CDS Spreads and Systemic Financial Risk

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Abstract

This paper investigates the information content of bond and Credit Default Swap prices of financial institutions for the measurement of systemic risk in the financial sector. Because CDS contracts involve counterparty risk, this is reflected in their prices. Then, the set of prices of all CDSs written by each member of the financial network on the other members - together with bond prices - reflects the risk-neutral probabilities of default of each institution and of each pair of institutions in the network. In the paper, I show how this information can be aggregated to construct bounds on the probability of systemic events. These bounds are proven to be the tightest ones possible given this information set. I apply my analysis to a group of 15 large American and European financial institutions, between January 2004 and March 2009. I show that markets perceived systemic risk to increase steadily after August 2007. Idiosyncratic default risk of some banks, more than systemic risk, spiked during the Bear Stearns and Lehman episodes. I also study how the contribution to systemic risk of each bank evolved over time. Interestingly, the analysis points out that the markets anticipated the default risk of a few dealers (Lehman Brothers and Merrill Lynch), reflecting not only high idiosyncratic risk but also the high joint default risk of subgroups of them.

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1 Introduction

Systemic risk in the financial sector has caused great concern during the recent crisis, both for agents in financial markets and for regulators. Systemic default - simultaneous default of multiple financial institutions - appeared to be a concrete possibility at several points. Consequently, different government interventions were enacted to reduce the risk of contagion of shocks across the highly interconnected financial network. To make decisions in this environment, being able to measure systemic risk is fundamental.

Measuring systemic default risk in the financial system, however, is a difficult task. Measures constructed looking directly at the books of financial institutions are backward-looking in nature and limited by the complexity of the positions and risks involved as well as by the availability of data\(^1\). On the other hand, measures that try to learn about systemic risks by looking at the historical distribution of their returns are generally unreliable because they involve estimating joint tail probabilities from limited time series, which is possible only under strong parametric assumptions\(^2\).

The most widely used measures of systemic risk try to circumvent these limitations by estimating default probabilities of financial institutions implied in the prices of traded securities such as equity options or Credit Default Swaps (CDSs), and then aggregating this information to learn about systemic risks\(^3\). These measures are forward-looking and reflect the information set of market participants. Figure 1 plots two examples of such market-based measures of default risk in the financial sector: the average yield spread and the average CDS spread of the top 15 financial institutions. The yield spread (the yield on a firm’s bonds in excess of the risk-free rate) and the CDS spread (the cost of insuring against the firm’s default) both reflect the probability that a firm defaults. The idea behind these measures is that an increase in systemic risk should cause the risk of default of each institution to increase, and therefore lead to an increase in both the average yield spread and CDS spread.

\(^1\)For example, see Demirguc and Detragiache (1998 and 1999), Gonzales-Hermosillo (1999, Cihak and Schaek (2007) and Poghoshan and Cihak (2009).

\(^2\)Examples of this method are Avesani, Pascual and Li (2006) and Lehar (2005). These papers assume Gaussian joint distribution of returns. Other measures have been constructed which look indirectly at the joint default distribution. For example, Kritzman et al. (2010) proxy for systemic risk using the fraction of total variance explained by the first L principal components of the covariance matrix of changes in CDS spreads.

\(^3\)See for example Huang, Zhou and Zhu (2009) and Segoviano and Goodhart (2009).
Unfortunately, the existing market-based measures can be misleading for two reasons. First, they involve strong modeling assumptions in aggregating the individual risks of financial intermediaries into estimates of systemic risk. For example, the CDS measure reported in Figure 1, which shows the average CDS spreads of a group of dealers, is only informative about the joint distribution of defaults if we make some assumptions about the relationship between marginal and joint default probabilities. Second, most measures based on the prices of over the counter (OTC) securities ignore counterparty risk. As I will show below, ignoring counterparty risk introduces a bias that increases precisely when the financial system is distressed.

In this paper I propose a novel measure of systemic risk, based on a combination of bond prices and CDS spreads, that in fact exploits the pricing of counterparty risk in CDS contracts. The measure improves over other market-based measures, by optimally aggregating the information contained in the prices of these securities into bounds on systemic risk - the probability of default of several institutions. These bounds are robust to modeling assumptions on the relationship between risks of individual institutions and systemic risk. Furthermore, the relation I derive can be used to show that assumptions about the extent of systemic risk have immediate consequences for counterparty risk pricing, and vice versa. Contrary to alternative measures, this measure allows a decomposition of movements in bond yields and CDS spreads into an idiosyncratic component and a systematic component. The decomposition shows that systemic risk has been steadily increasing since August 2007, and the observed spikes in bond yields and CDS spreads around March and September 2008, and then again in March 2009, can be attributed more to spikes in idiosyncratic risk than to increases in systemic risk. In addition, the measure I derive bounds the level of systemic risk (the probability of several correlated defaults) much below what is indicated by other measures. Finally, the method described in the paper can also be used to track the contribution of each individual institution to systemic risk, and to obtain a representation of the default probabilities across the network, in the scenario of worst systemic risk, at any point in time.

In particular, I derive an estimate of the maximum and minimum amount of systemic risk that is compatible with the observed prices of bonds and CDS securities written by the financial intermediaries to insure against the default of other intermediaries. This measure is a function of the liquidity premia in the bond market, and it becomes more precise as assumptions on the liquidity process are imposed. Because
I focus on bounds rather than point estimates, the measure does not depend on assumptions about the aggregation of the information contained in bond prices and CDS spreads. In addition, by writing the problem in a linear programming form, I show how these are the tightest bounds compatible with the observed pricing information, without being constrained to a specific model of higher-order correlation. These bounds, therefore, represent an alternative and robust market-based way to measure the amount of systemic risk.

The starting point for the analysis is the pricing of counterparty risk in CDS contracts. The price of insurance against the default on a bond sold by a financial institution (CDS spread) reflects both the probability of default of the bond issuer (called reference entity) and the correlation between its default and the default of the protection seller. In particular, the value of the insurance decreases as the aforementioned correlation increases. The price of a bond instead reflects only the marginal probability of default of the reference entity which issued the bond. Combining the price of the bond of a company with the spread of the CDS written on that company by a particular financial intermediary, we can learn about the correlation between the default of that company and the default of the protection seller. Therefore, it is possible in principle to learn about the default correlation of any two financial intermediaries by considering the protection sold by one dealer against the default of another one. However, this information set - that includes all marginal and pairwise default probabilities in the financial network - is not rich enough to completely pin down systemic risk. Usually, modeling assumptions are made to obtain a point estimate of the joint default probability given the available information set. In this paper, instead, I use the robust approach of obtaining optimal bounds consistent with the observed prices.

In the empirical analysis that follows, I construct bounds on the probability of multiple default events using the group of 15 financial institutions that have the largest nominal exposure to credit derivatives; these correspond to the main investment banks and broker/dealers in the American and European financial markets. In particular, I obtain the tightest bounds for the average monthly probability of correlated default of at least \( r \geq 1 \) institutions, for the period spanning January 2004 to March 2009, and under different assumptions on the liquidity process of the bond market.

Two limitations affect the construction of the bounds. First, the presence of an unobserved liquidity process in the bond market confounds the filtering of idiosyncratic
default probabilities out of CDS spreads, and therefore complicates the estimation of systemic risk. However, assumptions on the liquidity process allow a more precise measurement of systemic default risk. Second, for every reference entity, I observe only an average of the quotes posted by the main counterparties, so my information set is smaller than the ideal one. However, because they combine information from bond prices as well, the bounds I construct are still informative about asymmetries in the distribution of joint risks across the network. An additional caveat relates to the fact that this paper obtains risk-neutral, not objective, systemic default probabilities. These probabilities are interesting per se because they reveal the perception of the markets of the severity of these states of the world. In addition, it is reasonable to assume that they can be considered upper bounds on the objective default probabilities, and so the upper bound obtained in this way is also an upper bound on objective probabilities. However, it is important to keep in mind that the bounds capture variation in risk premia as well as variation in objective default probabilities4.

What emerges from this analysis is that very high systemic risk in financial markets, especially around the times when CDS spreads peaked, would not be consistent with the information contained in bond and CDS prices. This result suggests that throughout the crisis the level of systemic risk has been lower than other measures indicate. While the optimal bounds are a complex function of the constraints imposed by the set of observed prices, the intuition for the result is straightforward. If systemic risk in the financial markets were perceived to be particularly high, the price of insurance on large dealers - purchased from other large financial institutions - should have dropped considerably. But we can observe the average spreads (i.e. cost of insurance) that were actually posted to insure against each other’s default. The relatively high equilibrium spreads during this period impose an upper bound on the amount of systemic risk perceived in financial markets.

The paper proceeds as follows. Section 2 presents an introduction to Credit Default Swaps and counterparty risk. Section 3 presents the theory of the optimal probability bounds and discusses the main issues in estimating them. Section 4 describes the data, and Section 5 presents the results. Section 6 concludes.

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4Anderson (2009) makes this point comparing risk-neutral default processes obtained from CDS spreads with objective processes obtained using historical data on defaults.
2 Credit Default Swaps and Counterparty Risk

2.1 The Credit Default Swaps Market

Credit Default Swaps are credit derivatives that allow the transfer of credit risk of a firm between two agents. In a plain vanilla single-name CDS contract, the protection seller offers the protection buyer insurance against the default of an underlying bond issued by a certain company (called the *reference entity*). In the event of default by the reference entity, the seller commits to buy the bond for a price equal to its face value from the protection buyer\(^5\). In exchange for the insurance, the buyer pays a quarterly premium, called CDS *spread*, quoted as a percentage of the notional value insured. If default occurs, the contract terminates, and the quarterly payments are interrupted. If default does not occur during the life of the contract, the contract terminates at its maturity date.

While in general these contracts are traded over the counter and can be customized by the buyer and the seller, the contract with maturity of 5 years is relatively standardized. Its market is very liquid, in terms of low transaction costs to initiate a contract with a market maker on short notice, and quotes for this contracts are regularly obtained by financial data companies from the main dealers (see Blanco et al. (2003) and Longstaff et al. (2005)).

The CDS market has grown very quickly in the last few years. Notional exposures grew from about $5tr in 2004 to around $60tr at its peak in 2007, and despite the financial crisis the total exposure is still around $40tr. The main reason for this growth in gross terms is that, due to the high liquidity of the CDS market, the easiest way to adjust the exposure to credit risk has been to enter new CDS contracts (possibly offsetting the existing ones), rather than operating directly in the bond market or cancelling CDS agreements already in place. At the center of this network of CDS contracts, a few main dealers operated with very high gross and low net exposures, emerging as the main counterparties in the market. For example, the Fitch Ratings (2006) states that in 2006 the top 10 counterparties (all broker/dealers) accounted for

\(^5\)In practice, the terms of the CDS could involve physical delivery of the defaulted bond or cash settlement. In the former case, usually any bond of similar seniority can be delivered (for example, for the CDS written on a senior unsecured bond, any other senior unsecured bond of the firm could be delivered). In addition, the credit event could include restructuring and downgrade of the reference bond.
about 89% of the total protection sold. In addition, these dealers figure in the list of top reference entities for CDS contracts; they are also the firms on which most credit protection was sold. The picture that emerges is that of a very interconnected network composed of a few large intermediaries, linked to each other both by counterparty exposure and direct credit exposure.

2.2 Counterparty Risk

Traded over the counter, the CDS contract involves counterparty risk: the seller might default during the life of the CDS and therefore might not be able to comply with the commitments implied by the contract. In this case, the holders of CDS claims would still recover part of the expected payment they are due under the contract. Like other derivatives, CDS claims are treated pari passu with senior bonds, and in addition they are exempted from automatic stay of the assets of the firms, so that they can immediately seize any collateral that has been posted. In the case of early termination of the contract, the seller usually has to compensate the buyer for the replacement cost of the contract, i.e. the cost of initiating a new insurance contract with another protection seller.

