the portfolio decisions of domestic banks and foreign creditors, who may trade the debt with each other. But since the government understands the portfolio decisions and equilibrium conditions, it is methodologically equivalent to consider a problem where the government directly selects a combination \(C = (\alpha, D, B_f)\), subject to the condition that the combination selected is in fact realized in a rational expectations equilibrium.

The problem in period 2 is unchanged. The haircut function \( h = H(\alpha, D, B_f, \tilde{R}) \) is exactly the same as in the case with nontradable debt. This can be used to derive the bond price schedule \( Q(\alpha, D, B_f) \), just as before. Using similar techniques to those applied for the nontradable debt case, we derive the following representation of the government problem.

\[
V_1 = \max_{c_1,z} \mathbb{E} \{ u(c_1) + \beta V_2(z, \lambda) \}
\]

subject to
\[
c_1 = y_1 + z
\]
\[
c_1 \geq 0
\]
\[
\lambda = u'(c_1)
\]
\[
(z, \lambda) \in \tilde{G}
\]
for some set \( \tilde{G} \). We define \( V_2(z, \lambda) \) as follows:

\[
V_2(z, \lambda) = \max_{c_2,\alpha,D,B_f} \mathbb{E} \{ u(c_2) \}
\]

subject to
\[
c_2 = y_2 - (1-h)B_f + \tilde{R}f([(1-\alpha) + (1-h)\alpha]D)
\]
\[
h = H(\alpha, D, B_f, \tilde{R})
\]

For \( z < 0 \), combinations \( C = (\alpha, D, B_f) \) satisfy: \( \alpha = 0, B_f < 0 \).
For \( z = 0 \), combinations \( C = (\alpha, D, B_f) \) satisfy: \( B_f = 0, \alpha D = 0 \).
For \( z > 0 \), combinations \( C = (\alpha, D, B_f) \) satisfy:

\[
\frac{1}{1+r} \mathbb{E}(1-h) = \beta \mathbb{E} \left\{ \frac{u'(c_2) \lambda}{\lambda} \cdot (1-h) \cdot \left[ 1 + \tilde{R}f'([(1-\alpha) + (1-h)\alpha]D) \right] \right\}.
\]

(13)

Our notation suppresses the dependence of \( h \) on \((\alpha, D, B_f, \tilde{R})\). There exist many government
policy combinations such that $z = 0$, but we can ignore all but one of these. All of them are allocationally equivalent to the combination stated above: $B_f = 0, \alpha D = 0$.

There are now two state variables for the problem: the external debt level, $z$, and the marginal utility of period 1 consumption, $\lambda$. Relative to the nontradable debt case, we add the restriction that foreign creditors and the domestic representative consumer value the marginal unit of debt equally. Equation (13) is the equal marginal valuation restriction.

5.2 Optimal Government Default Decision

Proposition 12 (Optimal Haircut Decision) The optimal haircut decision $h = H(\alpha, D, B_f, \tilde{R})$ in the tradable debt case is exactly the same function as in the nontradable debt case.

Therefore, the remainder of the analysis of the tradable debt specification concentrates on the optimal government debt issuance decision.

5.3 Optimal Government Debt Issuance Decision

First, we state a broad result to explain conceptually how the feasible set of external debt levels $z$ differs from that in the nontradable debt specification.

Proposition 13 (Feasibility of External Debt) The feasible set of external debt levels depends not just on production side parameters, but also utility side parameters. External debt is only feasible if both domestic and foreign agents can be induced to hold positive quantities of debt.

The additional restriction (13) is imposed on the feasible set of external debt levels when debt is tradable between domestic and foreign agents in the period of issue, but not in the nontradable debt case. This extra restriction depends on preference parameters, such as the representative consumer’s discount factor and utility function. Therefore, preference parameters are relevant for the feasible set of debt levels $z$ and government policy combinations $(\alpha, D, B_f)$ in the tradable debt case. This is not true under the nontradable debt specification.

The preference parameters are important because they shape the relative incentives of domestic and foreign lenders to hold debt. The marginal valuation of debt by domestic banks may differ from the marginal valuation of foreign creditors for two reasons. Firstly, the expected return from holding debt is generally for domestic banks and foreign creditors, because domestic banks have the extra option of lending their total post-haircut assets to the firms, and they receive a loan rate from this market. Therefore, domestic banks receive a weakly higher gross return on debt than foreign creditors. Secondly, foreign creditors have a linear utility function, whereas the representative consumer’s utility function may be concave. With concave utility, the valuation of debt by the representative consumer depends on the implied consumption levels (and hence marginal utilities of consumption) in periods 1 and 2. This is not true for foreign creditors, whose marginal valuation of consumption is always unity.
For external debt to be feasible, i.e. $z = qB_f > 0$, there must exist combinations of government policies $(\alpha, D, B_f)$ such that both domestic banks and foreign creditors have positive holdings of debt. If domestic banks hold all the debt, foreign debt issuance $B_f$ is equal to zero, so external debt is $z = 0$. If foreign creditors hold all the debt, the government defaults fully on all debt in period 2. Debt has a price $q$ of zero in period 1, so again $z = 0$.

Subsection 5.3.1 provides analytical results for the special case where the representative consumer has linear utility: $u(c) = c$. We establish a restriction on the discount factor $\beta$ for external debt to be feasible at all, and we derive results regarding the feasibility and optimality of government policy combinations $(\alpha, D, B_f)$ for each external debt level $z$. Since the utility functions of domestic and foreign lenders are identical, the analysis draws heavily on the differences in the expected gross returns on government debt for domestic banks versus foreign creditors.

Subsection 5.3.2 considers the general case of the model where the representative consumer has concave utility: $u''(c) < 0$. Relative to the linear utility case, this framework adds the additional factor that a marginal unit of consumption is valued differently across different points in time and states of nature, depending on the consumption level. Numerical simulations are employed to characterize the solution of the model.

### 5.3.1 Special Case: Linear Utility Function

When the utility function is linear, i.e. $u(c) = c$, the state variable $\lambda$ is always equal to unity and is redundant. The government program reduces to a problem with only one state variable. Domestic banks are willing to purchase an infinite quantity of defaultable debt at any price less than

$$p_B = \beta \mathbb{E} \left\{ (1 - h) \cdot \left[ 1 + \bar{R}f' \left( \left( (1 - \alpha) + (1 - h)\alpha \right) D \right) \right] \right\}. $$

Foreign creditors are willing to purchase an unlimited quantity at any price below

$$q = \frac{1}{1 + r} \mathbb{E} (1 - h).$$

For the tradable debt case, $p_B = q$:

$$\frac{1}{1 + r} \mathbb{E} (1 - h) = \beta \mathbb{E} \left\{ (1 - h) \cdot \left[ 1 + \bar{R}f' \left( \left( (1 - \alpha) + (1 - h)\alpha \right) D \right) \right] \right\}. \quad (14)$$

This is the equal marginal valuation constraint in the linear utility case. The following proposition immediately follows.

**Proposition 14 (Feasibility of External Debt)** It is feasible for the sovereign to borrow from abroad if and only if the discount factor $\beta \in \left( 0, \frac{1}{1+r} \right)$. The maximum external debt level depends on $\beta$ and is weakly lower than in the nontradable debt case.

For $\beta = \frac{1}{1+r}$, equation (14) is inconsistent with a debt level $z > 0$. Consider any combination $C = (\alpha, D, B_f)$ with $B_f > 0$. External debt can be raised in period 1, i.e. $z > 0$, if there is
repayment in some states of nature in period 2. For those states of nature where there is repayment, it is also true that $Rf'(x) > 0$. Domestic banks receive a higher gross return than foreign creditors from holding debt. If they both discount future periods at the same rate, domestic debtholders value the defaultable debt more and therefore choose to hold all of the debt. Foreign agents end up with zero debtholdings because domestic banks purchase all of the debt from them. Therefore, no resources can be borrowed from abroad in period 1: $z = 0$. For $\beta = 0$, equation (14) is again inconsistent with external debt. Domestic banks place a zero value on the debt. At any positive price, they will not hold any of the debt. Therefore, there is no domestic output cost of default in period 2. The optimal policy for the government is to default on all of its debt in the final period, which means the price of debt $q$ in period 1 must be zero by rational expectations. It is not feasible for the government to issue debt at a positive price in the first period. Again, $z = 0$.

For $\beta \in \left(0, \frac{1}{1+r}\right)$, domestic banks can be persuaded to hold some, but not all, of the debt. Then, external debt is feasible in period 1. How much debt is feasible depends on the discount factor $\beta$.

Next, we describe some characteristics of the feasible set of government policies $(\alpha, D, B_f)$ that can support each level of external debt $z$. As in the nontradable debt case, if the government chooses to save, i.e. $z < 0$, it is constrained to issue only cash.

**Proposition 15 (Saving)** For $z < 0$, the government chooses: (i) $\alpha = 0$; (ii) $D = \bar{D}$; (iii) $B_f = (1 + r)z$.

Now consider the choice of positive external debt levels, i.e. $z > 0$. An amended version of Lemma 1 holds for the specification with tradable debt. This result allows us to focus on a restricted subset of $D$.

**Lemma 2** For any combination $C = (\alpha, D, B_f)$ such that $D > \bar{x}$, there exists some other combination $C' = (\alpha', D', B'_f)$ where $D' = \bar{x}$, such that $C'$ raises the same revenues as $C$ in period 1 and is equivalent to $C$ in terms of repayments abroad, output and hence consumption for all values of the productivity shock $\bar{R}$ in period 2. $C$ and $C'$ satisfy the Euler condition for defaultable debt for the same value of $\lambda$.

**Corollary 2** We can restrict our attention to combinations $C = (\alpha, D, B_f)$ such that $D \in [0, \bar{x}]$.

How does the feasible set of government policies $(\alpha, D, B_f)$ for each external debt level $z$ differ from the nontradable debt case? The next proposition establishes that the equal marginal valuation restriction reduces the size of the feasible set.

**Proposition 16 (Feasible Domestic Exposure Levels)** Fix $D = D'$. The feasible set of $(\alpha, z)$ values is a subset of the feasible set in the nontradable debt case.

Figure 9 illustrates this result. The bold lines are the boundaries of the feasible set in the nontradable debt case. For the tradable debt case, the line $\alpha_T(z)$ is the typical shape of the feasible
set. For some intuition, let us consider an example economy with a domestic level of cash and debt such that in period 2, the government repays debt fully in some states of nature and defaults on a portion of it in others. Conceptually, the exposure mechanism explained in the nontradable debt case also operates here. Therefore, for fixed $D = D'$, a reduction in the exposure level $\alpha$ causes an increase in the optimal haircut in those states of nature where the haircut was initially interior. This reduces the price that foreigners are willing to pay for the debt. It also reduces the price that domestic banks are willing to pay, but by a lesser amount. Therefore, in equilibrium the price $q$ falls and the fraction of debt held by foreigners, $\gamma = \frac{B_f}{\alpha D}$, declines. Unlike in the nontradable debt case, it is not possible to vary $\alpha$ and $\gamma$ independently, because of the equal marginal valuation constraint. Therefore, the feasible set is reduced. However, the relationship between exposure levels and the haircut is still given by the exposure mechanism described in the nontradable debt case.

![Figure 9: Feasible Set in the Tradable Debt Case](image)

**Proposition 17 (Optimal Domestic Exposure)** The optimal domestic exposure level $\alpha$ for any given external debt level $z$ depends on the risk aversion of the representative consumer.

How does the optimal domestic exposure level $\alpha$ change relative to the nontradable debt case? The same considerations apply as in the nontradable debt case, but as figure 9 illustrates, the changed feasible set must be taken into account. An immediate consequence of the equal marginal valuation restriction is that in terms of the optimal government debt issuance decision, the government cannot simply first set $D = \bar{x}$ and then decide the optimal exposure level $\alpha$. Both must be determined jointly.

**Proposition 18 (Total Domestic Debt)** Total domestic debt may be less than $\bar{x}$ at the optimum.
Proposition 8 does not hold in the version of the model with tradable debt. This result is a marked deviation from the prediction of the nontradable debt case, and it is proven by showing that a deviation from $D = \bar{x}$ may increase expected consumption in both periods 1 and 2. Suppose that we begin with total cash and domestically held debt set to $\bar{x}$. Let us explore how to increase expected consumption in period 2. There are two means by which the government can implement this. Firstly, it can keep both $D = \bar{x}$ and the proportion of debt held by foreign relative to domestic lenders $\gamma = \frac{B_f}{\alpha D}$ fixed, and reduce the exposure level of the domestic economy. This reduces repayments in period 2, hence increasing consumption in that period. However, this policy necessarily reduces consumption in period 1 by reducing the external debt that is raised by the government in that period. The desirability of such a policy depends on $\beta$.

Alternatively, the government can reduce both $D$ and $\gamma$. Let us consider a reduction in $D$ large enough to reduce output for the highest productivity state of nature. In this state of nature, it also increases the loans rate in the market for loans in period 2. This increases the return to holding debt for domestic banks in this state of nature, which means that the proportion of debt held by domestic banks rather than foreigners must rise. Therefore, $\gamma$ declines. This perturbation increases expected consumption in period 2. Moreover, in this case it is possible (although not certain) that the debt level $z$ rises due to a positive price effect on government debt. The higher domestic exposure level in period 1 reassures foreign creditors that repayments will be made by the government in period 2, and this increases the price $q$ of government debt sufficiently to increase the debt level $z$. Therefore, consumption in both periods may rise. If this is indeed true, then it is desirable (for any value of $\beta$) for the government to constrict output in the second period in order to induce domestic banks to hold debt, because such a policy reduces the interest rate on the debt.

Finally, we note that for the more general concave utility case, additional factors work to reduce the overall level of cash and domestically held debt $D$ below $\bar{x}$. Firstly, suppose that the perturbation above in fact reduces, rather than increases, consumption in period 1. Such a perturbation may still be desirable for a risk averse representative consumer, because it may increase consumption for the worst productivity shock values $\bar{R}$ in period 2. The perturbation then offers some level of insurance. Secondly, consider a reduction in $D$ that does not reduce output in period 2, because the optimal haircuts always remain strictly interior. Nevertheless, such a perturbation changes haircuts across different states of nature. For the linear utility case, this affects the valuation of debt by foreigners and domestic banks equipropotionately, so it does not change the proportion of debt in the hands of foreigners. If utility is concave, the marginal utility of domestic consumption varies across different states of nature, while the same is not true for foreign creditors. This means that the proportions of government debt held by foreign and domestic debtholders may indeed change, and this may be desirable.
5.3.2 General Case: Concave Utility Function

For the concave utility case, the equal marginal valuation constraint is reproduced below. The marginal unit of debt must be valued equally by domestic banks and foreign creditors:

\[
\frac{1}{1 + r} E(1 - h) = \beta E \left( \frac{u'(c_2)}{u'(c_1)} \cdot (1 - h) \cdot \left[ 1 + \bar{R}f' \left((1 - \alpha) + (1 - h)\alpha \right) D \right] \right).
\]

The difference from the linear utility case is that now, the valuation of debt by domestic banks depends on the implied consumption levels in periods 1 and 2. This is not true for foreign creditors. This alters the feasible set and makes the levels of the endowment \(y_1\) and \(y_2\) relevant for the restriction above, since they affect the marginal utility of consumption in the two periods. Therefore, these endowment levels are also important for the feasible sets of external debt levels \(z\) and government policy combinations \((\alpha, D, B_f)\).

The analysis of the linear utility case in the previous subsection provides some model-inspired intuition for the tradable debt specification. For the case with concave utility, we present numerical simulations rather than further analytical exercises. For the numerical exercises, we use the same functional forms and parameter values for the production function presented in subsection 4.3.2. We use a constant relative risk aversion utility function

\[
u(c) = \log(c)
\]

The discount factor is \(\beta = 0.8\). We set \(y_1 = 15\) and \(y_2 = 9\). For these parameters, it is not possible to borrow more than \(z_{\text{max}} = 0.3750\) units from abroad in period 1.

Panel a of figure 10 shows how the optimal exposure of the domestic economy varies with the asset level. For the region \(z < 0\), \(\alpha\) is set to zero and for \(z = 0\), the level of \(\alpha\) is irrelevant. For the region \(z > 0\), full domestic exposure \((\alpha = 1)\) is always optimal for the functional forms and parameter configuration detailed above. The optimal level of total domestically held cash and debt is shown in panel b. When the government decides to borrow from abroad, it finds it optimal to reduce the magnitude of total domestic debt below \(\bar{x}\). For the parameters chosen, the deviation does not actually reduce output in period 2. Nevertheless, it changes the optimal haircut in period 2 and this affects the composition of debtholders between domestic and foreign agents in an optimal manner. Finally, the optimal level of foreign debt issuance is monotonically increasing in the debt level for our simulated model.

How much does the government borrow from abroad? Figure 11 allows the initial endowment \(y_1\) to vary and displays the debt choice of the government as a function of this endowment. The government borrows more if the initial endowment is lower.
Figure 10: Optimal Debt Issuance Policy in the Tradable Debt Case

Figure 11: Optimal Debt Level as a Function of the Initial Endowment

6 Policy Implications

In this paper, we have developed a model and an analytical framework which may be used to evaluate the welfare implications of particular government policy recommendations.
6.1 Enhancing Commitment Power

The first best case is achieved if the government can commit in period 1 to a contingent schedule of haircuts in period 2, and if it can save abroad and issue debt at the same time. If the government can commit to repay its debt irrespective of the domestic exposure level of the banking system, there is no need for the economy to subject itself to default-induced domestic output costs in period 2.

What would repayments in the first best case look like relative to our model? Observed repayments in our framework are increasing in the productivity shock $\tilde{R}$. Therefore, the debt repayments vary across states of nature in a direction that we would expect from fully contingent debt. However, the first best case has higher welfare than the lack of commitment case, for two reasons. Firstly, the repayments received by foreign lenders are lower than the consumption cost of borrowing to domestic consumers, because borrowing in our model is associated with a schedule of deadweight welfare losses for the domestic economy. Because the government must “purchase commitment” to repay by exposing the domestic economy to its debt, realized haircuts in period 2 generate a schedule of output costs across states of nature, which do not benefit foreign lenders. Secondly, the contingency of repayments in our model does not in general match the first best case. For full commitment to repay and perfectly contingent debt, repayments in a particular state of nature would depend on the productivity realization $\tilde{R}$ and the maximum value of output $f(\bar{x})$. On the other hand, the repayment in our model depends on the marginal product schedule $f'(x)$, as well as the particular combination of government policies $(\alpha, D, B_f)$ chosen in period 1.

Whether the first best can be implemented in the prevailing legal environment is debatable. In essence, courts must be able to enforce contractual adherence by the government to any promised contingent repayment schedule. The government would issue debt with repayments that are court-enforceable and contingent on the productivity shock.

6.2 Improving Creditor Rights

Suppose that debt is non-contingent as in our model, but that the enforcement power of international courts is strengthened so that debtholders can compel the sovereign to repay the face value of the debt in full. This corresponds to an allocation where the period 2 haircut is zero for all values of the productivity shock $\tilde{R}$. Again, the government can credibly commit to repay debt without exposing the domestic economy to output costs in period 2.

Such a policy has several distinct effects on welfare. Firstly, suppose that debt is tradable between domestic and foreign agents in the period of issue, and that Proposition 18 holds so that it is optimal for the government to reduce $D$ below $\bar{x}$ in order to constrict the level of output in period 2. Such constriction is no longer optimal when there is no need to persuade domestic banks to hold any quantity of the debt. The economy benefits from being allowed to increase $D$ to $\bar{x}$. Secondly, the economy may also benefit because the maximum debt level in the full commitment case exceeds the maximum debt level in the case without commitment to repay. Thirdly, the
average consumption cost of borrowing falls to the gross riskless return $1 + r$: borrowing is no longer associated with a schedule of deadweight output losses borne by the domestic economy. The reduction in the average cost of borrowing benefits the domestic economy and tends to increase the level of borrowing in period 1. Finally, the policy may have one negative effect on welfare. In particular, the repayments on debt are no longer effectively contingent on the state of nature $\tilde{R}$. This hurts domestic consumers when they are risk averse, and this effect tends to decrease the level of borrowing in period 1.

In Figure 12, we present some numerical simulations to evaluate the effects of improving creditor rights. The parametrization of the model follows the specification used in subsection 5.3.2. Utility is higher in the full repayment case than in the nontradable and tradable debt specifications of the model developed in this paper. The level of borrowing by the government is substantially higher in the case with improved creditor rights. This shows that the reduction in the average consumption cost of borrowing dominates the removal of the contingency of debt repayments.

<table>
<thead>
<tr>
<th>Figure 12: Effect of Improving Creditor Rights</th>
<th>Full repayment</th>
<th>Nontradable</th>
<th>Tradable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Utility</td>
<td>4.789</td>
<td>4.786</td>
<td>4.786</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>/</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$D$</td>
<td>1</td>
<td>1</td>
<td>0.927</td>
</tr>
<tr>
<td>$B_f$</td>
<td>0.980</td>
<td>0.248</td>
<td>0.230</td>
</tr>
<tr>
<td>$z$</td>
<td>0.934</td>
<td>0.214</td>
<td>0.214</td>
</tr>
<tr>
<td>Average $h$</td>
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<td>0.094</td>
<td>0.022</td>
</tr>
<tr>
<td>Average $\rho$</td>
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<td>1.025</td>
<td>1.025</td>
</tr>
<tr>
<td>$z_{\text{max}}$</td>
<td>8.571</td>
<td>1.186</td>
<td>0.357</td>
</tr>
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</table>

### 6.3 Reduction in Domestic Exposure

Finally, international financial institutions may instruct the government to reduce the vulnerability of the domestic banking system to government debt. Figure 9 is the appropriate diagram to evaluate such a policy. For both nontradable and tradable debt cases, the domestic exposure level in our model is an optimal response of the economy to the lack of commitment of the sovereign, and a forced reduction in this policy variable may reduce the maximum level of external debt $z_{\text{max}}$ that may be raised by the government in period 1. Therefore, any recommendation by outside institutions to reduce the level of domestic exposure below the level chosen by the government is welfare-decreasing. It does not solve the market imperfections that are the underlying reason for the high observed domestic exposure level. Instead, it simply depresses the price of foreign held debt $q$, and pushes up the interest rates offered by foreign lenders.
Conclusion

In this paper we first construct a parsimonious model of an emerging market economy with two key imperfections: (i) the government’s inability to commit to repay debt; (ii) liquidity constraints in the domestic financial sector, due to a lack of insurance against shocks to bank capital. Sovereign default on debt reduces aggregate banking capital, and this leads to a contraction in domestic credit and production.

This model is then used to answer two sets of questions regarding optimal government policy. First, the optimal default decision is derived as a tractable function of inherited debt variables (determined by government policy in the previous period), current economic conditions and production side variables. The optimal haircut on debt is summarized in a simple diagram, which yields several comparative static predictions. The government imposes a low haircut if the proportion of government debt (as opposed to cash) in domestic banks’ portfolios is high, because the domestic economy is more exposed to the adverse consequences of default in this case. The total stock of government debt is not a good predictor of default risk at the optimum; rather, it is the ratio of sovereign debt claims that are held by foreign agents which is important. The higher is this ratio, the higher the risk of default by the government. This is important to remember when comparing emerging and advanced economies. Finally, the haircut on debt is countercyclical at the optimum, again matching the empirical evidence.

Second, we characterize the optimal sovereign debt issuance decision. When it issues debt, the government can influence three key variables: the total volume of government claims (cash and debt) issued, the proportion of debt (as opposed to cash) in domestic banks’ portfolios, and the ratio of debt held by foreigners relative to that held by domestic agents. Three key mechanisms play an important role in the debt issuance decision. The exposure mechanism captures the fact that the government can effectively “purchase commitment” to repay its debt in future periods by exposing the domestic economy to severe output costs ex post if the government reneges on its repayment obligations. Indeed, if the government wishes to borrow more resources from abroad, or if it wishes to raise the same amount more cheaply, it may find it necessary to increase the exposure of the domestic economy to assure foreign creditors of its intention to repay. We should expect that governments increase the fragility of the domestic economic environment, making output more vulnerable to the default decision, in order to be able to borrow more ex ante. This is an optimal response to the government’s lack of commitment. We should also observe that these output costs are in fact realized when the government is confronted by adverse productivity shocks. In states of nature ex post where the productivity shock realization is low, the government will find it optimal to default, because it can push some of the burden of the adverse shock onto foreign creditors. Default imparts some contingency unto debt repayments that are otherwise contractually non-contingent.