Usually, the collateral posted by the CDS seller is sufficient to cover this replacement cost. However, the higher is the current default probability of the reference entity, the higher is the replacement cost. If the protection seller defaults - or otherwise terminates the contract - in a state in which the default risk of the reference entity is high, the payment due for replacement will be high as well, and the collateral posted by the CDS seller may not be sufficient to cover it. In the extreme case, if the default of the seller occurs simultaneously with the default of the reference entity, the payment due under the contract would be equal to the fraction of the bond not recoverable in bankruptcy. At that point, it might be difficult or impossible to replace the contract. The main source of counterparty risk for the price of the CDS is then the double default of the seller and the reference entity. Indeed, it is possible that the default of the seller is triggered by the default of the reference entity itself, either directly (due to spillovers between the two companies) or indirectly, if the seller’s default is caused by the inability to pay the liability that emerged as a consequence of the credit event. The latter case

6There are now several proposal to create a centralized clearinghouse to reduce counterparty risk. For a detailed discussion, see Duffie and Zhu (2010).
could be very important - and has proved important during the recent financial crisis -
in all cases in which the protection seller has not hedged the CDS risk adequately.

A stylized two-period example of the pricing of bonds and CDS can be useful to see the role of counterparty risk in a simple way. We will consider two dealers, banks 1 and 2, which have issued a zero-coupon bond with a face value of $1 maturing at time 1, and the CDS contract written at time 0 by each of them against the default of the other one. Call $A_i$ the event of default of institution $i \in \{1, 2\}$ at time 1. Call $P(A_i)$ the probability of default of bank $i$, and $P(A_1 \cap A_2)$ the probability of double default at time 1. All probabilities are risk-neutral. Call $R$ the (certain) recovery rate on the bond in case of default, and suppose for now that in the event of default the CDS claim recovers the same as the bond (because their seniority level is the same). Finally, assume the risk-free rate between periods 0 and 1 is 0.

In this setting, the price of the bond issued by $i$, $p_i$, is determined as:

$$p_i = (1 - P(A_i)) + P(A_i)R = 1 - P(A_i)(1 - R)$$

If there is no counterparty risk in the CDS contract, the insurance premium $z_i$, or CDS spread, paid at time 0 to insure that bond is:

$$z_i = P(A_i)(1 - R)$$

It is then easy to see that between the bond and the CDS there is a theoretical arbitrage relation (which in fact holds beyond this simple example, see Longstaff et al. (2005)). Consider now the case in which there is counterparty risk in the CDS contract. Then, the spread paid to buy insurance from $j$ against $i$’s default will be:

$$z_{ji} = [P(A_i) - P(A_1 \cap A_2)](1 - R) + P(A_1 \cap A_2)(1 - R)R$$

$$= [P(A_i) - (1 - R)P(A_1 \cap A_2)](1 - R)$$

which decreases with the probability of both institutions defaulting $P(A_1 \cap A_2)$.

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7It is important to realize that the order of magnitude of counterparty risk could in theory be as high as the spread itself. While in models where defaults are independent we have $P(A_i \cap A_j) = P(A_i)P(A_j)$, most observers of the crisis would agree that defaults of major dealers are far from independent, and therefore the probability of the joint default can be of a much larger order of magnitude.
The arbitrage relation with the bond is broken.

Note that I use $P(A_1 \cap A_2)$ to indicate the event of double default. For a short enough time horizon, and for the group of largely interconnected financial intermediaries, it is reasonable to assume that conditional upon both intermediaries defaulting within a certain time interval, their defaults are clustered together, so that the loss from counterparty risk is comparable to that obtained in the event of double default. In this case, the probability of a double default event within that period is approximately the same as the probability that both institutions default within that time horizon. In the analysis below I assume that the reference time period is a month: whenever two large financial institutions default in the same month, their defaults will be clustered and the buyers of a CDS will be exposed to the double default loss.

2.3 Collateral Agreements and Pricing of Counterparty Risk

In order to protect against counterparty risk some contracts involve a collateral agreement, under which collateral calls are tied mechanically to downgrades of the rating of the reference entity and of the protection seller (for example, many contracts are tied to Fitch and S&P ratings). According to the ISDA Margin Survey 2008, only about 66% of the nominal exposure in credit derivatives (of which CDS are the most important type) had a collateral agreement in 2007 and 2008; this number was even lower in the years before. Of course, the extent to which collateral covers against counterparty risk is limited by the jump properties of the default events. As long as defaults are relatively anticipated, slow adjustment of collateral posted can basically remove all counterparty risk. However, especially for financial intermediaries, defaults often occur in very short horizons, so that the buyer does not have the possibility of obtaining enough collateral to cover all the losses in time. In addition to this, the collateral calls themselves, if large enough, can cause the default of the protection seller (this for example happened to AIG in September 2008). Therefore, the presence of collateral agreements improves but does not solve the problem of counterparty risk. Since this risk will be present in

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*The two defaults need not be exactly simultaneous. The same loss is incurred if - conditional of both dealers defaulting within the same time horizon - the default of the seller induces a large enough increase in the default probability of the reference entity, which causes a large jump in the replacement cost of the contract. A similar loss is also incurred if the default of the reference entity induces the successive default of the seller through the increase in the liabilities of the latter coming from the CDS claims.*
the final contracts, it should be priced in by rational agents.

Given that the incorporation of counterparty risk and collateral in the pricing of Credit Default Swaps is central for this study, it is worth discussing the related assumptions in detail, especially in reference to the observed events related to the credit crisis of 2008-2009. An initial question is whether counterparty risk was perceived at all by market participants. The growth of the percentage of OTC derivative contracts covered by some form of collateral (ISDA) indirectly confirms this. Besides, several documents from practitioners confirm that the issue was taken seriously throughout the crisis (for example, see Barclays (2008)).

A second issue involves the extent to which, given that agents were aware of counterparty risk, this risk was priced in the CDS spreads; after all, collateral might have been sufficient to eliminate most of it. There are several reasons to believe that this is not the case. First, we know from ISDA that only about 2/3 of the contracts were covered by collateral agreements. Besides, calculations by Singh and Aitken (2009) and Singh (2010) show that, at the end of 2009, large financial institutions still carried large under-collateralized derivative liabilities. In particular, they compute the total value of “residual derivative payables” - liabilities from derivative positions after netting under master netting agreements and in excess of the collateral posted. For the 5 largest US dealers this amount was more than $250bn. Even though these numbers include all derivative contracts, and not only CDSs, they suggest a general under-collateralization of derivative positions from these counterparties. Second, it is also known that most collateral requirements were imposed on hedge funds and smaller dealers, while large intermediaries often had much less stringent collateral requirements, if any at all. Third, the amount of collateral required to be posted (collateral calls) depends on the credit risk of the reference entity and of the protection seller. Even at the peak of the crisis, for example right before the weekend of September 15 2008, implied default probabilities of major dealers were relatively small and their credit ratings were still high - for example, Lehman’s senior bonds were rated A by S&P and A2 by Fitch. The realized default meant an approximately tenfold overnight change in default probabilities for several institutions at the time - though this change happened over the weekend, a non-trading day, and therefore collateral could not be adjusted for that change.

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9 For credit derivatives, as a percentage of exposure, collateralization went from 39% in 2004 to 58% in 2005, to 66% in 2007 and 2008.

10 In a document called “Why was Lehman Brothers rated ‘A’?”, issued on 9/24/2008, S&P reports: “we believe the downfall of Lehman reflected escalating fears that led to a loss of confidence—ultimately
In fact, the Lehman bankruptcy represents an excellent example to understand the counterparty risk of major financial dealers. Just before the weekend of the 13th and 14th of September 2008, many institutions were considered at risk (Lehman first among them), but the CDS spreads written on them did not indicate an extremely high likelihood of immediate default. For example, the Lehman 5 year CDS was trading at around 600bp per year, Merrill’s at 400bp, and the credit ratings of their debt had were still as high as 4 months before, with an implied default probability of less than 0.25% per annum. When Lehman went bankrupt, all CDS contracts written by Lehman were terminated. Its counterparties used their right to seize the collateral and they ended up incurring only small losses. At a first glance, it appears that counterparty risk proved to be very small during this episode.

However, this is due to the fact that thanks to the intervention of Bank of America and of the government (that saved Merrill Lynch on September 14th and AIG on September 16th, respectively), a double default event never occurred. Consequently, the CDS written by Lehman and owned by its counterparties were little in the money, because the value of the claims against Lehman was small. The collateral was barely enough to cover the replacement cost of the contracts when no double default occurred; clearly, it would have been vastly inadequate if another institution, for example AIG or Merrill Lynch, had been left to fail. Then, the CDS written by Lehman on AIG’s bond would have been in the money for the full amount insured, and the buyers of protection from Lehman would have recovered just a small fraction of their claim by seizing the posted collateral. This example, together with the other salient events of the financial crisis, highlights the limited role of collateral to protect against double defaults which, when they involve financial intermediaries, tend to happen in a sudden and correlated fashion.

Given that collateral is generally not enough to protect against losses in case of double default, we would expect counterparty risk to be priced in the CDS spreads. In particular, the spread on a CDS contract written by a dealer on another dealer’s default should be lower the higher the correlation of the default events of the two.

\footnote{Moody’s reports that the claims filed by the other major dealers related to derivative contracts were on the order of 1 to $2bn each.}

\footnote{The AIG contract was highly in the money on September 15th; its CDS spread spiked, but immediately came down the next day, when it was clear that the government would not let AIG default.}

\footnote{becoming a real threat to Lehman’s viability in a way that fundamental credit analysis could not have anticipated}. 
direct evidence on the extent to which counterparty risk was in fact priced in these contracts is extremely difficult, precisely because its effect on the spread relative to the bond yield can be confused with the effects of other variables, such as liquidity. Arora et al. (2009) document that dealers with high idiosyncratic default risk (as measured by the spread of the CDS written on them) did not post quotes systematically higher than the other dealers for the same reference entity. However, their analysis only looks at the variability of quotes around the daily average and cannot identify whether the average quote on a certain reference entity in fact reflects average counterparty risk. As I will discuss in detail below, the bounds I construct are based only on the average quote and therefore do not depend on the cross-sectional variation around it. Of course, as much as it is plausible to believe that agents, which realize the potential of losses in the case of double default, would price counterparty risk into CDS spreads, it is also possible that liquidity alone could explain the discrepancy between bond and CDS spreads. While the former view seems reasonable given the presence of documents from large dealers pointing to it (for example, Barclays 2008) and especially given the public discussion during the recent crisis, ruling out either view is impossible given the data currently available\textsuperscript{13}.

### 3 Probability bounds

This section develops the theory of the probability bounds on systemic default events for a network of $N$ institutions in which bond prices and CDS spreads are observed. I start with an introductory example that explains the main ideas. Then, I show how to use linear programming theory to solve the general problem, and I derive some properties of the optimal bounds and discuss details of the implementation.

\textsuperscript{13}Coval, Jurek and Stafford (2009) also calibrate the amount of counterparty risk in the CDX basis, using a structural model of default derived from Merton (1974) imposing the CAPM. The model finds little role for counterparty risk to explain the bond/CDX basis. In their model, correlation of defaults comes only from exposure to the market factor, and residual default risk is independent. This limits the amount of joint default probability the model can generate under reasonable calibrations, and therefore the amount of counterparty risk that can explain the basis.
3.1 Probability bounds on systemic risk: an introductory example

Suppose that the financial sector consists of only three intermediaries - banks 1,2 and 3. Since they are the only intermediaries in the market, protection against the default of \(i \in I \equiv \{1, 2, 3\}\) must be bought from a \(j \in I \setminus i\), i.e. one of the other two intermediaries. Following the simple pricing model presented above, we can write the bond prices \((p_i)\) and the CDS spreads sold by bank \(j\) on the bond of \(i\) \((z_{ji})\) as:

\[
p_i = 1 - (1 - R) \cdot P(A_i)
\]

\[
z_{ji} = (1 - R) \cdot (P(A_i) - (1 - R)P(A_i \cap A_j))
\]

where \(R\) is the (known) recovery rate on bonds and CDS and all probabilities are risk-neutral.

Note first that if we observe all bond prices \(p_i\) and all CDS spreads \(z_{ji}\), we can learn the implied marginal and pairwise risk-neutral probabilities of default. Bond prices reflect directly the marginal probability of default of each institution, while CDS spread allow to recover pairwise default probabilities after filtering out the marginal probabilities. Because this information set contains the default probabilities of one or at most two institutions, but contains no direct information on the probability of the default event of all three institutions, I call this a probability information set of order two. For example, suppose we observe:

\[
P(A_i) = 0.2 \quad \forall i
\]

\[
P(A_1 \cap A_2) = P(A_2 \cap A_3) = 0.07
\]

\[
P(A_1 \cap A_3) = 0.01
\]

In this paper I define systemic risk the probability of correlated default of at least \(r\) financial intermediaries, \(P_r\). With only three banks, we obtain the three following measures of systemic risk:

\[
P_1 = P(A_1 \cup A_2 \cup A_3)
\]

\[
P_2 = P((A_1 \cap A_2) \cup (A_2 \cap A_3) \cup (A_1 \cap A_3))
\]
\[ P_3 = P(A_1 \cap A_2 \cap A_3) \]

Note that all these definitions involve unions and intersections of all three defaults event, and therefore they are all \textit{probabilities of order three}, which is higher than the order of the information set presented above.

At first sight, one might think that if we observed \textit{all} bond prices and \textit{all} CDS spreads, thus learning \( P(A_i) \) and \( P(A_i \cap A_j) \) for each \( i \) and \( j \), we would be able to completely pin down the systemic probabilities \( P_1, P_2 \) and \( P_3 \). However, this is not the case: in general, an information set of order \( M \) cannot determine probabilities of order greater than \( M \). A simple graphical proof of this is reported in Figure 2, which uses Venn diagrams to represent probabilities. In that Figure, the area of each event is the same across the two panels, so the marginal probabilities of defaults are the same. The same is true for the pairwise default probabilities. However, it is easy to see that \( P_3 \), the intersection of all three events, is positive in the top panel and zero in the bottom panel.

Knowledge of the low-order probabilities, however, allows us to put bounds on higher-order probabilities, and therefore on systemic default risk. While finding the upper bound for \( P_3 \) is immediate (\( P_3 \leq 0.01 \)), finding the other bounds is more complicated - and especially so when there are more than three banks in the financial sector. The exact way to obtain tightest bounds is the object of the rest of this Section. When applied to this example, it yields the following bounds:

\[
0.45 \leq P_1 \leq 0.46 \\
0.13 \leq P_2 \leq 0.15 \\
0 \leq P_3 \leq 0.01
\]

This simple example already shows one of the main points of this analysis: the set of bonds and CDS, when counterparty risk is taken into account, represent a rich information set that can be used to learn about systemic risk by making \textit{no assumption} on the way these low-order probabilities aggregate at the higher-order level, but rather constructing bounds.

This simple example can be used to illustrate two additional concepts related to the measurement of systemic risk. The first one is that simply averaging the CDS spreads
of financial institutions can lead to an erroneous measure of systemic risk. Using the notation introduced above, such an index is:

\[
\frac{1}{6} \sum_{i} \sum_{j \neq i} z_{ij} = \frac{1}{6} \sum_{i} \sum_{j \neq i} (1 - R) \cdot (P(A_i) - (1 - R)P(A_i \cap A_j))
\]

\[
= (1 - R) \left[ \sum_{i} P(A_i) - (1 - R)P(A_1 \cap A_2) - (1 - R)P(A_2 \cap A_3) - (1 - R)P(A_1 \cap A_3) \right]
\]

If systemic risk increases while the marginal default probabilities remain the same (or do not increase enough), this index in fact decreases. The reason is that, because of counterparty risk, insurance gets worse but cheaper as a consequence of the increase in systemic risk, and therefore the average cost of insurance decreases. At least in some cases, then, this simple index fails to properly capture systemic risk.

The second idea that emerges from this simple example is the importance of using all the different prices available to construct the bounds, rather than using only the information contained in average bond and CDS spreads - which have been used before to construct alternative measures of systemic risk. While information on average default probabilities is enough to impose some bounds for the probability of systemic events, the additional restrictions given by the different prices can significantly improve on them. In the section below I prove that among all networks with the same average marginal and pairwise default probabilities, the widest bounds on systemic events (the least informative ones) are obtained when the network is symmetric, i.e. when all institutions have the same default probabilities. These bounds are in fact the same that are obtained if only average probabilities are observed. This tells us that the more asymmetric the financial network is in terms of observed marginal and pairwise default probabilities, the highest the gain of using all the individual prices and spreads in terms of precision of the bounds.

In the example above, suppose that instead of observing all the marginal and pairwise default probabilities we observed only the average probabilities (call this the partially aggregated information set):

\[
\frac{P(A_1) + P(A_2) + P(A_3)}{3} = 0.2
\]

\[
\frac{P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_1 \cap A_3)}{3} = 0.05
\]
In this case, it is easy to see that the upper bound on systemic risk is $P(A_1 \cap A_2 \cap A_3) \leq 0.05$, which in fact is attained by a symmetric network configuration in which $P(A_i \cap A_j) = 0.05$ for all $i$ and $j$. Using the full low-order information set rather than the partially aggregated one allows us to reduce the upper bound on systemic risk from 0.05 to 0.01. The same is true for the other definitions of systemic risk. The optimal bounds in this case are:

$$0.45 \leq P_1 \leq 0.50$$
$$0.05 \leq P_2 \leq 0.15$$
$$0 \leq P_3 \leq 0.05$$

which are wider than the ones obtained using all the low-order information available.

### 3.2 General theory of the probability bounds

In this section, I show how to represent the problem of constructing tightest bounds for probabilities of high-order events given a low-order information set. For now, I assume that we have already extracted the probability information set from the observed prices of traded securities. In the following sections, I discuss how to extract the marginal and pairwise default probabilities from the prices of bonds and CDS, and how limitations in the data I observe affect the estimation of the bounds.

Consider a finite set of basic events $A = \{A_1, ..., A_N\}$, which in this paper I interpret as the default events of a set of $N$ financial intermediaries. The relation between low-order probabilities (probabilities of unions and intersections of a few events in $A$) and higher-order ones (that involve many events in $A$) has long be studied in mathematics. In particular, it is known that the knowledge of low-order probabilities is not enough to pin down completely high-order ones, but is enough to put bounds to them. Two famous results in this direction are Boole’s and Bonferroni’s inequalities, which state that:

$$P\left(\bigcup_i A_i\right) \leq \sum_i P(A_i)$$

and

$$P\left(\bigcap_i A_i\right) \geq \sum_i P(A_i) - N - 1$$
and therefore bound the probabilities that respectively at least one and all of the basic events occur, using only information about the individual probabilities of the basic events. These bounds are not tight, in the sense that tighter inequalities can be written based on the same information set (the set of all marginal probabilities of events in $\mathcal{A}$).

In this paper, I define systemic event of order $r$ the default of at least $r$ out of the $N$ intermediaries. Systemic events are therefore of order $N$, because they involve unions and intersections of all the events in $\mathcal{A}$. I employ an estimation method that allows to obtain the tightest possible bounds for the probabilities of systemic events given the available information set, which includes bond prices and CDS spreads - these reflect order 1 and 2 default probabilities, because they depend on the occurrence of one or at most two elements in $\mathcal{A}$. The approach is based on linear programming (LP), and consists in writing the bounds as the solution to a LP problem\footnote{See Kwerel (1975). A LP problem is a constrained maximization problem in which both the objective and the constraints are linear in the maximization variable.}, which is easy to solve numerically even for a very large scale, though analytical solutions are difficult to find.

To see how the LP approach works, for a sample space $\Omega$ consider the finest partition of $\Omega$ created by unions and intersections of the basic events $A_1, ..., A_N$; call it $V$. Then, the probability of each union or intersection of the basic events can be expressed as the sum of the probabilities of some events in $V$. Since $V$ contains exactly $2^N$ elements, it is possible to represent the probability space by a vector with $2^N$ elements, each corresponding to the probability of an elementary event in $V$. Formally, the following proposition holds (see Boros and Prekopa (1989)):

**Proposition 1.** Call $\mathcal{F}$ the $\sigma$-algebra generated by the finite set of events $A_1, ..., A_N$ on a sample space $\Omega$. Call $V$ the finest partition of $\Omega$ that is included in $\mathcal{F}$. Then, $V$ has $2^N$ elements, and any probability system on $(\Omega, \mathcal{F})$ can be represented by a vector $p \in \mathbb{R}^{2^N}$, in the sense that $\forall A \in \mathcal{F}, \exists I_A \subseteq \{1, 2, 3, \ldots, 2^N\}$ s.t. $P(A) = \sum_{i \in I_A} p_i$.

In general, there are different vectors $p$ that represent the same probability system. In this paper, I use the one constructed according to the following Proposition.

**Proposition 2.** Define $\overline{B} \equiv \Omega \setminus B$, the complement of $B$. For every $i$ from 0 to $2^{N-1}$, consider its binary representation $b_i$, which consists of a vector of $N$ numbers, each either 0 or 1. Construct $p_{i+1}$ as follows:

$$p_{i+1} = P(A_1^i \cap A_2^i \cap \ldots \cap A_N^i)$$
where $A^*_j = A_j$ if element $j$ of $b_i$ is 1 and $A^*_j = \overline{A_j}$ if element $j$ of $b_i$ is 0. Then, $p$ represents a probability system on $F$ in the sense of Proposition 1.

A simple example can help illustrate this Proposition. In the case of three banks, we have three basic events: $A_1, A_2$ and $A_3$. The finest partition of the sample space obtained from unions and intersection of these events will have $2^3 = 8$ elements. Figure 3 shows the 8 elements of this partition. From the Figure, it is evident how one can express the probability of any union or intersection of the $A_i$ as the sum of the probabilities of a subset of these 8 elements. These 8 probabilities can be collected in a vector $p$ with 8 elements, and therefore the probability of any event $A'$ can be represented as the product $a'p$ for a certain vector $a$. The consistency of the probability system $p$ is assured by imposing $p \geq 0$ and $i'p = 1$, where $i$ is a vector of ones.

The ordering of the elements of $p$ is arbitrary, and Proposition 2 shows a way to construct a vector $p$ that leads to a unique choice for the order of its elements. The i-th element of the vector $p$ is obtained as follows. First, obtain the binary representation of the number $i - 1$, $b_i$. For example,

\[b_1 = [0 \ 0 \ 0]\]
\[b_2 = [0 \ 0 \ 1]\]
\[b_3 = [0 \ 1 \ 0]\]
\[b_4 = [0 \ 1 \ 1]\]
\[b_5 = \ldots\]
\[b_8 = [1 \ 1 \ 1]\]

Each of these vector can be interpreted as a vector of indicators of one of the three basic events. For example, $[0 \ 1 \ 1]$ represent the event in which $A_1$ does not occur, $A_2$ and $A_3$ occur. The element $i$ of $p_i$ will then be the probability of the event represented in this way by $b_i$. Therefore, we have:

\[p_1 = Pr\{\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}\}\]
\[p_2 = Pr\{A_1 \cap \overline{A_2} \cap \overline{A_3}\}\]
\[p_3 = Pr\{\overline{A_1} \cap A_2 \cap \overline{A_3}\}\]
\[p_4 = Pr\{A_1 \cap A_2 \cap \overline{A_3}\}\]

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\[ p_8 = Pr\{A_1 \cap A_2 \cap A_3\} \]

This is precisely the ordering represented in Figure 3.