Considerations of domestic fragility are also important in the consideration of the optimal ratio of foreign to domestically held debt. The higher is this ratio (over some range), the higher the volume of resources the country can borrow from abroad. However, a higher ratio also induces a
larger average haircut in the final period, and therefore lower average output and consumption in that period. These considerations affect the total level of external debt in the first period, and the corresponding distribution of government debt between domestic and foreign holders.

The above mechanisms apply whether the defaultable debt is tradable or not in the period of issue. When debt is tradable in the period of issue, an additional restriction is imposed on the feasible set for debt issuance. In particular, the government can no longer force the domestic banking system to hold the quantities of cash and defaultable debt that the government chooses. Banks choose their portfolios optimally. Specifically, there is an equal marginal valuation restriction: both domestic and foreign lenders must hold debt and value the marginal unit of debt equally, if external debt is to be sustained. If domestic banks value the debt more highly than foreign creditors, they purchase most of the debt and it is difficult for the government to raise resources from abroad. If domestic banks are unwilling to hold much of the debt, then foreign creditors will anticipate high defaults by the government in the subsequent period, and the price of government debt will be low. This again constrains the ability of the government to raise resources from abroad.

Whether the nontradable or tradable debt specifications are more accurate descriptions of reality is debatable. If governments are able to mandate that domestic banks hold a certain fraction of government or other assets in their portfolio, then the nontradable debt specification may be more appropriate. In general however, if domestic banks have unhindered access to secondary markets where they can purchase and sell government debt, it is more desirable to pay heed to the predictions of the tradable debt case. Philosophically, the assumption of active secondary markets for government debt in the period of issue sits better with our assumption (throughout the bulk of this paper) that haircuts on domestic and foreign debtholders are equal. Possible justifications for such equal haircuts on foreign and domestic creditors include that the government cannot observe who is holding the debt at the moment of repayment, or that the existence of secondary markets imposes constraints on the government’s ability to effectively execute different haircuts on different groups of lenders.

Since the exposure of the domestic economy is an optimal response to the underlying problem of the sovereign’s lack of commitment, recommendations by international financial institutions to reduce the exposure of the economy to default may have the counterintuitive side-effect of a reduction in the ability of the government to borrow from abroad, and in general may reduce welfare if the advice is binding. This feature of the model derives from the benevolent government setup, since such a government already decides to impose ex post costs on the domestic economy in an optimal manner ex ante. Improvements in creditor rights reduce the average cost of borrowing because deadweight losses in output are no longer necessary to induce repayment by the government. However, if only non-contingent debt contracts are available, then the lack of contingency in repayments is inferior in welfare terms to the first best case.

The two period model on this paper can be extended to an infinite horizon framework. In Basu (2009), we maintain the assumption that sovereign default affects the domestic economy, but does not lead to reduced access to international capital markets. In this extended model, optimal default
in a particular period leads to lower contemporaneous output levels, but in fact higher consumption in current and future periods. Average output levels increase in the future if the ratio of foreign to domestic claims is reduced as a result of the default decision. Finally, default on external debt leads to an enhanced ability to borrow new resources from abroad because it reduces the debt burden taken forward into the future, without any effect on the feasible set of external debt levels in future periods. These predictions are in some ways at odds with the previous literature on sovereign debt, where default typically leads to higher consumption in the period of default, but lower welfare in the future due to lost capital market access. Our take on these contrasting predictions is as follows. On the one hand, we do believe that our predictions match the experiences of some recent defaulter nations, who have managed to rebound economically after cutting their external debt stocks (Sturzenegger and Zettelmeyer 2006). On the other hand, we freely acknowledge that default decisions do often lead to disruptions on international capital markets in ways that are not the focus of our model. Therefore, we believe that the approach in this paper and that in the bulk of the infinite horizon literature are complementary components of a more general theory.
8 Appendix

A. Proofs of Results in the Main Text


Let us consider conditions (a)-(f) of Definition 1. Since there is a continuum of banks, competition between banks for the savings of the consumers will result in zero profits for the banks ($\Pi_B = 0$), and all the gains from the banks’ investments accrues to the consumers. Combining the consumer and bank problems:

\[
p_A u'(c_1) = \beta \mathbb{E}\{u'(c_2) \cdot \rho\} \quad \text{for } A_d > 0
\]

\[
\geq
\]

\[
\text{for } A_d = 0
\]

\[
p_B u'(c_1) = \beta \mathbb{E}\{u'(c_2) \cdot (1-h) \cdot \rho\} \quad \text{for } B_d > 0
\]

\[
\geq
\]

\[
\text{for } B_d = 0
\]

The consumer budget constraints (2) and (3) hold with equality, and $c_1, c_2 \geq 0$. For $u''(c) < 0$, the consumer problem is convex and these conditions are necessary and sufficient for a maximum. For $u''(c) = 0$, banks wish to purchase an infinite quantity of cash if the price $p_A$ is less than $\beta \mathbb{E}\{\rho\}$, and none if the price exceeds this level. They wish to purchase an infinite quantity of defaultable debt if the price $p_B$ is less than $\beta \mathbb{E}\{(1-h) \cdot \rho\}$, and none if the price exceeds this level.

As described in Section 2, equilibrium in the market for loans in period 2 establishes that the loan rate is a function of the total post-haircut assets of the banking system:

\[
\rho = 1 + \tilde{R} f'(X)
\]

Firm profits in equilibrium are given by

\[
\Pi_F = X + \tilde{R} f(X) - \left\{1 + \tilde{R} f'(X)\right\} X
\]

Substitute the two government budget constraints, the expression for savings and firm profits into the consumer budget constraint:

\[
c_1 \leq y_1 + qB_f
\]

\[
c_2 \leq y_2 - (1-h)B_f + \tilde{R} f(x)
\]

where

\[
x \leq A_d + (1-h)B_d
\]

Given bank investments and firm profits, substituting the government budget constraints into the consumer budget constraint yields the resource constraints. If we include the government budget
constraint and the resource constraint, we can drop the consumer budget constraint from the problem.

Equation (7) determines the price $q$, and is included as the rational expectations constraint on the problem.

Now let us apply Definition 2. It is an optimum for the government budget constraints to hold with equality, so the resource constraints above will hold with equality. We do require that the representative consumer’s Euler conditions hold. However, the prices $p_A$ and $p_B$ do not appear in any other equation, and therefore the Euler equations are redundant constraints for the problem. In particular, it is possible to solve the government problem and then derive the prices $p_A$ and $p_B$ as residuals of the exercise. Finally, it is not necessary to keep track of $T_1$ and $T_2$ for the problem (these quantities can be calculated as residuals from the solution to the government program), so we drop the equations that define them. This yields the first specification of the government program presented in the main text.

The second specification is derived from the first as follows. We assume that $y_2$ is large enough so that constraint (9) is never binding. As mentioned in the text, a sufficient condition on the production function to ensure that this approach is valid is: $\lim_{x \to 0} xf'(x) = 0$.

Let us consider the determination of the haircut $h$ in period 2. Notice that the haircut appears only inside the $u(c_2)$ expression in the final period. The first order condition with respect to $h$ in this period yields:

$$u'(c_2) \left[ B_f - \tilde{R} f' \left( \left[ (1 - \alpha) + (1 - h)\alpha \right] D \right) \cdot \alpha D \right] = 0.$$ 

If $h$ is interior, it must satisfy this first order condition. It is also possible for the haircut $h$ to be at the boundaries 1 or 0 if the expression inside the square brackets is always positive or negative, respectively. The equation above provides the expression for the haircut in the main text, expression (10). The key result is that the haircut may be written in the form

$$h = H(\alpha, D, B_f, \tilde{R}).$$

Therefore the bond price can be derived:

$$Q(\alpha, D, B_f) = \frac{1}{1+r} \mathbb{E} \left\{ 1 - H(\alpha, D, B_f, \tilde{R}) \right\}.$$ 

Replace the haircut decision and the bond price with the above expressions. Next we reduce the number of state variables. Use the law of iterated expectations and define $\hat{U}_1 = \mathbb{E}U_1$, $\hat{U}_2(q, B_f) = \mathbb{E}U_2(q, B_f)$. The government problem in period 1 can be written:

The government problem may be written as follows. In period 1:

$$\hat{U}_1 = \max_{c_1, q, B_f} \mathbb{E} \left\{ u(c_1) + \beta \hat{U}_2(q, B_f) \right\}$$

48
subject to
\[ c_1 = y_1 + qB_f \]
\[ c_1 \geq 0 \]
\[ (q, B_f) \in \hat{G}, \]
where the expression \( \hat{U}_2 (q, B_f) \) is defined by the program:
\[
\hat{U}_2 (q, B_f) = \max_{c_2, \alpha, D} \mathbb{E} \{ u (c_2) \}
\]
subject to
\[ c_2 = y_2 - (1-h)B_f + \tilde{R} \left[ ((1-\alpha) + (1-h)\alpha] D \right) \]
\[ h = H \left( \alpha, D, B_f, \tilde{R} \right) \]
\[ q = Q (\alpha, D, B_f) \]
\[ B_f < 0 \Rightarrow \alpha = 0 \]
where \( \hat{G} \) is the set of feasible \((q, B_f)\) pairs generated by the \( q = Q (\alpha, D, B_f) \) relation. Notice that the second subproblem involves the government choosing \((\alpha, D)\) before the productivity shock in the second period is realized.

Finally, define
\[
V_2 (z) = \max_{q, B_f} \hat{U}_2 (q, B_f)
\]
subject to
\[ qB_f = z. \]
Then we derive the second version of the government program presented in the main text. The set of feasible asset values \( G \) is unbounded below. It is bounded above by the maximum value that can be achieved by the function \( Q (\alpha, D, B_f) \cdot B_f \). This value will depend upon the specification of the production function and the range of possible values for the productivity shock. 

**Proof of Formulations of the Government Problem in Subsection 5.1.**

This first specification of the government program is derived in a similar manner as above, but with the new restriction: \( p_B = q \). The representative consumer’s Euler equation for defaultable debt is not redundant because the price in this equation \( q \) appears elsewhere in the government problem. The representative consumer’s Euler equation for cash is redundant.

The second version of the government program in the main text follows the approach for non-tradable debt, but again with the new restriction: \( p_B = q \). It is important to check the cases where the defaultable debt is held entirely by either domestic agents (in which case \( B_f = 0 \)) or foreign
creditors (in which case \( \alpha D = 0 \)). We obtain the following government program.

\[
V_1 = \max_{c_1, z} \mathbb{E} \{ u(c_1) + \beta V_2 (z, \lambda) \}
\]

subject to

\[
c_1 = y_1 + z \\
c_1 \geq 0 \\
\lambda = u'(c_1) \\
(z, \lambda) \in \tilde{G}
\]

for some set \( \tilde{G} \), where the expression \( V_2 (z, \lambda) \) is given by

\[
V_2 (z, \lambda) = \max_{c_2, \alpha, D, B_f} \mathbb{E} \{ u(c_2) \}
\]

subject to

\[
c_2 = y_2 - (1 - h)B_f + \tilde{R} f \left( \left[ (1 - \alpha) + (1 - h)\alpha \right] D \right) \\
h = H \left( \alpha, D, B_f, \tilde{R} \right)
\]

If \( z < 0 \), combinations \( C = (\alpha, D, B_f) \) satisfy: \( \alpha = 0, B_f < 0 \).