Propositions 1 and 2 imply that bounds on the probability of a (systemic) event \(A'\) in \(F\) subject to constraints on low-order probabilities can be rewritten as a linear programming problem. In particular, the following Corollary holds:

**Corollary.** The upper bound for the probability of \(P(A')\), i.e. the solution to:

\[
\begin{align*}
\text{max } P(A') \\
\text{s.t.} \\
P(A_i) &= a_i \\
&\ldots \\
P(A_i \cap A_j) &= a_{ij}
\end{align*}
\]

can be found as the solution to the problem:

\[
\begin{align*}
\text{max } c'p \\
\text{s.t.} \\
p &\geq 0 \\
in'p &= 1 \\
Ap &= b
\end{align*}
\]

for \(c, A, b\) depending only on the available information. The lower bound is obtained by solving the corresponding minimization problem.

**Proof.** The corollary is an immediate consequence of the fact that the probability of every union or intersection of events in \(A\) can be expressed as a product \(a'p\) for some \(a\). The exact formulas for \(A, b \text{ and } c\) in the general case are reported in the Appendix.

As mentioned before, the definition of systemic event I employ is indexed by \(r\): “at least \(r\) institutions default” within a specified time period. This allows to capture
systemic events of different degrees of severity, from the likelihood that at least one institution defaults (the union of all \( A_i, r = 1 \)) to the likelihood of default of all institutions (the intersection of all \( A_i, r = N \)). Since these different events are all within the \( \sigma \)-algebra generated by the basic events \( A_1, \ldots, A_N \), their probability can be represented by the product of \( p \) with some vector \( c_r: c_r'p \).

For the simple case of three banks reported above, it is easy to verify looking at Figure 3 that the probabilities \( P_r \) of at least \( r \) institutions defaulting can be expressed as

\[
P_1 = [0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1] \cdot p \\
P_2 = [0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1] \cdot p \\
P_3 = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1] \cdot p
\]

and that the constraints can be rewritten as:

\[
P(A_1) = [0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1] \cdot p = a_1 \\
P(A_1 \cap A_2) = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1] \cdot p = a_{12}
\]

and so on. The LP formulation of this problem, with \( r = 2 \), has:

\[
c_2 = [0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1]
\]

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
\]
and

\[
b = \begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_{12} \\
a_{23} \\
a_{13}
\end{bmatrix}
\]

The problem derived in the Corollary is a standard LP program and while difficult to solve analytically, it is easy to solve numerically even as the scale of the problem gets large. Besides, the linearity of the problem guarantees that the global optimum is always found when solving it numerically, and that if the LP problem is infeasible then a solution to the original problem does not exist. Finally, Farkas’ lemma guarantees that it is always possible (and numerically feasible) to prove whether the system has no solution\textsuperscript{15}, which is important to confirm the convergence of the numerical algorithm.

3.3 Probability bounds: implementation

Estimating the bounds on the probability of systemic events described above requires conditioning on the marginal and pairwise default probabilities of financial institutions. In this paper, I extract risk-neutral probabilities from bond yields and CDS spreads. In order to do so, I need to specify a pricing model for these securities, that takes into account not only default risk, but also other important determinants of prices. In particular, several studies (such as Huang and Huang (2003) or Longstaff et al. (2005)) have established that liquidity premia in the bond market are, with default risk, a major component of the yield spread. I will then pay particular attention to the liquidity factor. In addition, the implementation of the bounds is affected by the availability of CDS data. The theoretical bounds presented above condition on all pairwise default probabilities, and could be estimated directly if I observed, for each pair of dealers \(i\) and \(j\), the quote that \(j\) asks to sell CDS protection against \(i\)’s default. Unfortunately, I only observe an average of the quotes reported by different dealers, and therefore the bounds I am in practice able to estimate condition on a smaller information set. I now

\textsuperscript{15}Farkas’ lemma states that one and only one of the following is true: 1) There exists an \(x \in \mathbb{R}^N\) such that \(Ax = b\) and \(x \geq 0\). 2) There exists a \(y \in \mathbb{R}^M\) such that \(A'y \geq 0\) and \(b'y < 0\). Finding a solution to the second system of inequalities immediately proves that the probability bounds problem is infeasible given the constraints.
discuss each of these issues separately.

### 3.3.1 Pricing bonds

To price bonds, I use a simple pricing model of the reduced-form class, as in Duffie (1998), Lando (1998), Duffie and Singleton (1997, 1999), and Hull and White (2000, 2001), with constant risk-neutral hazard rates of default.

Three elements are crucial in determining the price of a bond: credit risk, the recovery process and the liquidity premium. Assume that the recovery process is independent of all other processes, and call the expected recovery rate \( R \). Call \( T \) the maturity of the bond and \( r^F_t \) the riskless rate process. The reduced-form approach consists in specifying a risk-neutral hazard of default process \( h(t) \), such that \( h(t) \) is the risk-neutral probability of default within \( dt \) conditional on survival until \( t \). In addition, I incorporate a potentially time-varying liquidity process \( \gamma_t > 0 \).

In this framework, the time 0 price of a risk-free bond of face value $1 at time \( t \) is:

\[
\delta(0, t) = E^Q_0 (e^{-\int_0^t r_s ds})
\]

where \( E^Q \) indicates the expectation taken under the risk neutral probability measure.

The price of a liquid bond of face value $1, maturity \( T \), coupon rate \( c \) and recovery equal to a fraction \( R \) of the value of a Treasury zero-coupon bond of comparable maturity is:

\[
B(0, T) = E^Q_0 \left[ c \int_0^T e^{-\int_0^t (r_s + h_s + \gamma_t) ds} dt + e^{-\int_0^T (r_t + h_t + \gamma_t) dt} \right]
\]

Following Duffie (1999), one can incorporate a time-varying liquidity process \( \gamma_t > 0 \), and the price of the bond becomes:

\[
B(0, T) = E^Q_0 \left[ c \int_0^T e^{-\int_0^t (r_s + h_s + \gamma_t) ds} dt + e^{-\int_0^T (r_t + h_t + \gamma_t) dt} + R \int_0^T e^{-\int_0^T r_s ds e^{-\int_0^t h_s ds h_t dt}} \right]
\]

In theory, it is possible to write down a parametric version of the (potentially not independent) processes that govern the evolution of \( r_t, h_t, \gamma_t \), and additionally a time-varying recovery rate. Examples of this can be found in Duffie and Singleton (1997)
and Longstaff, Mithal and Neis (2005). In what follows, I use a simplified pricing model that assumes that at each time, looking forward from $t$ up to the maturity of each bond, $h_{t+s}$ and $\gamma_{t+s}$ are constant. I discretize the model to a monthly horizon, and I assume that coupons are paid monthly. The choice of a month is motivated by the relative reference period for the CDS spreads discussed in Section 2.

The discretized formula for the price at time 0 of a T maturity bond under these assumptions is:

$$B(0, T) = c \left( \sum_{t=1}^{T} \delta(0, t)(1 - h)^t(1 - \gamma)^t \right) +$$

$$+ \delta(0, T)(1 - h)^T(1 - \gamma)^T + R \left( \sum_{t=1}^{T} \delta(0, T)(1 - h)^{t-1}(1 - \gamma)^{t-1}h \right)$$

where $h$ is the average monthly probability of default, $\gamma$ is the monthly liquidity process, $c$ is the monthly coupon paid and $R$ is the expected recovery rate.

### 3.3.2 Pricing CDSs

As explained in Section 2, a component of the spread of the CDS is counterparty risk\(^{16}\). Counterparty risk arises because in some states of the world the protection seller cannot pay the buyer the full amount owed. A fraction of that amount can still be recovered thanks to collateralization and to the seniority of CDS claims in bankruptcy.

While it is possible to build continuous-time models of CDS prices that take into account the exact dynamics of events and the timing of defaults, the availability of pricing data on CDS is so limited that it would not allow extracting joint default processes in a complex model with several free parameters. Note that several models of joint default that appear in the literature are based on the assumption that default intensities are independent conditional on the realization of a state vector; in these models, counterparty risk at short horizon is small by construction, because it is proportional to the product of the marginal default probabilities. Therefore, these models would not be the most appropriate ones for the case in which defaults are not independent even at short horizons. A valid alternative would be a model of correlated default intensi-

\(^{16}\)Liquidity in the CDS market could also be present, but it is much less likely to be an issue because of several reasons, among which the fact that they require much less capital at origination and they are not in fixed supply. For an additional discussion of this and on the supporting evidence, see Blanco, Brennan and Marsh (2003,2005).
ties, as introduced by Jarrow and Yu (2001): there, defaults are independent over very short horizons of time, but the default of one institution increases the default intensity of the others. While I do not explicitly use this model in my empirical analysis, my discrete-time pricing formulation is compatible with a model in which defaults within a certain period of time (for example a month) are correlated due to spillovers from one institution to the other. However, conditional on one institution surviving for enough time after the default of the other, the spillover effect on its default intensity is small.

As with bonds, I discretize the model to one month intervals, and I assume that both marginal and joint hazard rates are constant from the perspective of the time of the pricing until the maturity of the CDS. I assume that the payoff of the CDS for the following month is as follows. If the seller does not default within the month but the reference entity defaults, the payment is made in full. Therefore, a month is considered an amount of time sufficient for the seller to establish whether to default or not on the CDS obligation. If the seller defaults within the month but the reference entity does not, the contract terminates. As discussed in Section 2, it is reasonable to assume that in this single-default case the recovery rate of the CDS is close to 100%. Finally, if both the seller and the reference entity default in the same month, I assume that the two defaults happen in a connected way and only an amount $S$ of the full payment is recovered - this case corresponds to the double-default case, in which counterparty related losses are important.

The discretized CDS pricing equation can be written as:

$$
\sum_{s=1}^{T} \delta(0, s-1)(1 - P(A_i \cup A_j))^{s-1}z_{ji} =
$$

$$
= \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1}(P(A_i) - (1 - S)P(A_i \cap A_j))\delta(0, T)(1 - R)
$$

where $P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_i \cap A_j)$, and $z_{ji}$ is the spread of the CDS written by $j$ to insure against $i$'s default.

After a linear approximation discussed in the Appendix, it is possible to rewrite the spread as:

$$
z_{ij} = (P(A_j) - (1 - S)P(A_i \cap A_j))\frac{T\delta(0, T)(1 - R)}{\sum_{s=1}^{T} \delta(0, s-1)}
$$
Note that this equation is a restriction on the marginal and pairwise default probabilities of $i$ and $j$, and as such it could be used to obtain a constraint on $P(A_i \cap A_j)$ for given $P(A_i)$. However, it can also be imposed directly as a constraint in the LP program presented above, since it is linear in the marginal and pairwise default probabilities.

### 3.3.3 Implementation of the bounds

Once the pricing models for bonds and CDSs are specified, the implementation of the bounds requires two further steps. The first one is discussing what assumptions to impose on the bond liquidity process $\gamma_i$. The second is modifying the linear programming problem to take into account the limitations of the CDS data.