If \( z = 0 \), combinations \( C = (\alpha(s), D(s), B_f(s)) \) satisfy one of:

(i) \( B_f = 0, \alpha D = 0 \);

(ii) \( B_f = 0, \alpha D > 0 \) with

\[
\frac{1}{1 + r} \mathbb{E} (1 - h) \leq \beta \mathbb{E} \left\{ \frac{u'(c_2)}{\lambda} \cdot (1 - h) \cdot \left[ 1 + \tilde{R} f' \left( \left[ (1 - \alpha) + (1 - h)\alpha \right] D \right) \right] \right\}
\]

(iii) \( B_f > 0 \) with

\[
\frac{1}{1 + r} \mathbb{E} (1 - h) = 0
\]

If \( z > 0 \), combinations \( C = (\alpha, D, B_f) \) satisfy:

\[
z = \frac{B_f \mathbb{E} (1 - h)}{1 + r}
\]

\[
z\lambda = \beta B_f \mathbb{E} \left\{ u'(c_2) \cdot (1 - h) \cdot \left[ 1 + \tilde{R} f' \left( \left[ (1 - \alpha) + (1 - h)\alpha \right] D \right) \right] \right\}.
\]

Our notation suppresses the dependence of \( h \) on \( (\alpha, D, B_f, \tilde{R}) \).

Now consider cases (ii) and (iii) for the debt level \( z = 0 \). Case (ii) can be replicated by issuing no defaultable debt and by issuing \( D \) units of cash instead. Case (iii) can be replicated by issuing zero defaultable debt. Therefore, without loss of generality, we may ignore these cases and assume that no defaultable debt is issued when the sovereign wishes to raise zero debt. This yields the
formulation in the main text.

Proof of Proposition 1.

In the first best case, the economy has access to contingent debt. Assume that $y_2$ is sufficiently high so that the government wishes to make net repayments abroad in every state of nature in period 2. Then the ability to save and borrow at the same time in period 1 is redundant, and the government chooses only to issue debt. Also assume that $y_2$ is sufficiently high so that the government always has enough resources to repay all of its debt if it so wishes. Consider the nontradable debt case. Output in period 2 is maximized by setting $A_d \geq \bar{x}$, $B_d$ to any value and the government problem reduces to the form written in the proposition. Results 2 and 3 of the proposition follow immediately.

For the tradable debt case, an additional constraint is added to the problem which reduces the feasible set:

$$\frac{1}{1+r} \mathbb{E}(1-h) \geq \beta \mathbb{E} \left\{ \frac{u'(c_2)}{u'(c_1)} \cdot (1-h) \cdot \left[ 1 + \tilde{R} f'(x) \right] \right\}.$$ 

This ensures that domestic banks do not purchase all of the defaultable debt, so foreign creditors hold some of the debt and the economy can borrow resources from abroad. It can be verified that the first best optimum in the nontradable debt case remains in the feasible set of the tradable debt case, now for some specific $B_d \geq 0$. The government sets $B_d$ at this value and issues total defaultable debt $B = B_d + B_f$. ■

Proof of Proposition 2.

Assume that $y_2$ is sufficiently high so that the government always has enough resources to repay all of its debt if it so wishes. Consider the nontradable debt case. The government can set $A_d \geq \bar{x}$ to maximize output in period 2 and $B_d$ is set to any value. The government problem for $B_f$ reduces to the form written in the proposition. Results 2 and 3 of the proposition follow immediately. For the tradable debt case, the additional constraint on the feasible set is now:

$$\frac{1}{1+r} \geq \beta \mathbb{E} \left\{ \frac{u'(c_2)}{u'(c_1)} \cdot \left[ 1 + \tilde{R} f'(x) \right] \right\}.$$ 

The optimum for the nontradable debt case remains feasible in the tradable debt case, now for some specific $B_d \geq 0$. The government sets $B_d$ at this value and issues total defaultable debt $B = B_d + B_f$. ■

Proof of Lemma 1.

The optimal haircut by the government for any realization of the productivity shock $\tilde{R}$ is derived by the maximization of the expression for consumption in period 2. This helps us to understand the output and consumption profiles in period 2. For any realization of the productivity shock $\tilde{R}$,
output is given by \( \tilde{R}f(x) \), where \( x = [(1 - \alpha) + (1 - h)\alpha]D \). Consumption is given by the formula

\[
c_2 = y_2 - (1 - h)B_f + \tilde{R}f(x).
\]

Consider a combination \( C = (\alpha, D, B_f) \) such that \( D > \bar{D} \), or equivalently, a combination \( (A_d, B_d, B_f) \) such that \( [A_d + B_d] > \bar{D} \). We consider the possible cases.

Case 1: \( 0 \leq A_d \leq \bar{D} \).

Keep \( A_d \) unchanged. Reduce the magnitudes of \( B_d \) and \( B_f \) equipropotionately until \( [A_d + B_d] = \bar{D} \). This combination \( C' \) raises the same revenues as \( C \) in period 1 and is equivalent to \( C \) in terms of repayments abroad, output and hence consumption for every realization of the productivity shock \( \tilde{R} \) in period 2.

Case 2: \( A_d > \bar{D} \).

Set \( A_d = \bar{D} \) and \( B_d = B_f = 0 \). This combination \( C' \) raises the same revenues as \( C \) in period 1 and is equivalent to \( C \) in terms of repayments abroad, output and hence consumption for every realization of the productivity shock \( \tilde{R} \) in period 2.

Proof of Corollary 1.

This immediately follows from Lemma 1.

Proof of Proposition 3.

The formula in this proposition follows from the following equation:

\[
\frac{B_f}{RB_d} = f'([A_d + (1 - h)B_d]) \quad (15)
\]

and the restriction established by Corollary 1. Given the assumptions on the production function, \( (f')^{-1} [\gamma] \) is strictly decreasing in the argument \( \gamma \). This yields the comparative statics listed.

Proof of Proposition 4.

The first order condition with respect to the haircut \( h \) in period 2 yields equation (15), which is illustrated in figure 3. This characterizes the solution for interior \( h \). The upper bound for the haircut is binding when

\[
(f')^{-1} \left[ \frac{B_f}{RB_d} \right] < A_d.
\]
This condition may hold for states of nature with lower productivity realizations \( \tilde{R} \). Denote these states by the set \( S_1 = \{ R^*, ..., R^* \} \). The zero bound for the haircut is binding when

\[
(f')^{-1} \left[ \frac{B_f}{R B_d} \right] > A_d + B_d.
\]

This condition may apply for states of nature with higher productivity realizations \( \tilde{R} \). Denote these states by the set \( S_3 = \{ R^{**}, ..., \tilde{R} \} \).

The first order condition determines the haircut when it is interior, i.e., for states between \( R^* \) and \( R^{**} \) (non-inclusive). Let us denote these states by the set \( S_2 \). The above argument establishes the formula for the total real resources raised by the government as a function of the combination \((A_d, B_d, B_f)\):

\[
z = \frac{1}{1 + r} \sum_{R \in S} (1 - h) B_f \cdot \Pr(\tilde{R})
\]

\[
= \frac{1}{1 + r} \sum_{R \in S_2} \gamma \cdot \left( (f')^{-1} \left[ \frac{\gamma}{R} \right] - A_d \right) \Pr(\tilde{R}) + \frac{1}{1 + r} \sum_{R \in S_3} B_f \cdot \Pr(\tilde{R})
\]

where we define \( \gamma = \frac{B_f}{B_d} \).

The final step of the proof is to show that formula (16) can yield \( z > 0 \) for some choice of configuration \((A_d, B_d, B_f)\). Set \( A_d = 0 \) and \( B_d = \bar{D} \). Equation (16) reduces to

\[
z = \frac{\gamma}{1 + r} \sum_{R \in S} (f')^{-1} \left[ \frac{\gamma}{R} \right] \Pr(\tilde{R}).
\]

Choose any \( \gamma > 0 \). Then \( (f')^{-1} \left[ \frac{\gamma}{R} \right] \in (0, \bar{x}) \). The absolute value of borrowing is positive, as required.

**Proof of Proposition 5.**

For negative debt values \( z < 0 \), i.e., saving abroad, there is no default by foreigners because foreign institutions credibly commit to fully repay. The government can only issue cash to domestic residents. Therefore:

\[
f \left( [(1 - \alpha) + (1 - h)\alpha] D \right) = f (D)
\]

\[
z = \frac{B_f}{1 + r}.
\]

It is straightforward to see that the proposition follows.
Proof of Proposition 6.

Step 1: Attainment of a maximum $z$ for each $(A_d, B_d)$ pair.

Equation (16) yields

$$
z = \frac{1}{1+r} \sum_{\tilde{R} \in S_2} \gamma \cdot \left\{ (f')^{-1} \left[ \frac{\gamma}{\tilde{R}} \right] - A_d \right\} \Pr(\tilde{R}) + \frac{1}{1+r} \sum_{\tilde{R} \in S_3} B_d \cdot \Pr(\tilde{R})$$

$$= \frac{\gamma}{1+r} \sum_{\tilde{R} \in S_2} (f')^{-1} \left[ \frac{\gamma}{\tilde{R}} \right] \Pr(\tilde{R}) - \frac{\gamma}{1+r} \cdot A_d \left[ \sum_{\tilde{R} \in S_2} \Pr(\tilde{R}) \right] + \frac{\gamma}{1+r} \cdot B_d \left[ \sum_{\tilde{R} \in S_3} \Pr(\tilde{R}) \right]. \tag{17}$$

Fix $A_d$ and $B_d > 0$ and treat this expression as a function of $\gamma \in [0, \infty)$. It is continuous in $\gamma$. First, consider $A_d > 0$. It can be shown that there exists a value $\gamma_M(A_d, B_d)$ such that for all $\gamma > \gamma_M(A_d, B_d)$, the sets $S_2$ and $S_3$ are empty and therefore $z = 0$. This feature means that the supremum value of $z$ must be achieved for $\gamma$ within the compact set $[0, \gamma_M(A_d, B_d)]$. Apply the Weierstrass Theorem for the function defined on this compact set. This proves that expression (17) attains a maximum for some $\gamma$ in this set. All values of $z$ between 0 and the maximum value can be achieved.

Next, consider $A_d = 0$ and $B_d > 0$. A different approach is required. The set $S_1$ is now empty, and equation (17) reduces to:

$$z = \frac{\gamma}{1+r} \sum_{\tilde{R} \in S_2} (f')^{-1} \left[ \frac{\gamma}{\tilde{R}} \right] \Pr(\tilde{R}) + \frac{\gamma}{1+r} \cdot B_d \left[ \sum_{\tilde{R} \in S_3} \Pr(\tilde{R}) \right]. \tag{18}$$

Consider each of the functions for values of the productivity shock $\tilde{R}$:

$$g(\gamma, \tilde{R}) = \gamma \cdot (f')^{-1} \left[ \frac{\gamma}{\tilde{R}} \right].$$

An increase in $\gamma$ corresponds to a reduction in $x$. The function $g(\gamma, \tilde{R})$ can be written as:

$$g(\gamma, \tilde{R}) = \tilde{R} f'(x) \cdot x \text{ for } x = (f')^{-1} \left[ \frac{\gamma}{\tilde{R}} \right].$$

We assume that $\lim_{x \to 0} x f'(x) = 0$. This establishes that $\lim_{\gamma \to \infty} g(\gamma, \tilde{R}) = 0 \forall \tilde{R}$, and hence that $\lim_{\gamma \to \infty} \mathbb{E}g(\gamma, \tilde{R}) = 0$.

Whatever the value of $B_d$, as $\gamma$ increases there eventually comes a point when the set $S_3$ becomes empty. Therefore, the limit of the expression (18) as $\gamma \to \infty$ is equal to the limit of the same expression without the second term.

$$\lim_{\gamma \to \infty} z(\gamma) = \lim_{\gamma \to \infty} \mathbb{E}g(\gamma, \tilde{R}) = 0.$$
There exists debt for this case \( A_d = 0, B_d > 0 \). Choose any value \( \gamma > 0 \), and this yields a positive value of debt, say \( z_1 \). Let us pick a value of \( \varepsilon \in (0, z_1) \). This value of \( \varepsilon \) corresponds to a number \( \gamma_M (0, B_d) \) such that \( z \) is lower than \( \varepsilon \) for all \( \gamma \geq \gamma_M (0, B_d) \). Then we know that the supremum level of debt that can be attained lies in the set \( [0, \gamma_M (0, B_d)] \). Apply the Weierstrass Theorem for the continuous function (18) over this compact set. This establishes that the maximum is attained. Furthermore, all values of \( z \) between 0 and the maximum value can be achieved.

For \( B_d = 0 \), \( z = 0 \) irrespective of the value of \( A_d \).