I deal with the problem of unobserved liquidity in the bond market in the following way. For each intermediary $i$, I obtain from bond prices an estimate of the risk-neutral marginal probability of default, given an assumed liquidity process $\gamma_i$, using the pricing model presented above. This results in a function $h_i(\gamma_i)$ for each $i$. Note that this function is decreasing in $\gamma_i$, and since it is reasonable to assume $\gamma_i \geq 0$, it has its maximum for $\gamma_i = 0$: $h_i(0)$. Assuming a zero liquidity premium gives us the highest possible implied default probability, for given bond prices. The function $h_i(\gamma_i)$ is estimated separately at each time $t$ using the cross-section of senior unsecured bonds outstanding for firm $i$ (the details are reported below in Section 4).

Once the function $h_i(\gamma_i)$ is estimated, there are two ways to proceed in order to construct the bounds on the probability of systemic events. The first is to assume a certain value for $\gamma_i$, say $\tilde{\gamma}_i$ and impose the constraint in the LP problem:

$$P(A_i) = h_i(\tilde{\gamma}_i)$$

The problem with this method is that since the process $\gamma_i$ is unobserved, it is difficult to make plausible assumptions about it. If one instead is more confident about imposing a lower bound for $\gamma_i$, call it $\underline{\gamma}_i$, but does not want to impose anything else, then the constraint could be replaced by the less stringent constraint:

$$P(A_i) \leq h_i(\underline{\gamma}_i)$$

In other words, if we know that the liquidity process $\gamma_i$ is at least as large as $\underline{\gamma}_i$, this
implies that the default probability cannot be higher than $h_i(\gamma_i)$, since the function $h$ is decreasing in $\gamma_i$. Of course, we can always impose the least binding constraint of all, 

$$P(A_i) \leq h_i(0)$$

which means imposing no assumptions at all about the liquidity process (apart of its nonnegativity).

In the analysis below I will start from computing bounds that only assume $\gamma_i \geq 0$ and then study how they vary as more stringent lower bounds on $\gamma_i$ are imposed. At first, the level of $\gamma_i$ will be calibrated to the level observed in 2004 under the assumption that the spread between bond yields and CDS spreads was entirely due to liquidity premia (as opposed to counterparty risk). Then, I will show how the results change if $\gamma_i$ is increased further.

The second factor to take into account when implementing the LP problem is the fact that for each institution $i$ I do not observe the spread written on its bond by every other institution $j$, but only an average of the quotes provided by the $N - 1$ counterparties:

$$z_i = \frac{1}{N - 1} \sum_{j \neq i} z_{ji}$$

This means that when I compute the bounds, instead of the set of constraints (one for each pair $i \neq j$)

$$z_{ji} = [P(A_i) - (1 - S)P(A_i \cap A_j)] \frac{T\delta(0,T)(1 - R)}{\left[\sum_{s=1}^{T} \delta(0, s - 1)\right]}$$

I will only be able to impose the constraints (one for each $i$):

$$z_i = \left[ P(A_i) - (1 - S) \left( \frac{1}{N - 1} \sum_{i \neq j} P(A_i \cap A_j) \right) \right] \frac{T\delta(0,T)(1 - R)}{\left[\sum_{s=1}^{T} \delta(0, s - 1)\right]}$$

It is important to note that this constraint assumes that the spread $z_i$ is obtained by averaging across the spreads quoted by the other $N - 1$ institutions in the group considered. Unfortunately I do not observe exactly which dealers contributed to the quotes, nor do I observe the exact weighting scheme employed, which presumably gives
more weight to more active institutions. My solution is to consider a group of firms that most likely represents the sample from which the quotes come from. Since this market is very concentrated, and the top 10 firms alone account for about 90% of the protection sold, I believe that including the top 15 dealers should make sure that the average spread reflects the average counterparty risk of this group of dealers. Besides, since all of these institutions are very active in the CDS market, equal weighting seems a reasonable approximation to the true weighting scheme. Of course, it is possible that the spread partly reflects quotes obtained from financial institutions outside the group I consider, or that some of the dealers in the group do not post quotes at all times. In both cases, I would likely underestimate counterparty risk. In the former case, because quotes may be obtained from smaller institutions for which the recovery rate of the CDS in case of double default could be lower. In the latter case, because if the institutions that are not posting a quote are the riskiest ones, the average spread observed would be biased upwards. However, as long as these problems affect only a few institutions at a time, the effect on the average spread should be small.

Finally, to be able to construct the bounds, I need to specify expected recovery rates. The baseline expected recovery rate is 40% for senior unsecured bonds, which is standard in the literature. For CDS, I assume a recovery rate of 40% as well, which implies that CDS and senior unsecured bond claims on a firm are of the same level of seniority. Since CDS share the additional layer of seniority with the other derivatives, and since at least a fraction of CDS contracts have a collateral agreement, this expected recovery rate might in practice be higher, which would imply higher counterparty risk for given CDS spreads. For the reasons discussed in Section 2, it is unlikely that collateral can provide sufficient protection against the double default case, so even taking collateralization into account the recovery rate in that case would likely be not much higher than that of senior bonds (instead, I am assuming that the recovery rate for CDSs is 100% in the case of single default).

Following the steps described above to obtain a feasible maximization problem given the data available, the maximization problem for the probability of at least $r$ out of $N$ dealers defaulting can now be stated as follows:

$$
\max P_r
$$

s.t.

$$
P(A_i) \leq h_i(\gamma_i) \ \forall i
$$

(1)
\[
\left[ P(A_i) - (1 - S) \left( \frac{1}{N - 1} \sum_{i \neq j} P(A_i \cap A_j) \right) \right] \left( \frac{T\delta(0, T)(1 - R)}{\sum_{s=1}^{T} \delta(0, s - 1)} \right) = \bar{z}_i \ \forall i \quad (2)
\]

which, as shown above, can be represented in linear form as:

\[
\max_p d_r^r p
\]

s.t.

\[
\begin{align*}
p & \geq 0 \\
\iota'p &= 1 \\
Cp & \leq d \\
Ep &= f
\end{align*}
\]

where the constraint (3) corresponds to the set of constraints (1), and the constraint (4) corresponds to (2).

In the empirical analysis reported below, I estimate the upper and lower bounds for systemic risk separately for every trading day \( t \) between 2004 and 2009. To do this, I estimate \( h_i(\gamma_i) \) separately for each \( t \) and use daily observations of \( z_i \). This way, I obtain a time-series for both the upper and lower bounds that allows me to study how market perceptions of systemic risk changed over time.

### 3.4 Properties of the bounds

Because the probability bounds are the solution to a LP problem, it is difficult to fully characterize them algebraically. However, some important properties can be derived appealing to the linear structure of the problem. First, I prove in Proposition 3 that if the maximization problem is symmetric to the ordering of the default events (which is for example the case if we only observe average CDS and bond information), then the network configuration that attains the bounds will be symmetric - all banks have the same probability of default and all pairs of banks have the same pairwise probability of default. Second, I show that as a consequence of this result, I can apply some results from the LP literature (e.g. Boros and Prekopa (1989)) to derive closed form solutions for the bounds when the network is symmetric. In particular, I can study how the width of the bounds varies as the underlying low-order probabilities of default change.
3.4.1 Symmetry of the probability system

**Definition.** Consider the vector \( p \in \mathbb{R}^{2N} \) representing a probability system on the \( \sigma \)-algebra generated by the basic events \( A_1, \ldots, A_N \), as in Propositions 1 and 2. Consider a permutation \( J \) of the indices of the basic events: \( A_{J_1}, \ldots, A_{J_N} \), and call \( J \) the set of permutations. Define \( p_J \in \mathbb{R}^{2N} \) the vector representing the probability system generated by \( A_{J_1}, \ldots, A_{J_N} \) that corresponds to \( p \), constructed as in Proposition 2.

**Definition.** A linear combination of the elements of \( p \) defined by the vector \( c \) is *symmetric* with respect to the generating events \( A_1, \ldots, A_N \) if \( c'p = c'p_J \forall J \in J \).

For example, take two events \( A_1 \) and \( A_2 \). A vector \( p \) representing the probability system constructed as in Proposition 2 would have 4 elements, corresponding to:

\[
\begin{align*}
P(A_1 \cap A_2) &= p_1 \\
P(\overline{A}_1 \cap A_2) &= p_2 \\
P(A_1 \cap \overline{A}_2) &= p_3 \\
P(\overline{A}_1 \cap \overline{A}_2) &= p_4
\end{align*}
\]

In this case, only one additional permutation of the generating events is possible, \( J = \{2, 1\} \), and we have:

\[
\begin{align*}
p_{J_1} &= p_1 \\
p_{J_2} &= p_3 \\
p_{J_3} &= p_2 \\
p_{J_4} &= p_4
\end{align*}
\]

An example of symmetric weighting vector \( c \) is the one corresponding to the probability of the union of the events, \( c = [1 1 0 1]' \), since \( c'p = c'p_J = P(A_1 \cup A_2) \).

**Definition.** A probability system \( p \) is *symmetric* if every event in \( V \) (the finest partition of the sample space generated by the basic event) has the same probability in all permutations of the generating events.
For example, with three generating events \((N = 3)\), the probability system is symmetric if \(P(A_1) = P(A_2) = P(A_3)\) and \(P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_1 \cap A_3)\).

**Definition.** A linear programming problem

\[
\begin{align*}
\max & \quad c'p \\
\text{s.t.} & \quad Ap \leq b
\end{align*}
\]

is *symmetric* if \(c\) and all rows of \(A\) are symmetric with respect to the generating events \(A_1, \ldots, A_N\).

We can now state the following proposition:

**Proposition 3.** Suppose that the probability bounds correspond to a symmetric LP problem. Then, the bounds are attained by a symmetric probability system.

**Proof.** See Appendix.

**Corollary.** The bounds on systemic events of the type “at least \(r\) institutions default” given a symmetric constraint set (for example, constraints on the average marginal and pairwise default probabilities) are attained by a symmetric probability system.

The bounds obtained in a symmetric network in which we observe all marginal and pairwise probabilities will always be at least as wide as those obtained in an asymmetric network with the same averages of the low-order probabilities. The difference between the bounds obtained in the two cases captures precisely the extent to which asymmetry in the network shape affects the probability of systemic events.

### 3.4.2 Width of the bounds

The Corollary to Proposition 3 also implies that the analytical formulation of the bounds derived by Boros and Prekopa (1989) for the case in which only the average marginal and pairwise default probabilities are known applies to the case in which all low-order probabilities are observed, as long as the network is symmetric with respect to the latter. This is an interesting benchmark to understand some of the properties of the bounds. The following Proposition holds:
Proposition 4. Denote with \( p_r \) the probability of the occurrence of at least \( r \) events. If the probability system is symmetric with respect to marginal and pairwise default probabilities, i.e.

\[
P(A_i) = q_1 \quad \forall i
\]

\[
P(A_i \cap A_j) = q_2 \quad \forall i, j
\]

then the upper and lower bounds for \( p_r, \ r \geq 3 \), out of \( N \) events, have the following properties:

- For given \( q_1 \)
  - if \( q_1 < \frac{r-1}{N} \), the lower bound is 0 for low \( q_2 \) and is increasing in \( q_2 \) for higher \( q_2 \).
  - if \( q_1 > \frac{r-1}{N} \), the lower bound is first decreasing and then increasing in \( q_2 \).

- For given \( q_1 \)
  - if \( q_1 < \frac{N-2}{N(N-2)-r+1} r \), the upper bound is first increasing and then decreasing in \( q_2 \).
  - if \( q_1 > \frac{N-2}{N(N-2)-r+1} r \), the upper bound is first 1 and then decreasing in \( q_2 \).

- Given \( q_1 \), the width of the bounds is 0 for the lowest possible \( q_2 \), then increases with \( q_2 \) and then decreases to 0 for \( q_2 = q_1 \). Therefore, there is a point of maximum width in the interior of the space for \( q_2 \).

- Similar results hold when \( q_1 \) varies and \( q_2 \) is held fixed.

**Proof.** See Appendix. \( \square \)

Proposition 4 implies that the tightness of the bounds on systemic risk varies in a very precise way with changes in the low-order risks. When pairwise default probabilities \( q_2 \) are either very high or very low compared to the marginal default probabilities \( q_1 \), the structure of the network is pinned down very precisely and there is little uncertainty about systemic default risk, based on bond and CDS prices alone; different parametric models that aggregate in different ways this low-order information will agree on an estimate of systemic risk. For intermediate values of \( q_1 \) and \( q_2 \), however, low-order
probabilities are less informative about systemic events. The scope for modeling assumptions in aggregating low-order probabilities is greater.

Therefore, if direct information on systemic risks is scarce and agents learn about them by combining low-order information with some parametric model, the width of the bounds captures the scope for disagreement based on modeling assumptions. Starting from a situation of low risks (low $q_1$ and $q_2$), increases in the marginal and pairwise risk correlations will coincide with an increase in the scope for disagreement, at least up to the point where the bounds narrow again (and everybody agrees on the severity of the systemic risks). This might have important consequences for the stability of the financial system, because it allows for more extreme disagreement about systemic risks even for fully rational agents facing the common information set, precisely when low-order risks increase.

4 Data

4.1 Choice of the set of intermediaries

The focus of this paper is on systemic events that affect the core of the global financial network, represented by the group of largest and most connected financial institutions. Motivated by the literature on contagion in financial networks\textsuperscript{17}, which emphasizes the role of the gross risk exposures of intermediaries for the propagation of shocks, I focus on the set of dealers with the highest gross exposures to the CDS market. In particular, I choose a group of 15 institutions that includes the largest American and European broker/dealers, but excludes large firms (like AIG) which have higher net exposure to credit risk but lower gross exposure. The list contains 8 American and 7 European investment banks, commercial banks and broker/dealers: Bank of America, Citigroup, Goldman Sachs, Lehman Brothers, JP Morgan, Merrill Lynch, Morgan Stanley, Wachovia, Abn Amro, Bnp Paribas, Barclays, Credit Suisse, Deutsche Bank, HSBC, UBS.

Several sources confirm that this group of banks is really at the center of the financial network, at least as far as CDS exposures are concerned. Fitch Ratings, referring to

\textsuperscript{17}see for example Allen and Gale (2000), Giesecke and Weber (2004), Adrian and Brunnermeier (2009)
different years (2006, 2008), reports that the top 5 dealers account for more than 80% of
the value of outstanding CDS contracts. Credit Derivatives Research\(^{18}\) claims that the
group of 15 institutions mentioned above covers 90% of the volume of CDS protection
sold. Not only are these firms the largest issuers and buyers of CDS protection: they are
also consistently among the top entities on which protection is sold. In other words,
these institutions are at the same selling protection to each other against external
default events, and exchanging protection against the default of other members of the
same group. For example, Fitch reports that the top 5 sellers of CDS protection also
appear in the set of top 20 reference entities.

The data cover, with daily frequency, the period from January 2004 to March 2009,
during which time the CDS market was already well developed and liquid.

4.2 Data description

4.2.1 Credit Default Swaps

Since Credit Default Swaps are derivatives traded over the counter, they are in general
not standardized. However, the 5-year CDS based on the ISDA format (which standard-
dizes credit event definitions and collateral requirements) has emerged as the reference
contract, which allowed it to achieve high liquidity. For this CDS, Bloomberg reports
mid-market quotes that are obtained by averaging the quotes reported by different deal-
ers (through CMA) - euro-denominated for European banks and dollar-denominated for
US banks. The series contain a few missing values, which are filled by interpolation.

Table 1 reports summary statistics on CDS spreads. While CDS spreads between
2004 and 2009 have usually been quite low, on the order of 50bp, they reached levels
higher than 1000bp in some periods. Figure 4 reports the time-series of CDS spreads
for some banks in the sample (thin line)\(^{19}\). From the figure, it is easy to see that while
movements in CDS spreads seem to be highly correlated across dealers, there is a certain
degree of heterogeneity, especially since 2007. The highest spreads are reached in the

\(^{18}\)Credit Derivative Research publishes the Counterparty Risk Index, which tracks the average CDS
spread of these 15 dealers.

\(^{19}\)Note that after 15 September 2008 both Lehman Brothers and Merrill Lynch drop out of the
group; Lehman brothers went bankrupt while Merrill Lynch was acquired by Bank of America. While
Wachovia merged with Wells Fargo, this happened in July 2009 and therefore after the end of the
sample considered here.
few months after Lehman’s collapse, but not for all banks (for example, not for HSBC, whose spread remains relatively low).

### 4.2.2 Corporate Bonds

As discussed in Section 3, an important step of the construction of the bounds is the estimation, from bond prices, of the risk-neutral probability of default given a certain value for the liquidity process: $h_i(\gamma_i)$. For every trading day $t$, I estimate the risk-neutral probability $h_i$ as a function of $\gamma_i$ for each firm. Given the set of bonds outstanding for firm $i$ at time $t$, I estimate $h_i$ from their prices using least absolute deviations, rather than OLS, to reduce the impact of outliers\(^{20}\).

For each of the 15 institutions considered, I search Bloomberg for senior unsecured zero and fixed coupon bonds with maturity less than 10 years. I exclude callable, puttable, sinkable, and index-linked bonds, since their prices reflect the value of the embedded options. To eliminate some outliers, I remove all observations with prices above $2 (for bonds whose face value is $1). I consider dollar-denominated bonds for US firms and euro-denominated bonds for European firms. Since the availability of Bloomberg data on European bonds is fairly limited, I integrate it with bonds data from Datastream; in general, it appears that the quality of these data is lower than Bloomberg (the frequency of outliers and stale prices is higher), so I avoid using them for US firms. The main results of the paper are robust to the use of Bloomberg data alone, or using both Datastream and Bloomberg data for all firms.

Table 2 reports some statistics on the availability of bond data. In the first column we can see the average number of bonds available for each institution, for the days in which that institution is in the sample (i.e. excluding Lehman Brothers and Merrill Lynch after September 15th 2008). For example, the default probability for Bank of America $h$ is estimated using on average 14 bonds each day. The next columns break down this number by year. For several European dealers, bond data is missing in some periods. These dealers will still be part of the empirical analysis below, but they will be excluded for the periods with missing bond data. Since the end of 2007, all dealers have valid bonds.

\(^{20}\)In fact, the results are robust to the use of OLS.
4.2.3 Risk-free rate

As a reference risk-free rate, I use US and European government bond yields, constructed from the generic Bloomberg yield curves, as is customary for reduced-form models. An alternative would be to use swap rates, which are less affected by tax and liquidity issues. Their use for investment-grade bonds is advocated by Howeling and Vorst (2005). However, swap rates contain some credit and counterparty risk (see Sundaresan (1991) and Duffie and Huang (1996)); given the centrality of counterparty risk in my analysis, I prefer to employ government bonds.

4.3 The CDS basis

In the absence of counterparty risk and liquidity premia, CDS spreads, the risk-free rate and bond yields are related by an approximate arbitrage relation\(^1\). Intuitively, one could buy the bond and completely insure it against default, or invest in the risk-free security of the same maturity. The deviation from this arbitrage, and in particular the difference between the CDS spread and the corresponding corporate yield spread is called the basis of the CDS trade, and has been extensively studied in the literature (see for example Blanco, Brennan and Marsh (2005)). I report in Table 1 some descriptive statistics on the basis of the 15 dealers in my dataset\(^2\), and I plot the interpolated yield on a 5-year bond yield spread (thick line) and the CDS spread (thin line) in Figure 4 for some banks. As the Table and the Figure show, the basis is usually negative, because the CDS spread is lower than the corresponding bond yield spread.

Three important facts emerge from Figure 4. First, the basis has greatly increased in absolute value for most banks during the crisis. Second, movements in the basis are correlated across dealers, but there is heterogeneity especially during the recent crisis. As discussed in Section 3, this in turn results in a higher degree of asymmetry in the marginal and pairwise default probabilities implied by bond and CDS prices. Finally, Figure 4 and Table 1 show that in some cases the basis drops to zero (and can even switch sign). This typically occurs when, in response to bad news, CDS spreads jump

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\(^1\) The approximation would be perfect if floating notes were considered and is only approximate using fixed-coupon rates, as explained by Duffie (1999)

\(^2\) Yields for the 5 year bond were linearly interpolated from the available yields, and approximated with the yield of the bond of closest maturity in the case in which bonds bracketing a maturity of 5 years were not present.
upwards immediately while bond yields react with a lag, because of the lower liquidity of the bond market. Of course, one limitation of the methodology presented in this paper is that such events will be interpreted as a drop in counterparty risk, at least until the bond yields adjust. However, since this phenomenon typically occurs for a few days, it will not affect the general trends of the bounds. Besides, smoothing the bounds with a filter could reduce the problem.

The standard explanation for a negative basis in the literature has since been liquidity in the bond market. Since bonds are less liquid than Credit Default Swaps, the argument goes, bond yields are higher than the corresponding CDS spreads because they reflect liquidity premia in addition to credit risk. In fact, Longstaff, Mithal and Neis (2005) have exploited this explanation to estimate liquidity premia in the bond market. More recently, Garleanu and Pedersen (2010) calibrate with the CDS-bond basis an equilibrium model with differential margin constraints across securities (where bonds have higher capital requirement and therefore command a higher yield). Note that while bond liquidity can explain a negative basis, the cases in which the basis is positive are problematic under this interpretation as well. It is difficult to explain why the liquidity component of the yield spread would drop during the times of crisis in which typically this phenomenon occurs.

Once counterparty risk in the CDS market is taken into account, it is not clear any more that the wedge between CDS spreads and bond yields is caused by liquidity. The presence of counterparty risk lowers the value of insurance, and therefore the CDS spread, producing a negative basis.

5 Results

5.1 Bounds with no restrictions on liquidity

I begin the analysis imposing no restrictions on the liquidity process for bonds, i.e. using the constraint

$$P(A_i) \leq h_i(0)$$

in computing the bounds. As explained in Section 3, this imposes the least binding restrictions on the implied risk-neutral marginal probabilities of default. They could
be as low as zero (the whole bond/CDS basis is explained by liquidity), or as high as $h_i(0)$, the probability obtained from bond prices under the assumption that the liquidity process $\gamma_i$ is equal to zero. Note that at the same time, the constraints derived from CDS spreads implicitly impose a lower bound on the marginal default probabilities. This is due to the fact that counterparty risk cannot be negative, and since

$$\left[ P(A_i) - (1 - S) \left( \frac{1}{N - 1} \sum_{i \neq j} P(A_i \cap A_j) \right) \right] \frac{T \delta(0, T)(1 - R)}{\sum_{s=1}^{T} \delta(0, s - 1)} = z_i \quad (6)$$

we must have

$$P(A_i) \frac{T \delta(0, T)(1 - R)}{\sum_{s=1}^{T} \delta(0, s - 1)} \geq z_i$$

Figure 5 plots the bounds for the average monthly probability of default of at least one institution ($P_1$, upper panel), at least three ($P_3$, middle panel), and at least five ($P_5$, bottom panel).