**Step 2: Comparative statics with respect to \( \alpha \), given \( D = D' \).**

Fix \( D = D' \). Consider an increase in \( A_d \) and an equal reduction in \( B_d \). Furthermore, change \( B_f \) so as to preserve the value of the ratio \( \gamma = \frac{B_f}{B_d} \). This corresponds to a reduction in \( \alpha \). The perturbation considered reduces the value of the function in expression (17) for any given value of \( \gamma \), both through a fall in the second term and a possible reduction in the set of states \( S_2 \) for which the upper bound on the haircut is not binding. Therefore, the maximum value of the expression given by equation (17) must be lower (the maximum is still attained, by repeated application of the Weierstrass Theorem). This establishes that \( \alpha (z) \) is weakly increasing in \( z \).

For \( z = 0 \), we may set \( \gamma = 0 \). The value of \( A_d \) does not matter. Therefore \( \alpha (0) = 0 \). From the argument above, the highest value of the expression (17) is achieved when \( A_d = 0 \). Therefore \( \alpha (\tilde{z}_{\text{max}} (D')) = 1 \).

**Proof of Proposition 8.**

There are two steps of the proof.

**Step 1: The zero bound for the haircut is never binding at the optimal combination \( C = (\alpha, D, B_f) \).**

Proof by contradiction.

**Case 1: \( B_f > 0 \).**

Suppose that the optimal combination \( C = (\alpha, D, B_f) \) satisfies \( D < (f')^{-1} \left[ \frac{B_f}{RB_d} \right] \). The zero bound for the haircut is binding for states in the set \( S_3 = \{ R^{**}, ..., \bar{R} \} \). For these states:

\[
(f')^{-1} \left[ \frac{B_f}{RB_d} \right] > D
\]

\[
\iff \frac{B_f}{RB_d} < f' (D).
\]

Consider the following perturbation: an infinitesimal equiproportionate increase in the magnitudes of \( B_d \) and \( B_f \) that preserves the value of the ratio \( \gamma = \frac{B_f}{B_d} \). This perturbation is feasible. Since \( B_d \)
and \( B_f \) take positive values, the perturbation involves \( dB_d, dB_f > 0 \).

For states of nature in the set \( S \setminus S_3 \), the perturbation does not change repayments abroad, output or consumption. For states of nature in the set \( S_3 \), the perturbation does have an effect. Note that

\[
c_2 \left( \alpha, D, B_f, \tilde{R} \right) = y_2 - (1 - h)B_f + \tilde{R}f \left( (1 - \alpha) + (1 - h)\alpha \right) D
\]

where

\[
h = H \left( \alpha, D, B_f, \tilde{R} \right).
\]

We suppress the dependence of \( c_2 \) on \( \left( \alpha, D, B_f, \tilde{R} \right) \) in our notation. For values of the productivity shock \( \tilde{R} \in S_3 \), the perturbation has the following effect on \( c_2 \):

\[
dc_2 = -d \left[ (1 - h)B_f \right] + \tilde{R}df
\]
\[
= -\frac{B_f}{B_d} dB_d + \tilde{R}f' (D) dB_d
\]
\[
> -\frac{B_f}{B_d} dB_d + \tilde{R} \frac{B_f}{RB_d} dB_d = 0.
\]

So the perturbation increases consumption \( c_2 \) and increases repayments abroad for these states of nature. The second effect increases \( z \) in the initial period, which means higher consumption in period 1.

Therefore, the perturbation considered increases consumption in period 1 and weakly increases consumption for every value of the productivity shock \( \tilde{R} \) in period 2. The original combination cannot have been optimal. This argument means that we choose \( D \) such that:

\[
D \geq (f')^{-1} \left[ \frac{B_f}{\tilde{R} \alpha D} \right].
\]

**Case 2: \( B_f = 0 \).**

Suppose that the optimal combination \( C = (\alpha, D, B_f) \) satisfies \( D < \tilde{D} \). The optimal government policy in period 2 is not to default at all. Consider the following perturbation: an infinitesimal increase in the magnitude of \( D \). This is feasible, and leaves the optimal policy unchanged. The perturbation has the following effect:

\[
dc_2 = \tilde{R}df
\]
\[
= \tilde{R}f' (D) dD > 0, \quad \forall \tilde{R}.
\]

So the perturbation increases consumption \( c_2 \) for all values of the productivity shock \( \tilde{R} \) in the final period, while leaving consumption in the initial period unchanged. The original combination cannot have been optimal.
Step 2: Completion of Proof.

This immediately follows from the above. ■

Proof of Proposition 9.

We optimally set \( D = \bar{D} \), so the set \( S_3 \) is empty.

The approach in this proof is to take a given combination \( C = (\alpha, \bar{D}, B_f) \) which raises debt \( z \) such that \( \alpha > \alpha(z) \), and to ask whether this combination is optimal. In particular, one perturbation we consider is to reduce the level of domestic exposure \( \alpha \) and adjust \( B_f \) appropriately so that the same level of debt \( z \) is raised in period 1.

Step 1: Optimal size of \( B_f \) for any given exposure level.

Consider two combinations \( C = (\alpha, \bar{D}, B_f) \) and \( C' = (\alpha, \bar{D}, B'_f) \) which raise the same level of debt \( z \). The exposure level is the same but for combination \( C' \), the ratio \( \gamma' = \frac{B'_f}{\alpha \bar{D}} \) is larger than \( \gamma = \frac{B_f}{\alpha \bar{D}} \). It can be shown that the combination \( C' \) is not optimal. For any given level of domestic exposure, we choose the lowest value of \( B_f \) that raises any given level of debt \( z \). For the basic idea behind this part of the proof, see the proof of Proposition 10. The corollary of this result is that if we plot the debt level \( z \) as a function of \( \gamma \) for a given level of \( \alpha \), a necessary condition for a combination \( C = (\alpha, \bar{D}, B_f) \) to be optimal is that it lies either (i) on the upward-sloping portion of the graph, or (ii) at a local (non-global) maximum of the graph. We assume this condition holds.

Step 2: Perturbation that reduces \( \alpha \) and increases \( \gamma = \frac{B_f}{\alpha \bar{D}} \).

Let us increase \( A_d \) by an infinitesimal amount, so that the exposure level \( \alpha \) falls. This corresponds to a downward shift in the graph of \( z \) against \( \gamma \) described above, with a greater downward shift for higher \( \gamma \) values. From the information in Step 1 of this proof, it is therefore only feasible to raise the same level of debt \( z \) by increasing \( \gamma \). The increase in \( \gamma \) is marginal if the combination was initially on the upward-sloping portion of the graph, and it is a discrete jump in \( \gamma \) if the combination was initially at a local (non-global) maximum. In the latter case, there is a discrete fall in consumption and the perturbation is not optimal. The remainder of this proof concentrates on the former case. We split the perturbation into two stages.

Increase \( A_d \), and reduce \( B_d \) and \( B_f \) equiproportionately to keep \( D = \bar{D} \) and \( \gamma = \frac{B_f}{\alpha \bar{D}} \) constant. For states of nature \( \bar{R} \in S_1 \), repayments abroad in period 2 are unaffected, but output (and hence consumption) is affected in the final period:

\[
dc_2 = \bar{R}f'(A_d) dA_d > 0.
\]
For states of nature $\tilde{R} \in S_2$, output is unaffected, but repayments fall:

$$dc_2 = \frac{B_f}{B_d} dA_d > 0.$$ 

This perturbation reduces the debt level $z$.

**Increase $B_f$ until the debt level $z$ rises to the initial level.** For states of nature $\tilde{R} \in S_1$, repayments abroad, output and consumption in period 2 are unaffected.

For states of nature $\tilde{R} \in S_2$, repayments rise and consumption falls. Adapting the expression in the proof of Proposition 10:

$$dc_2 = - \left\{ (f')^{-1} \left[ \frac{\gamma}{R} \right] - A_d \right\} \cdot d\gamma < 0.$$ 

Now combine the two components of the perturbation described above, and set $dA_d$ and $d\gamma$ so that the level of debt raised $z$ is unchanged as a result of the perturbation. For states of nature $\tilde{R} \in S_1$:

$$dc_2 = \tilde{R} f' (A_d) dA_d > 0.$$ 

For states of nature $\tilde{R} \in S_2$:

$$dc_2 = \sum_{\tilde{R} \in S_2} \frac{d\gamma}{Pr(\tilde{R})} \cdot \left[ \sum_{\tilde{R} \in S_2} \left\{ (f')^{-1} \left[ \frac{\gamma}{R} \right] + \frac{\gamma}{R} \left[ (f')^{-1} \left[ \frac{\gamma}{R} \right] \right]' \right\} Pr(\tilde{R}) \right] - d\gamma \cdot \left\{ (f')^{-1} \left[ \frac{\gamma}{R} \right] - A_d \right\}.$$ 

It is straightforward to prove that:

$$\mathbb{E}_{\tilde{R} \in S_2} \{dc_2\} < 0, \quad \frac{d}{d\tilde{R}} \{dc_2\} < 0 \quad \forall \tilde{R} \in S_2.$$ 

Consider the set $S_2$. Such a deviation depresses average consumption in the set $S_2$, but consumption is depressed less in states of nature which receive a worse productivity shock. For permissible production functions $f(x)$, it is possible that the combination of deviations actually increases consumption for the worst productivity shock realizations within $S_2$ in period 2. This may apply even if set $S_1$ is empty. If $S_1$ is not empty, of course, consumption necessarily increases for the worst realizations of the productivity shock $\tilde{R}$.

It follows that there exist some utility functions $u(c)$ with sufficiently high risk aversion such that the representative consumer finds this perturbation optimal. The sovereign finds it optimal to reduce the level of domestic exposure because this perturbation enables the sovereign to purchase some insurance across states of nature against the productivity shock realization $\tilde{R}$. ■
Proof of Proposition 10.

Step 1: Proof that it is optimal to choose the lowest level of $\gamma$ that achieves a given debt level $z$.

We set $D = \bar{D}$, and let $u(c)$, $f(x)$ and $y_2$ be specified such that $\alpha = 1$ is the optimal level of exposure for all levels of debt. Then sets $S_1$ and $S_3$ are empty. Consider any combination $C = (1, \bar{D}, B_f)$ and define $\gamma = \frac{B_f}{\bar{D}}$. What is the effect of an infinitesimal increase in $\gamma$ on consumption in period 2?

$$dc_2 = -d[(1 - h)B_f] + \bar{R}df$$

where

$$-d[(1 - h)B_f] = -d[(1 - h)B_d \cdot \gamma]$$

$$= -\gamma \cdot d[(1 - h)B_d] - (1 - h)B_d \cdot d\gamma$$

$$= -\gamma \cdot \left( [(f')^{-1}]' \left[ \frac{\gamma}{R} \right] \frac{d\gamma}{R} \right) - \left( (f')^{-1} \left[ \frac{\gamma}{R} \right] \right) \cdot d\gamma$$

and

$$df = f' \left( (f')^{-1} \left[ \frac{\gamma}{R} \right] \right) \left( [(f')^{-1}]' \left[ \frac{\gamma}{R} \right] \frac{d\gamma}{R} \right)$$

$$= \frac{\gamma}{R} \cdot \left( [(f')^{-1}]' \left[ \frac{\gamma}{R} \right] \frac{d\gamma}{R} \right) \cdot d\gamma.$$ 

Therefore

$$dc_2 = - (f')^{-1} \left[ \frac{\gamma}{R} \right] \cdot d\gamma < 0. \tag{19}$$

We have presented full details of the mathematical derivation, but since the haircut decision is always interior and set in an optimal manner, the final result can be reached more concisely via application of the Envelope Theorem. The expression $dc_2$ takes the same sign for all $\gamma \in [0, \infty)$.

Thus an increase in the ratio $\gamma$ always reduces consumption $c_2$ for all values of the productivity shock $\bar{R}$ in period 2. Now suppose that the same level of $z$ can be achieved for two levels of the ratio $\frac{B_f}{\bar{D}}$ given by $\gamma_1$ and $\gamma_2 > \gamma_1$. Both values of $\gamma$ achieve the same level of consumption in period 1, but $\gamma_2$ results in lower consumption than $\gamma_1$ in period 2. Therefore the lower value of $\gamma$ must be optimal.

Step 2: Completion of proof.