The thin line represents the upper and lower bounds obtained using only bond prices, i.e. imposing only the constraints (5) but not constraints (6) in the maximization (minimization) problem. Therefore, these bounds ignore the information on counterparty risk contained in the bond/CDS basis. The thick line represents the bounds obtained using all available information. In this sense, they are the optimal bounds conditional on the prices of bonds and CDSs that we observe. Finally, the dotted line plots bounds obtained averaging each set of constraints (5) and (6) across $i$’s, therefore obtaining only two constraints: one coming from bond prices and one from CDS spreads. These bounds are approximately those one would obtain if only the average bond yield and the average CDS spread were observed. Note that the bounds that use all the information available are in fact significantly tighter than the ones that use only bond prices. This underlines the importance of including restrictions on CDS spreads to study systemic risk.

Several interesting facts emerge from this Figure. All bounds agree on a significant increase in the maximum possible amount of systemic risk, starting August 2007. The upper bounds were at very low levels before then, and they started gradually increasing afterwards$^{23}$. The lower bound on $P_1$ increased as well over this period, even though

$^{23}$I only plot the bounds starting in 2007 to focus on the recent events. Between 2004 and 2007, the bounds are at the same levels as in the first months of 2007.
at a slower pace. While the lower bounds in the bottom two panels (for $P_3$ and $P_5$) are all zero, the pattern overall is suggestive of a gradual increase in systemic risk.

A second point relates to the width of the bounds. In all the three panels, the bounds become wider and wider over time, with some fluctuations. This signals an increase in uncertainty about the magnitude of systemic risk, conditional on the information contained in bond and CDS prices. This result is in line with the anecdotal evidence, as well as with the theoretical analysis discussed in Section 3. While the theoretical results presented there refer to the case of a perfectly symmetric network, the general point seems to be valid here as well. In times in which the underlying low-order default probabilities increase, uncertainty about high-order probabilities increases. The width of the bounds keeps increasing at least until the probabilities are so high that there is agreement on the severity of the risk, and from that point on the width of the bounds diminishes.

Maybe the most interesting result that emerges from Figure 5 is the decomposition of the movements in bond yields and CDS spreads into idiosyncratic and systemic default risk. By looking only at the bounds that do not make full use of the information available (thin line and dotted line), one would conclude that besides increasing between 2007 and 2009, systemic risk spiked in March 2008, September 2008 and finally March 2009. This account is in fact consistent with the one pictured in Figure 1 using traditional measures of systemic risk. However, looking at the optimal bounds that make use of all available bond and CDS prices (thick line), a different story emerges. What accounts for the spikes in bond yields and CDS spreads observed in those episodes is more the idiosyncratic default risk of a few banks than systemic risk. This is evident comparing the optimal bounds in the top panel with the two bottom panels. The other measures of systemic risk are not able to pick up this important difference because they do not take into account the full extent of the asymmetry of the network.

The approach presented in this paper also allows to quantify the amount of systemic risk, in terms of risk-neutral probabilities. For example, the bounds presented in Figure 5 (which make no assumptions on the liquidity process in bond prices) imply that at the end of 2009 the monthly probability of at least one default was between 1% and 4%, the probability of at least 3 defaults was 0 to 1.5%, and that of at least 5 defaults 0 to 1%. Naturally, the level of the bounds depends directly on the pricing model used, and so different models might estimate different levels.
Finally, note that imposing no assumptions on the liquidity premium in bonds, as we have done here, implies that there is a wide range of possible marginal default probabilities that are compatible with bond prices. This of course results in relatively wide bounds. Below I introduce assumptions on the liquidity process, that restrict the range of possible marginal probabilities.

5.2 Bounds with assumptions on liquidity

As explained in Section 3, if we assume that the liquidity-driven cost of holding a bond is at least $\gamma_i$, the upper bound on each of the marginal default probabilities decreases to $h_i(\gamma_i)$. In turn, this means that the bounds on systemic risk become tighter. In addition, the effect of liquidity is different across dealers at each point in time, and therefore tighter assumptions on liquidity increase the asymmetry of the network. For banks whose yield spread is just slightly above the CDS spread, a small increase in $\gamma_i$ implies that the counterparty risk priced in the CDS contract on $i$ decreases proportionally by a large amount. For banks for which the difference is very high, the same increase in $\gamma_i$ will have little effect on the amount of counterparty risk compatible with CDS spreads. For similar reasons, the effect of liquidity assumptions on the measure of systemic risk varies over time.

To start from a reasonable value of the liquidity process, I calibrate $\gamma_i$ for each bank so that all the difference between the yield spread and the CDS spread (the bond/CDS basis) in 2004 is attributed to the liquidity premium. This implies a liquidity component of the yield spread of about 50-60%. This is in line with the calibrations of Huang and Huang (2003) and Longstaff, Mithal and Neis (2005). I then look at the bounds for minimum values of $\gamma_i$ proportional to this starting point.

Figure 6 plots the bounds on the monthly probability of at least five institutions defaulting when $\gamma_i = 0$ (top panel), when $\gamma_i$ is calibrated to match the bond/CDS basis in 2004 (middle panel) and twice that amount\(^24\) (bottom panel). Therefore, the top panel replicates the bottom panel of Figure 5. Lowering the upper bound on the marginal probabilities of default has three effects on the measures of systemic risk. First, it directly rules out some very high values for the default probabilities of each

\(^24\)While it is difficult to know precisely by how much liquidity premia increased over the recent crisis, a liquidity cost twice that of 2004 seems quite plausible given the anecdotal evidence about the state of the bond markets during the crisis.
single bank. Second, it indirectly lowers the maximum amount of counterparty risk present in CDS contracts. This is because in order to match observed CDS spreads, high counterparty risk must be balanced by high marginal default probabilities (see equation (6)). Finally, to the extent that, as discussed before, the effect is different across dealers, it increases the asymmetry of the network and therefore the gain in using all the information available.

The bounds constructed using bond prices but not CDS spreads reflects only the first effect. As tighter assumptions on liquidity are imposed, the upper bound shifts down relatively uniformly. From the top to the bottom panel, the average decrease in the monthly probability of at least 5 institutions defaulting is about 0.3%.

The bounds that only use average information (dotted line) reflect a combination of the first two effects: the direct effect on the marginal probabilities and the indirect effect on average counterparty risk. However, the second effect is the most important when we look at the default probability of 5 institutions or more. This upper bound in fact does not shift down uniformly over time, but it does so more in certain periods. Interestingly, imposing liquidity assumptions eliminates the spike of systemic risk attributed by this bound to the Bear Stearns episode, but not the subsequent Lehman episode or in March 2009. This is a sign that asymmetry across the network was more important after September 2008 than it was before.

Finally, the optimal bound (thick line) reflects all three effects, including the effect of increased asymmetry in the network. This makes it react more to changes in liquidity assumptions, and especially so around the three episodes of the crisis mentioned above.

Imposing even mild liquidity assumptions shifts the bounds down, and especially so the optimal one. This means that once we take liquidity into account, systemic risk can be bounded to relatively low values, especially during high turbulence periods when CDS and bond yields spiked. The story Figure 6 tells is in contrast with several accounts of the crisis that underlined the fear of a systemic event around those periods. In a sense, the bounds point out an internal inconsistency of many of these accounts. High systemic risk in the financial sector requires low CDS spreads (relative to bond yields) for contracts written by large dealers against the default of other dealers. Since the counterparties to those contracts are in fact the same dealers that are at the core of the financial network, a high risk of systemic default ought to reduce the value of insurance contracts bought from them.
5.3 Individual contributions to systemic risk

The method described in this paper to obtain the bounds on systemic risk also allows to study the evolution of the default risk of each bank and its relation with the rest of the network. In particular, by solving for the probability system that *attains* the upper bound, we can learn about the marginal and pairwise probabilities of default in the most interconnected scenario. Besides, we can also study the contribution of each bank to systemic risk. This allows us to better understand the dynamics underlying the bounds of Figures 5 and 6.

We can start by looking at the configuration of the financial network at the upper bound of systemic risk\(^{25}\), in terms of marginal and pairwise probabilities. Figure 7, for example, plots a snapshot of part of the network as of 08/19/2008, about a month before Lehman’s collapse. The nodes of the diagram represent the individual banks and are associated with monthly marginal probabilities of default. The segments that connect the nodes report the joint default probability of the two intermediaries. The graph reflects the market expectations about the risks of default of these banks individually and in pairs.

In decreasing order of marginal default risk we find Lehman Brothers, Merrill Lynch, Morgan Stanley, Citigroup, and Goldman Sachs. In addition, we can see that the pair at highest risk of default is Merrill Lynch with Lehman Brothers. It is evident from this graph how the market was perceiving a high joint default risk of these two banks a while before the weekend in which both underwent severe problems. Excluding these two, most of the other pairwise default probabilities are similar across banks, and cross-sectional variation in the marginal default probabilities reflects different idiosyncratic default risk more than joint default risk.

Using a similar approach, for each pair of banks \(i\) and \(j\) we can track the evolution of \(P(A_i), P(A_j)\) and \(P(A_i \cap A_j)\) over time. Figure 8 plots these probabilities for three different pairs (all combinations of Lehman, Merrill Lynch, and Citigroup). This graphs confirms the high degree of heterogeneity in marginal default probabilities across banks, but also that pairwise default probabilities were significantly higher for some pairs of banks than others. For example, the pairwise default probability of Lehman Brothers and Merrill Lynch was about twice that of Merrill and Citigroup, and it also increased

\(^{25}\)Here I consider the solution for the maximization of the probability that at least 5 institutions default \((P_5)\). Similar results obtain for different definitions of systemic risk.
much more starting in mid 2008.

By finding the probability system that attains the bounds we can also to study
the contribution of each institution to systemic risk and track its evolution over time.
One way to capture this is to compute bounds for the probability that institution \(i\) is
involved in a multiple default event:

\[ Pr\{\text{at least 5 default} \cap \text{i defaults}\} \]

Figure 9 plots the upper bound for this probability for four banks (Citigroup,
Lehman Brothers, Merrill Lynch and HSBC) as well as the average across the other 11
banks (to improve readability, I plot a 7 day moving average). Some interesting results
emerge from this graph. First of all, there is again considerable heterogeneity across
firms, both in the levels and in the changes. While the contribution to systemic risk
increases for all banks after August 2007, the growth is much faster for Lehman and
Merrill Lynch than for the other banks. For example, HSBC ’s contribution remains
relatively low throughout the sample (lower than the average across other banks for
most of the sample).

Besides, the picture shows clearly that markets perceived an increased systemic
risk of Lehman and Merrill Lynch way before the weekend of September 13th 2008,
capturing the idea that they were not only high-risk banks but fundamentally systemic.
Interesting as well is the sharp increase in Citigroup’s contribution to systemic risk after
Lehman’s collapse and then again around the beginning of 2009. These sharp increases
are not matched by HSBC, whose contribution to systemic risk remained more stable
even during the periods of highest turbulence, nor they are matched by the average of
the other dealers.

Figures 7 to 9 also allow us to shed some additional light on the mechanics of the
bounds on systemic events reported in Figure 5. Looking at that Figure, we concluded
that systemic risk increased steadily since August 2007, with spikes in the probability
of default of at least one dealer \(P_1\), but not as much for \(P_3\) and \(P_5\). These additional
figures show us where this result is coming from. Even in the scenario of highest
systematic risk, for most banks the contribution to systemic risk is relatively small.
Only few banks, among which Lehman and Merrill Lynch, were perceived to be at a
high risk of default. An increase in their idiosyncratic risk of default was accompanied
by a concurrent risk of correlated default with a few other institutions. Since these
spikes in default risk involved a few banks at a time, they did not directly result in a
large generalized increase in systemic risk across the financial network, especially where
many banks would be involved. At the same time, the increase in correlated risk for
some dealers also made the network more asymmetric. This explains the gains in terms
of width of the bounds when using all the information available.