Equation (17) reduces to

$$z = \frac{\gamma}{1 + r} \sum_{R \in S} (f')^{-1} \left[ \frac{\gamma}{R} \right] \Pr(R).$$
The above expression is continuous in $\gamma \in [0, \infty)$. Imposing that $\lim_{x \to 0} xf'(x) = 0$, the relevant set for $\gamma$ is a compact set of the form $[0, \gamma_{\text{max}}]$ and the maximum value of $z$ is attained on this set. Step 1 of the proof implies that the optimal $B_f$ value will lie within the region $[0, \gamma_{\text{max}}]$. Continuity of the $z(\gamma)$ function and application of the result in step 1 of the proof yields the result that the optimal $B_f$ is increasing in $z$.

From result 1 of Proposition 3, an increase in $B_f$ increases the optimal haircut for all values of the productivity shock $\bar{R}$, and hence the interest rate on government debt. ■

**Proof of Proposition 11.**

We set $D = \tilde{D}$. Let $u(c)$, $f(x)$ and $y_2$ be specified such that $\alpha = 1$ is the optimal level of exposure for all levels of debt. The condition $\lim_{x \to 0} xf'(x) = 0$ means that the set of feasible debt levels takes the form $[0, z_{\text{max}}]$. Note also that the expression for $z$ as a function of $\gamma$ is continuous and differentiable. As the desired level of debt $z$ varies, the optimal level of $\gamma$ (and hence $B_f$) may exhibit discontinuities. The Euler condition needs to take this into account. The left derivative takes the form:

$$
\left( \frac{\Delta c_2}{dz} \right)_- = -(1 + r) \frac{(f')^{-1} \left[ \frac{z}{\bar{R}} \right]}{\mathbb{E} \left\{ (f')^{-1} \left[ \frac{z}{\bar{R}} \right] + \frac{\gamma + \Delta \gamma}{\bar{R}} (f')^{-1} \left[ \frac{z}{\bar{R}} \right]' \right\}} < 0,
$$

which approaches $\infty$ for $\gamma$ corresponding to a point of discontinuity. The right derivative takes the above form at points when the optimal $B_f$ schedule is continuous in $z$, but the following form at points of discontinuity:

$$
\left( \frac{\Delta c_2}{dz} \right)_+ = -(1 + r) \frac{(f')^{-1} \left[ \frac{\gamma + \Delta \gamma}{\bar{R}} \right]}{\mathbb{E} \left\{ (f')^{-1} \left[ \frac{\gamma + \Delta \gamma}{\bar{R}} \right] + \frac{\gamma + \Delta \gamma}{\bar{R}} (f')^{-1} \left[ \frac{\gamma + \Delta \gamma}{\bar{R}} \right]' \right\}} - \int_{\gamma}^{\gamma + \Delta \gamma} (f')^{-1} \left[ \frac{z}{\bar{R}} \right] \cdot dz < 0,
$$

where $\Delta \gamma$ is the jump in $\gamma$ at the point of discontinuity.

The above argument proves the first claim in the proposition. The third claim follows from the formulae presented above and equation (19), which can be used to prove that a downward jump in $\gamma$ improves consumption more for higher realizations of the productivity shock $\bar{R}$.

The second claim in the proposition is proven as follows. First, consider the left derivative and define

$$
\Upsilon_-(z, \bar{R}) = \frac{(f')^{-1} \left[ \frac{z}{\bar{R}} \right]}{\mathbb{E} \left\{ (f')^{-1} \left[ \frac{z}{\bar{R}} \right] + \frac{\gamma + \Delta \gamma}{\bar{R}} (f')^{-1} \left[ \frac{z}{\bar{R}} \right]' \right\}}.
$$

We prove that:

$$
\mathbb{E} \left\{ \Upsilon_-(z, \bar{R}) \right\} = \frac{\mathbb{E} \left\{ x \right\}}{\mathbb{E} \left\{ x \right\} + \mathbb{E} \left\{ \frac{f'(x)}{f''(x)} \right\}} > 1,
$$

60
which establishes the result desired. A similar argument applies for the right derivative, after taking the points of discontinuity into account. ■

Proof of Proposition 12.

This is identical to the proof of Proposition 3. ■

Proof of Proposition 14.

Domestic banks are willing to purchase an infinite quantity of defaultable debt at any price less than
\[
p_B = \beta \mathbb{E} \left\{ (1 - h) \cdot \left[ 1 + \tilde{R} f' \left( [(1 - \alpha) + (1 - h)\alpha] D \right) \right] \right\}
\]
Foreign creditors are willing to purchase an unlimited quantity at any price below
\[
q = \frac{1}{1 + r} \mathbb{E} (1 - h)
\]
For the tradable debt case, \( p_B = q \):
\[
\frac{1}{1 + r} \mathbb{E} (1 - h) = \beta \mathbb{E} \left\{ (1 - h) \cdot \left[ 1 + \tilde{R} f' \left( [(1 - \alpha) + (1 - h)\alpha] D \right) \right] \right\}
\]
Equation (20) does not hold, so debt is not feasible.

Let \( \beta = 0 \). Whatever the combination \( C = (\alpha, D, B_f) \), domestic banks are not willing to purchase debt at any positive price. Debt cannot be sustained.

If:

Let \( \beta \in \left( 0, \frac{1}{1 + r} \right) \). We prove the existence of debt by construction. For this case, there exists a finite \( \gamma^* = \frac{B_f}{B_d} > 0 \) such that:
\[
1 = \beta (1 + r) [1 + \gamma^*].
\]
Set \( A_d = 0, B_d = \bar{D} \) and \( B_f = \gamma^* \cdot \bar{D} > 0 \). For any value of the productivity shock in period 2, the optimal haircut decision follows:
\[
\gamma^* = \tilde{R} f' \left( [A_d + (1 - h)B_d] \right).
\]
If conditions (21) and (22) hold, it is immediate that equation (20) is satisfied. It is feasible for the sovereign to issue debt.

As required. ■

Proof of Proposition ??.

In the nontradable debt case, the maximum debt level $z_{\text{max}}$ is achieved for $\alpha = 1$ and $D = \bar{D}$. We set $\gamma = \frac{B_f}{D}$ so as to maximize the debt level for these values of $\alpha$ and $D$. Denote this level as $\hat{\gamma}$. For the tradable debt case, it is possible to show that the maximum level of $\gamma$ is $\gamma^*$, as defined in the proof of Proposition 14. If $\gamma^* < \hat{\gamma}$, the maximum level of debt in the tradable debt case is clearly strictly lower than $z_{\text{max}}$. If $\gamma^* > \hat{\gamma}$, then it can be proven that a lower level of $\gamma$ can only be achieved for exposure level $\alpha = 1$ if the total level of domestic debt $D$ is strictly less than $\bar{D}$. In fact, it is achieved for a value of $D$ such that the zero bound for the haircut is binding. Again, the maximum debt level is strictly lower than $z_{\text{max}}$. Only if $\gamma^* = \hat{\gamma}$ is the maximum level of debt equal for the nontradable and tradable debt cases.

Since $\gamma^*$ depends on $\beta$, it follows the maximum level of debt in the tradable debt case depends on the same parameter. ■

Proof of Proposition 15.

For $z < 0$, foreigners credibly commit not to default on the government’s savings abroad. The government issues cash to domestic residents, and it chooses the value of $D$ that maximizes domestic production. The claims in the proposition follow immediately. ■

Proof of Lemma 2.

For any combination $C = (\alpha, D, B_f)$ such that $D > \bar{D}$, Lemma 1 establishes that there exists some other combination $C' = (\alpha', D', B'_f)$ where $D = \bar{D}$, such that $C'$ raises the same revenues as $C$ in period 1 and is equivalent to $C$ in terms of repayments abroad, output and hence consumption for all values of the productivity shock $\bar{R}$ in period 2. We construct $C'$ from $C = (\alpha, D, B_f)$ in the same manner as described in the proof of Lemma 1. It remains to prove that $C$ and $C'$ satisfy the Euler condition for defaultable debt for the same value of $\lambda$.

Case 1: $z \leq 0$.

Construct $C'$ from $C = (\alpha, D, B_f)$ in the same manner as in the proof of Lemma 1. None of the defaultable debt is issued, so no Euler condition needs to be satisfied for this debt.

Case 2: $z > 0$.

Apply Lemma 1. Since $C'$ raises the same revenues as $C$ in period 1 and is equivalent to $C$ in terms of repayments abroad, output and hence consumption for all values of the productivity shock $\bar{R}$ in
period 2, it also follows that the expression $f'(x)$ is also the same for $C$ and $C'$ for all values of the productivity shock $\tilde{R}$. These results are sufficient to show that $C$ and $C'$ satisfy the equation

$$\lambda \cdot \frac{1}{1 + \rho} \mathbb{E}(1 - h) B_f = \beta \mathbb{E} \left\{ u'(c_2) \cdot (1 - h) B_f \cdot \left[ 1 + \tilde{R} f'(\frac{(1 - \alpha) + (1 - h)\alpha}{D}) \right] \right\}$$

for the same value of $\lambda$. ■

**Proof of Corollary 2.**

This immediately follows from Lemma 2. ■

**Proof of Proposition 16.**

Fix $D = D'$ such that the zero bound for the haircut is binding for some values of the productivity shock $\tilde{R}$ in period 2. Consider an increase in $A_d$ and an equal reduction in $B_d$. This corresponds to a reduction in $\alpha$. Furthermore, change $B_f$ so as to preserve the value of the ratio $\gamma = \frac{B_f}{B_d}$. This perturbation was feasible in the nontradable debt case, but it is no longer feasible in the tradable debt case. The exposure mechanism means that the haircut increases for those states of nature where the haircut was initially interior. The price that foreigners are willing to pay for the debt declines by more than the price that domestic banks are willing to pay, because domestic banks still receive a high loans rate in those states of nature when the debt is fully repaid in period 2. The right hand side of equation (20) exceeds the left hand side. It can only be satisfied with equality again if the ratio $\gamma$ falls.

In the nontradable debt case, any value of $\gamma$ is achievable for given $\alpha$ and $D$. This means that for any given $\alpha$ and $D$, the entire set of debt levels $[0, z_{\text{max}}(D')]$ can be achieved by varying $\gamma$. In the tradable debt case, this is no longer true. The corollary of this result is that only a restricted set of $z$ values is achievable for any given $\alpha$ and $D$. ■

**Proof of Proposition 18.**

Set $D$ equal to the smallest value such that the zero bound on the haircut is not binding for any value of the productivity shock $\tilde{R}$ in period 2, i.e., $D = \tilde{D}$, where

$$\tilde{D} = \left( f' \right)^{-1} \left[ \frac{B_f}{RB_d} \right].$$

The only value of $\gamma = \frac{B_f}{B_d}$ consistent with this value of $D$ and equation (20) is $\gamma^*$, as given in the proof of Proposition 14. Let $\gamma = \gamma^*$. Now consider alternative perturbations that increase expected consumption in period 2. It can be shown that there are only two possible perturbations. We consider the effect of each on consumption in periods 1 and 2.
Perturbation 1: Increase $A_d$, keeping $D = \bar{D}$ and $\gamma = \gamma^*$ fixed. Step 2 of the proof of Proposition 9 establishes that this perturbation increases consumption for states of nature in the sets $S_1$ and $S_2$. Set $S_3$ is empty. Therefore, expected consumption in period 2 increases. The debt level $z$ is affected:

$$dz = -\frac{\gamma}{1 + r} \left[ \sum_{\tilde{R} \in S_2} \Pr(\tilde{R}) \right] dA_d < 0,$$

which means that consumption in period 1 falls by this same amount.

Perturbation 2: Reduce $B_d$ and $\gamma$, keeping $A_d$ fixed. $B_d$ and $\gamma$ must be reduced in a manner such that equation (20) is still satisfied. The reduction in $B_d$ has no effect on consumption for any value of the productivity shock. It does not affect output or the volume of debt repayments for any values of the productivity shock lower than $\bar{R}$. For the highest productivity shock value, it reduces output and repayments equally at the margin, leaving consumption unchanged. Step 2 of the proof of Proposition 9 establishes that the reduction in $\gamma$ improves consumption for all states of nature in the sets $S_1$ and $S_2$. It can be verified that this is also true for states of nature in the set $S_3$. Therefore, expected consumption in period 2 increases. Let it increase by the same amount as in perturbation 1. The effect on the debt level $z$ is ambiguous. For the debt level $z$ to rise, the condition that must be satisfied is:

$$\left[ \sum_{\tilde{R} \in S_2} \left\{ (f')^{-1} \left[ \frac{\gamma}{\bar{R}} \right] + A_d + \frac{\gamma}{\bar{R}} \left[ (f')^{-1} \right]' \left[ \frac{\gamma}{\bar{R}} \right] \right\} \Pr(\tilde{R}) \right] + B_d \Pr(\bar{R}) + \frac{\gamma}{\bar{R} \left| f''(\bar{D}) \right|} \sum_{\tilde{R} \in S_2} (1 - h) \Pr(\tilde{R}) < 0.$$

The term in the square brackets is ambiguous in sign. Let it be negative. The interpretation of this assumption is that in this range, an increase in the proportion of debt held by domestic banks increases the price of debt sufficiently to increase the total repayments for all states of nature in the set $S_2$. If $\Pr(\bar{R})$ is very small, and $\bar{R}$ and $\left| f''(\bar{D}) \right|$ are very large, then the condition above may be satisfied. ■

B. Division of Debt Categories into Cash and Defaultable Debt

In the environment studied in the paper, the government can issue two types of debt: cash (type $A$) and defaultable debt (type $B$). This section of the appendix shows that this delineation of debt types is an equilibrium outcome of a slightly more general model. In this general model, the government can issue three types of debt. Debt type $A$ can be held by domestic residents only, debt type $B$ can be held by both domestic and foreign lenders, and debt type $M$ can be held by foreign creditors only. The government can choose different haircuts for the three different types of debt: $h_A$, $h_B$ and $h_C$ respectively. We prove that the government will choose to fully repay all of debt $A$, to repay none of debt $M$, and to repay the fraction $1 - h$ of debt type $B$. These results establish that the more general model reduces to the model presented in the main text of the paper.
Modifications to the Model

The maximization problem of domestic agents now takes into account that all debt is defaultable. Specifically, the Euler equations of the representative consumer will be modified in the appropriate manner to take the haircuts into consideration. The government budget constraints are altered to the following:

\[ T_1 \leq p_A A_d + p_B B_d + q_B B_f + q_M M_f, \]  \hspace{1cm} (23)  
\[ T_2 \geq (1 - h_A)A_d + (1 - h_B)\{B_d + B_f\} + (1 - h_M)M_f \]  \hspace{1cm} (24)

Prices of foreign-held debt follow the equations:

\[ q_B = \frac{1}{1 + r} \mathbb{E}\{1 - h_B\} \]  \hspace{1cm} (25)  
\[ q_M = \frac{1}{1 + r} \mathbb{E}\{1 - h_M\}. \]  \hspace{1cm} (26)

The definition of equilibrium follows.

**Definition 3** A **Rational Expectations Equilibrium** for this economy comprises sequences for allocation rules \( \{c_1, s_1, c_2, \{\chi\}, x\} \), prices \( \{p_A, p_B, q_B, q_M, \rho\} \) and policies \( \{A_d, B_d, B_f, M_f, h_A, h_B, h_M, T_1, T_2\} \) that satisfy:

(a) Consumers choose \( \{c_1, s_1, c_2\} \) to maximize utility (1) subject to the budget constraints (2), (3) and the nonnegativity constraints on consumption (4), taking prices, bank contract offers, government policies and the endowment as given.

(b) Banks offer contract schedules \( \chi : s_1 \rightarrow S(s_1, \tilde{R}) \) in period 1 to maximize expected profits, taking prices and government policies in period 2 as given.

Banks choose lending quantity \( x \) in period 2 to maximize profits, taking the loan rate \( \rho \) as given.

(c) Firms choose borrowing level \( x \) to maximize profits (8), taking the loan rate \( \rho \) as given.

(d) Government chooses \( \{h_A, h_B, h_M, T_2\} \) in period 2 to satisfy the government budget constraint (24) in that period, taking \( \{A_d, B_d, B_f, M_f\} \) and the shock \( \tilde{R} \) as given.

Government chooses \( \{A_d, B_d, B_f, M_f, T_1\} \) in period 1 to satisfy the government budget constraint (23) in that period, taking the price functions \( \{p_A(A_d, B_d, B_f, M_f), p_B(A_d, B_d, B_f, M_f), q_B(A_d, B_d, B_f, M_f), q_M(A_d, B_d, B_f, M_f), \rho(x)\} \) and government policies in period 2, \( h_A(A_d, B_d, B_f, M_f, \tilde{R}), h_B(A_d, B_d, B_f, M_f, \tilde{R}), h_M(A_d, B_d, B_f, M_f, \tilde{R}) \) and \( T_2(A_d, B_d, B_f, M_f, \tilde{R}) \), as given.

(e) All markets clear for the economy. In particular, the markets for cash, defaultable debt, goods and loans clear.
(f) Bond prices for foreign debt follow rational expectations: 
\[ q_B(A_d, B_d, B_f, M_f) = \frac{1}{1+r} \mathbb{E}\{1 - h_B\} \]
and 
\[ q_M(A_d, B_d, B_f, M_f) = \frac{1}{1+r} \mathbb{E}\{1 - h_M\} \]
taking the government policies 
\[ h_B\left(A_d, B_d, B_f, M_f, \tilde{R}\right) \]
and 
\[ h_M\left(A_d, B_d, B_f, M_f, \tilde{R}\right) \]
in period 2 as given.

The goods market clearing condition yields the resource constraints:
\[ c_1 \leq y_1 + q_B B_f + q_M M_f \]
\[ c_2 \leq y_2 - (1 - h_B)B_f - (1 - h_M)M_f + \tilde{R}f((1 - h_A)A_d + (1 - h_B)B_d) \]

Now we turn to the optimal policy problem for the government.

Definition 4 The Government Problem is to maximize utility (1) over time consistent rational expectations equilibria. In particular, we must satisfy not only the equilibrium conditions above but also the additional optimization decisions:

(g) Government chooses \( \{h_A, h_B, h_M, T_2\} \) in period 2 to maximize \( u(c_2) \) given \( \{A_d, B_d, B_f, M_f\} \) and the shock \( \tilde{R} \).

Government chooses \( \{A_d, B_d, B_f, M_f, T_1\} \) in period 1 to maximize \( u(c_1) + \beta \mathbb{E}u(c_2) \), taking the price functions \( \{p_A(A_d, B_d, B_f, M_f), p_B(A_d, B_d, B_f, M_f), q_B(A_d, B_d, B_f, M_f), q_M(A_d, B_d, B_f, M_f), \rho(x)\} \) and government policies in period 2, \( h_A\left(A_d, B_d, B_f, M_f, \tilde{R}\right), h_B\left(A_d, B_d, B_f, M_f, \tilde{R}\right), h_M\left(A_d, B_d, B_f, M_f, \tilde{R}\right) \) and \( T_2\left(A_d, B_d, B_f, M_f, \tilde{R}\right) \), as given.

We consider two different scenarios. In the first specification, defaultable debt is not tradable between domestic banks and foreign creditors in the period of issue. In the second specification, defaultable debt is tradable in the period of issue. In the latter case, we impose the additional restriction:
\[ p_B = q_B. \]

We adopt the same approach to the problem described in subsections 4.1 and 5.1.

Nontradable Debt

Apply the methodology for the results in the main text, to derive the following formulation. In period 1:
\[ U_1 = \max_{c_1, \alpha, D, B_f, M_f} \left\{ u(c_1) + \beta \mathbb{E}U_2\left(\alpha, D, B_f, M_f, \tilde{R}\right) \right\} \]
subject to
\[ c_1 = y_1 + q_B B_f + q_M M_f \]
\[ c_1 \geq 0 \]
\[ q_B = \frac{1}{1+r} \mathbb{E}\{1 - h_B\left(\alpha, D, B_f, M_f, \tilde{R}\right)\} \]
\[
q_M = \frac{1}{1 + r} \mathbb{E} \left\{ 1 - h_M \left( \alpha, D, B_f, M_f, \tilde{R} \right) \right\}
\]

\[B_f < 0 \Rightarrow \alpha = 0\]

\[t = 2:\]

\[
U_2 \left( \alpha, D, B_f, M_f, \tilde{R} \right) = \max_{c_2, h_A, h_B, h_M} u(c_2)
\]

subject to

\[
c_2 = y_2 - (1 - h_B)B_f - (1 - h_M)M_f + \tilde{R}f \left( - \left[ (1 - h_A)A_d + (1 - h_B)B_d \right] \right)
\]

\[c_2 \geq 0\]

\[y_2 \geq (1 - \alpha)D + (1 - h) \left[ \alpha D + B_f \right] \tag{27}\]

\[0 \leq h_A \leq 1\]

\[0 \leq h_B \leq 1\]

\[0 \leq h_M \leq 1.\]

We again assume that \(y_2\) is large enough so that the expression (27) never binds. Let us now focus on the haircut decisions in period 2. Firstly, raising \(h_M\) to 1 improves the objective function without violating any constraints. So it is optimal to set \(h_M = 1\) for all values of the productivity shock \(\tilde{R}\). The government never makes repayments on debt type \(M\). It immediately follows that \(q_M = 0\) and that the quantity of debt issuance \(M_f\) in the initial period is payoﬀ irrelevant for the representative consumer. We may set \(M_f = 0\) without loss of generality. Secondly, lowering \(h_A\) to 0 improves the objective function without violating any constraints. It is optimal to set \(h_A = 0\) for all values of the productivity shock \(\tilde{R}\). The government never defaults on any portion of debt type \(A\).

So the only debt type with potentially variable haircuts is debt type \(B\), held by both domestics and foreigners. Let us deﬁne \(h = h_B\). Then the above program reduces to the program in the main text of this paper.

** Tradable Debt**

The argument for the nontradable debt case can be modiﬁed for this case. The same results follow.

**C. Concurrent Saving and Borrowing**

In the environment studied in the paper, the government cannot save in foreign assets abroad and concurrently issue defaultable debt. \(B_f < 0\) corresponds to saving abroad, and \(B_f > 0\) captures the issuance of defaultable debt to foreign creditors. This section of the appendix relaxes this restriction and allows the government to both save abroad and issue defaultable debt at the same
time. This is achieved by considering a slightly amended model where the sovereign has access to a richer set of assets. It can still issue defaultable debt, which is denoted by \( B_f \geq 0 \). In addition, it can save abroad in a third asset, \( J_f \leq 0 \), which yields a gross return of \( 1 + r \) in every state of nature of the final period. The sovereign may choose any concurrent combination of defaultable debt issuance and saving in foreign assets. The main result of this section is that the feasible set of debt values \( G \), and in particular the maximum level of debt sustainable in a rational expectations equilibrium, remain unchanged from the case in the main text.

**Modifications to the Model**

The maximization problem of the consumers remains unchanged, and domestic banks have access to the same set of assets as in the main text. The government budget constraints are altered to the following:

\[
\begin{align*}
T_1 & \leq p_A A_d + p_B B_d + q B_f + \frac{1}{1+r} J_f, \\
T_2 & \geq A_d + (1-h) [B_d + B_f] + J_f
\end{align*}
\]

The definition of equilibrium follows.

**Definition 5** A *Rational Expectations Equilibrium* for this economy comprises sequences for allocation rules \( \{c_1, s_1, c_2, \{x\}, x\} \), prices \( \{p_A, p_B, q, \rho\} \) and policies \( \{A_d, B_d, B_f, J_f, h, T_1, T_2\} \) that satisfy:

(a) Consumers choose \( \{c_1, s_1, c_2\} \) to maximize utility (1) subject to the budget constraints (2), (3) and the nonnegativity constraints on consumption (4), taking prices, bank contract offers, government policies and the endowment as given.

(b) Banks offer contract schedules \( \chi : s_1 \rightarrow S(s_1, \bar{R}) \) in period 1 to maximize expected profits, taking prices and government policies in period 2 as given.

Banks choose lending quantity \( x \) in period 2 to maximize profits, taking the loan rate \( \rho \) as given.

(c) Firms choose borrowing level \( x \) to maximize profits (8), taking the loan rate \( \rho \) as given.

(d) Government chooses \( \{h, T_2\} \) in period 2 to satisfy the government budget constraint (6) in that period, taking \( \{A_d, B_d, B_f, J_f\} \) and the shock \( \bar{R} \) as given.

Government chooses \( \{A_d, B_d, B_f, J_f, T_1\} \) in period 1 to satisfy the government budget constraint (5) in that period, taking the price functions \( \{p_A (A_d, B_d, B_f, J_f), p_B (A_d, B_d, B_f, J_f), q (A_d, B_d, B_f, J_f)\} \) and government policies in period 2, \( h (A_d, B_d, B_f, J_f, \bar{R}) \) and \( T_2 (A_d, B_d, B_f, J_f, \bar{R}) \), as given.
(e) All markets clear for the economy. In particular, the markets for cash, defaultable debt, goods and loans clear.

(f) Bond prices for foreign debt follow rational expectations: $q(A_d, B_d, B_f, J_f) = \frac{1}{1+r}\mathbb{E}\{1-h\}$, taking the government policy $h(A_d, B_d, B_f, J_f, \tilde{R})$ in period 2 as given.

The goods market clearing condition yields the resource constraints:

$$c_1 \leq y_1 + qB_f + \frac{1}{1+r}J_f$$

$$c_2 \leq y_2 - (1-h)B_f - J_f + \hat{R}f(A_d + (1-h)B_d)$$

Now we turn to the optimal policy problem for the government.

**Definition 6** The Government Problem is to maximize utility (1) over time consistent rational expectations equilibria. In particular, we must satisfy not only the equilibrium conditions above but also the additional optimization decisions:

(g) Government chooses $\{h, T_2\}$ in period 2 to maximize $u(c_2)$ given $\{A_d, B_d, B_f, J_f\}$ and the shock $\tilde{R}$.

Government chooses $\{A_d, B_d, B_f, J_f, T_1\}$ in period 1 to maximize $u(c_1) + \beta\mathbb{E}u(c_2)$, taking the price functions $\{p_A(A_d, B_d, B_f, J_f), p_B(A_d, B_d, B_f, J_f), q(A_d, B_d, B_f, J_f), \rho(x)\}$ and government policies in period 2, $h(A_d, B_d, B_f, J_f, \tilde{R})$ and $T_2(A_d, B_d, B_f, J_f, \tilde{R})$, as given.

In the first specification, defaultable debt is not tradable between domestic banks and foreign creditors in the period of issue. In the second specification, defaultable debt is tradable in the period of issue. In the latter case, we impose the additional restriction:

$$p_B = q$$

Again, we adopt the approach to the problem described in section 4.1.

**Nontradable Debt**

Apply the methodology applied to derive the formulation in the main text. The government problem may be derived:

$$V_1 = \max_{c_1, z} \mathbb{E}\{u(c_1) + \beta V_2(z)\}$$

subject to

$$c_1 = y_1 + z$$

$$c_1 \geq 0$$
where the expression $V_2 (z)$ is defined by

$$V_2 (z) = \max_{c_2, \alpha, D, B_f, J_f} \mathbb{E} \{ u(c_2) \}$$

subject to

$$c_2 = y_2 - (1 - h)B_f - J_f + \tilde{R}f \left( [(1 - \alpha) + (1 - h)\alpha] D \right)$$

$$h = H \left( \alpha, D, B_f, \tilde{R} \right)$$

$$z_1 = Q \left( \alpha, D, B_f \right) \cdot B_f \geq 0$$

$$z_2 = \frac{1}{1 + r} J_f < 0$$

for some set $\tilde{G}$. Our notation suppresses the dependence of $h$ on $\left( \alpha, D, B_f, \tilde{R} \right)$ in the consumption equation.

Let us briefly interpret this program. The government’s problem may again be decomposed into two components. The intertemporal component of the problem concerns how much to borrow in period 1, $z$. The intratemporal component uses the functional form for the default decision $h$ in the final period in order to calculate the optimal combination $(\alpha, D, B_f, J_f)$ for the chosen $z$ value.

The intratemporal decision is more complicated in this model than in the main text. The debt issuance decisions $(\alpha, D, B_f)$ correspond to a gross debt position, i.e., $z_1 \geq 0$. The saving decision $J_f$ corresponds to a gross asset value $z_2 \leq 0$. The net asset position of the economy at the beginning of the final period is the summation of these two gross positions: $z = z_1 + z_2$.

The feasible set of values for $z_1$ is the same as the positive region of the feasible set $G$ from the model of the main text. The feasible set for $z_2$ is the non-positive real line. It follows that the feasible set for $z$ is the same as in the model studied in the main text of the paper: $\tilde{G} = G$. One corollary of this result is that the maximum feasible debt level is the same for this model and the environment studied in the main text. Another corollary is that the maximum level of debt $z_{\text{max}}$ is achieved with the same values of $(\alpha, D, B_f)$ as in the model of the main text, and $J_f$ set to zero.

** Tradable Debt**

The arguments provided above for the nontradable debt case can be adapted for the specification with tradable debt. The feasible set of debt is the same as in the model in the main text.

**D. Justification for Equal Haircuts**

Throughout this paper, we assume that the government defaults equally on domestic and foreign lenders. The appendix considers possible justifications for this setup.
**Unobservability of the debtholder**

If the government cannot observe the residence of debtholders at the moment of repayment, it cannot discriminate and offer different haircuts to different categories of lenders, domestic or foreign. Even if the government can observe purchases of its own bonds in period 1, the existence of secondary markets for government debt means that defaultable debt may change hands over time. In period 2, the composition of the debt holders may be quite different.

**Legal Restrictions**

Governments may issue several different categories of debt, with each category classified as having different risk characteristics and awarding different legal rights to the creditors. In this case, it may be possible for the government to default on different debt categories in a differential manner, but legal constraints may force the sovereign to treat all debtholders within an asset class equally.

**Equal Haircuts as an Equilibrium Outcome**

The government may wish to execute different haircuts on different debtholders, but the existence of secondary markets may make this impossible or ineffective.

In the environment studied in the paper, the government cannot distinguish whether the holders of the defaultable debt are domestic or foreign. Let us now consider a model where the sovereign can indeed discern whether the debt holders are domestic banks or foreign creditors at the point of repayment, but where it is still impossible for it to direct transfers to the domestic productive sector except by repaying government debt. The government has the option to select different haircuts for debt owed to domestic banks and foreign creditors.

The timing of the model considered in this section is captured in Figure 13. The stages of the model highlighted in bold are the steps which are not present in the model in the main text of this paper. There is lack of commitment between periods: the government cannot credibly make a commitment in period 1 regarding the extent of repayment of debt in period 2. However, the process of default in period 2 has a particular structure. After the stochastic productivity shock and the deterministic endowment are realized, the government announces haircuts of $h_d$ and $h_f$ on debt held by domestic and foreign residents respectively. The government conditions the haircut not on the holder of the debt at the point of the announcement, but at the point of repayment. For example, $h_f$ is the haircut on debt held by foreigners at the time of execution of the haircut, not on debt held by foreigners at the time of the announcement of the haircut. Following this announcement, domestic and foreign holders of defaultable debt can trade it with each other on secondary markets. After such trading is completed, the government must enforce the haircuts that it announced earlier in the same period. In other words, the government has short-term (within-period) commitment: when it comes to the execution of default, the government must follow the haircut announcements that it has made at the beginning of the period. Domestic and foreign lenders settle their secondary trading accounts before the loans market opens in period 2.
Figure 13: Amended Model Timeline

**Period 1**
- Endowment $y_1$ realized.
- Government issues debt $A_d, B_d, B_f$ and transfers proceeds $T_1$ to consumers.
  - Consumers consume $c_1$ goods and save $s_1$ in banks.
  - Banks invest in government debt $A_d, B_d$.
  - Foreigners purchase government debt $B_f$.

**Period 2**
- Productivity shock $\bar{R}$ realized.
- **Government announces** $h_d, h_f$.
- **Secondary market trades between domestic banks and foreigners.**
- **Government imposes lump sum taxes** $T_2$ and applies pre-announced haircuts $h_d, h_f$ on debt $B_d$.
- **Domestic banks and foreigners settle their secondary trading positions.**
  - Banks lend $x$ to firms.
    - Firms borrow and produce $F(x, \bar{R}) = x + \bar{R}f(x)$.
  - Consumers consume $c_2$ goods.

Let us specify the facilities available in secondary market trading. All holders of defaultable government debt have access to secondary market trading accounts. These accounts allow debtholders to borrow unlimited funds from abroad in order to purchase government bonds from other debtholders, but these funds must be fully repaid before the loans market opens, at a gross (within-period) interest rate of one. This feature of the secondary market trading account means that it is possible for domestic (or, indeed, foreign) debtholders to purchase all the government debt between the announcement and execution of haircuts, at a price equal to $1 - h$, where $h$ is the haircut that corresponds to the purchaser of the debt. Foreign debtholders may also purchase all of the debt. After the execution of the haircuts, the secondary market trading markets must be settled, i.e., receipts from government debt repayments must be used to repay all borrowed funds from abroad. Then the domestic productive sector produces output.

This timing of events preserves the liquidity constraint on the domestic production sector in the event of non-repayment of sovereign debt.

Let us now analyze the response of domestic and foreign debtholders to government haircut announcements. There are 3 cases to consider.

**Case 1:** $h_f > h_d$.

Foreigners value government debt at $1 - h_f$, which is lower than $1 - h_d$, the value of the debt to domestic agents. Foreign creditors are willing to sell their debt holdings at any price above $1 - h_f$. Therefore the supply of bonds $S_f$ takes the shape in Figure 14. Domestic debtholders are willing to purchase the debt at any price less than or equal to $1 - h_d$. Their demand for debt $D_d$ is horizontal.
at this price. Therefore, the secondary market price of debt is $1 - h_d$, as shown in the figure.

![Figure 14: Secondary Markets for Debt](image)

At the time of the execution of the haircut, all the debt is in the hands of domestic debtholders. The haircut of $h_d$ is applied. Domestic debtholders make no profit on secondary market trades, since they must repay exactly this quantity to settle the secondary market trading account. Foreigners receive $1 - h_d$ for their debt. So all agents suffer a haircut of $h_d$ on the debt. The government achieves this haircut on all of its debt.

**Case 2:** $h_f < h_d$.

An analogous argument to that above establishes that after the announcement and before the execution of the haircuts, foreigners purchase all of the debt from domestic debtholders at the price $1 - h_f$. The haircut applied on all of the debt, and therefore the haircut achieved by the government, is $h_f$.

**Case 3:** $h_f = h_d$.

In this case, domestic and foreign lenders value the debt at the same price. Therefore whether they purchase it from each other or not is irrelevant for the haircut imposed on the debt, and for the payoffs of domestic and foreign debtholders. The haircut applied on all of the debt is $h_f = h_d$.

Therefore, no matter the configuration of the haircuts announced by the government, both domestic and foreign debtholders effectively suffer the same haircut on the debt $h = \min\{h_d, h_f\}$, and the government achieves this haircut $h$ on all its debt. For the rest of the model, it does not matter who ends up holding the debt after secondary market trading. Thus, the above argument proves the following result.
Lemma 3 Consider any equilibrium with configuration of haircuts $\mathcal{H} = \{h_d, h_f\}$ announced by the government. Let $h = \min \{h_d, h_f\}$. There exists another equilibrium where the government announces the haircuts $\mathcal{H} = \{h, h\}$, which achieves the same payoffs for domestic and foreign debtholders and for the government.

We conclude this section with a short discussion of the applicability of this result. The lag between haircut announcements and execution is designed to capture the fact that in reality, secondary markets are nearly always open for trading. Typically in the event of default, the institutional structure requires that governments announce haircuts in advance of making (partial) repayments. Secondary debt markets are always active, and in particular they will be open in the time between announcement and execution of haircuts (no matter how long this interval is in practice, especially if the secondary market is liquid). Therefore, debt can change hands in this interval. This is what we need for the mechanism in this version of the model to be functional. Given this mechanism and the tradability of debt between domestic and foreign debtholders, haircuts are equalized across different categories of lenders.

References


2. Alessandro, Mauro. 2007. “Domestic Financial Development and Foreign Credit to Private Firms.” Macro Lunch Presentation, MIT.