These Figures also point out some limitations of the approach used in this paper
to measure systemic risk. Around the Bear Stearns episode (March 2009), for a few
days the bond/CDS basis of both Lehman and Merrill Lynch was extremely low. This
imposes a more stringent upper bound to the counterparty risk present in the CDS
contracts written on them. In turns, this implies that for a brief amount of time these
institutions appeared less correlated to the rest of the network. Most likely, this is due to
a lag in the incorporation of information in bond prices relative to CDS spreads, rather
than to a true decrease in counterparty risk. The method with which the bounds are
constructed is naturally affected by this problem, because it compares contemporaneous
bond prices and CDS spreads. However, as long as the information gets reflected in
bond prices relatively quickly, it should not affect the main conclusions of the analysis,
which looks at movements at lower frequency.

6 Extension and robustness tests

6.1 Assumptions on the recovery rate of CDSs

Since the bounds constructed in this paper rely on the pricing of counterparty risk
in Credit Default Swaps, it is important to explore the robustness of the results to
assumptions about the recovery rate of the CDSs in case of double default, $S$. As
discussed in Section 2, this variable depends on the severity of the default state, the
seniority of CDS claims relative to other claims against the defaulted counterparty, the
availability of collateral and the ability to seize it.

Remember that the pricing formula for CDSs is:

$$z_{ji} = [P(A_i) - (1 - S)P(A_i \cap A_j)] \frac{T\delta(0, T)(1 - R)}{\sum_{s=1}^{T} \delta(0, s - 1)}$$
A recovery rate of zero \((S = 0)\) means that counterparty risk has the highest impact on CDS spreads, while a recovery rate of 1 means that counterparty risk has no effect on the spreads.

To understand the effect of assumptions on \(S\) on the measure of systemic risk presented in this paper, remember that the upper bound on systemic risk is attained by the most correlated probability system that satisfies the constraints:

\[
P(A_i) \leq h_i(\gamma_i)
\]

\[
P(A_i) - (1 - S) \left( \frac{1}{N-1} \sum_{i \neq j} P(A_i \cap A_j) \right) = b_i
\]

where \(b_i = z_i \left[ \sum_{s=1}^{T} \delta(0,s-1) \right] \) is given\(^{26}\).

Intuitively, for given \(S\), one would obtain the most correlated probability system by setting \(P(A_i)\) as high as possible (up to the constraint \(h_i(\gamma_i)\)) for all banks, and then increasing \(\left( \frac{1}{N-1} \sum_{i \neq j} P(A_i \cap A_j) \right)\) to match CDS spreads. A higher recovery rate in case of double default means that to match the same CDS spread a higher probability of double default is needed. It would then follow that a higher \(S\), given the constraints, allows us to construct a more correlated probability system, and therefore a higher level of systemic risk. This intuitive reasoning, however, does not take into account the internal restrictions of consistency of the probability system. The next subsection explores them in the case of a symmetric network. I will then show how these mechanisms are amplified in the case of an asymmetric network.

### 6.1.1 The case of a symmetric network

For a symmetric network, using the notation of Section 3, call the marginal probabilities of default \(q_1\) and the pairwise probabilities of default \(q_2\). The previous constraints

\(^{26}\)I focus on the upper bound for the probability of at least \(r > 1\) events occurring. Following the analysis reported in section 3, the same argument holds for the lower bound for the probability that at least 1 institution defaults, since that is achieved for a very correlated system. It is easy to see why the results for the lower bound for \(r > 1\) and the upper bound for \(r = 1\) do not depend on \(S\): these bounds look for the least correlated system, which can always be obtained by setting the marginal default probabilities at the levels implied by the CDS spreads and attributing the bond/CDS basis entirely to liquidity. Also, in an asymmetric network the solutions to the problems for different \(r\) will not be the same, even though they all present a high degree of pairwise correlations.
become:

\[
q_1 \leq h \\
q_1 - (1 - S)q_2 = b
\]

where \( h \) is the (common) upper bound on the marginal probability of default and \( b \) is the (common) \( b_i \).

To maximize systemic risk, we would intuitively set

\[
q_1 = h
\]

and then

\[
q_2 = \frac{q_1 - b}{1 - S}
\]

so that the entire difference between bond yields and CDS spreads is explained by counterparty risk. For given \( q_1 \), \( q_2 \) is increasing in \( S \), and systemic risk with it. This captures the intuition that a higher recovery rate implies that higher counterparty risk is needed to explain the same bond/CDS basis.

This effect in fact is at play when \( S \) is small enough. As \( S \) grows, however, \( q_2 \) keeps increasing faster and faster, and at some point it will reach the level \( q_2 = q_1 = h \). At that point, the internal consistency of the probability system kicks in, which requires:

\[
q_2 \leq q_1
\]

What happens then as \( S \) increases further? The only way to satisfy the constraints is to lower \( q_1 \) below \( h \): for \( q_1 = h \) there exists no probability system able to satisfy both constraints. Then, \( q_2 \) will be equal to \( q_1 \) and:

\[
q_2 = q_1 = \frac{b}{S}
\]

which is decreasing in \( S \). As \( S \) reaches 1, the only way to obtain a consistent probability system is setting \( q_1 = b \), and therefore \( q_2 = b \).

This is the reason for why the bounds on systemic risk will first increase and then decrease with \( S \) in a symmetric system. The economic intuition behind this result is as
follows. If $S$ is low, it is possible to assume that the whole bond/CDS basis is explained by counterparty risk, rather than liquidity. Therefore, the marginal probabilities of default are at the upper bound (determined by bond prices), and counterparty risk is responsible for the observed wedge between bond-implied and CDS-implied default probabilities. At the same time, because CDSs are very risky (recovery in case of double default is low), even a small probability of joint default is enough to explain the basis: therefore, joint default probabilities cannot be very high.

As $S$ increases, it is at first possible to maintain the hypothesis that there is no liquidity premia in bond markets and the whole basis is explained by counterparty risk. However, because the recovery rate in case of double default is larger, to explain the same basis a higher probability of joint default is needed. Therefore, systemic risk increases with $S$.

However, for very high $S$, the joint default probability required to match the whole bond/CDS basis is impossibly high given the upper bounds on the marginal default probabilities. The only way to match both bond and CDS prices is to admit that some of the basis is explained by liquidity. Then, only the remaining part of the basis needs to be explained by counterparty risk. But this means that now marginal probabilities of default have to decrease, even at the upper bound. In turn, these lower the maximum amount of joint default probability and therefore the maximum probability of joint default is lower. In the limit, when $S = 1$, counterparty risk has no effect on the CDS spread. The only way to match both bond prices and CDS spreads is to assume that the whole basis is explained by liquidity, and set $q_1 = b$. In turn, this implies $q_2 = q_1 = b$ at the upper bound, which might be lower than the case of $S = 0$.

These results explain why the highest upper bound is in general achieved for a value of $S$ between 0 and 1, and why the results on systemic risk are robust to the choice of $S$.

### 6.1.2 Asymmetric networks

In an asymmetric network, restrictions to the probability system across banks with different marginal default probabilities are tighter, and therefore make the upper bound on systemic risk decline even faster with high $S$. To see why this is the case, we can
look at a case of two banks, 1 and 2. We have:

\[
P(A_1) \leq h_1
\]

\[
P(A_2) \leq h_2
\]

\[
P(A_1 \cap A_2) = \frac{1}{1 - S}[P(A_1) - b_1]
\]

\[
P(A_1 \cap A_2) = \frac{1}{1 - S}[P(A_2) - b_2]
\]

Suppose now \( S = 0 \). When the network is asymmetric, the upper bound on marginal probabilities might not be attained by all banks at the upper bound for systemic risk, even for this case of low recovery rate. To see this, suppose that \( h_2 < h_1 \) but \( b_1 = b_2 = 0 \). Then, at the maximum of counterparty risk, \( P(A_1 \cap A_2) = h_2 = P(A_2) \). But of course this means that \( P(A_1) < h_1 \). This explains why, in an asymmetric network, the upper limit on the marginal default probability might not be binding for all banks, even when \( S = 0 \). Low-risk banks constrain the marginal probabilities of more risky banks, through consistency restrictions. In fact, it is as if \( P(A_1) \) was bounded by \( h_2 \). It is easy to see how this effect can operate in larger networks, so that consistency restrictions can determine an even higher reduction of the bounds on systemic risk as \( S \) increases, compared to the symmetric case.

\section{Conclusion}

This paper shows that bond prices and CDS spreads represent a rich information set for learning about joint probabilities of default of intermediaries in the financial network. This information set can be used to construct bounds on the probability of systemic events, defined as the probability of several institutions defaulting within a certain time horizon. These bounds are the best possible ones given the available information, and they are obtained as the solution to a linear programming problem. Even without imposing assumptions on the liquidity premia in bond markets, the bounds obtained in this way are significantly tighter than the ones obtained from bond prices alone. The bounds also improve over bounds constructed using only aggregate information about bond and CDS spreads, but only when the structure of the network is sufficiently asymmetric. More stringent liquidity assumptions, which affect different institutions
in different ways, and therefore constrain the network to be more asymmetric, increase
the gain of using the full information set.

The paper shows how it is possible to decompose the changes in bond and CDS
spreads across the network into an idiosyncratic and a systematic component of default
risk. The method described here can also be used to to track the contribution to sys-
temic risk of the institutions in the financial network, as well as obtain a representation
of the network in the scenario of highest systemic risk.

Using the full information set available from observed prices, the bounds I construct
are able to show that some of the recent spikes in the prices of bonds and CDS of
financial institutions correspond to increases idiosyncratic default risk (that affected a
small number of banks) more than systemic risk. Besides, we can see that systemic
risk has been increasing gradually since August 2007. Finally, markets seemed to have
anticipated quite well the systemic nature of the exposure to risk of some banks (Lehman
and Merrill Lynch), as well as the links between them.

The approach used in this paper has several limitations. First, it is capturing market
perceptions about probabilities rather than the true probabilities. This means that
just as securities can be mispriced, the bounds can reflect various imperfections and
mispricing that happen in financial markets, including slow incorporation of information
and underestimation of risks. Besides, basing the measure on prices also implies that
the bounds presented here refer to risk-neutral probabilities, not true probabilities.27
Second, while the approach is built to minimize the number of assumptions about
the correlation structure of the network, especially regarding high-order joint risks,
some assumptions are still necessary to obtain the bounds. For example, extracting
risk-neutral marginal and pairwise default probabilities from prices requires imposing
a pricing model and taking a stand on liquidity in bond markets. Finally, since it is
difficult to distinguish the effect of counterparty risk on the bond/CDS basis from other
factors that may affect it (for example liquidity), it is impossible to rule out alternative
explanation of the basis, that attribute a less important role to counterparty risk.
These alternative explanations might have different implications for the measurement
of systemic risk.

However, to the extent that we believe counterparty risk plays a role in determining

27Note however that the risk-neutral probabilities for these events can be considered an upper bound
on the true probabilities and therefore still quite informative about the latter.
CDS spreads, the method presented in this paper can help understand the evolution of systemic risk in the financial markets during the recent crisis. Because of the possibility of constructing the bounds in real time, this measure could also be used in the future by agents and regulators to complement other measures in monitoring the market perceptions of systemic risk.
References


35. Huang, Jing-zhi, and Ming Huang, 2003, “How much of the corporate-Treasury yield spread is due to credit risk?”, Working paper, Penn State University.


Appendix

CDS pricing in discrete time

Start from the discretized pricing equation with constant hazard rate and risk-free rate:

\[ T \sum_{s=1}^{T} \delta(0, s-1)(1 - P(A_i \cup A_j))^{s-1}z_{ji} = \]

\[ = \left[ \sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1}(P(A_i) - (1 - S)P(A_i \cap A_j)) \delta(0, T)(1 - R) \right] \]

We can rewrite the equation as:

\[ z_{ij} \delta(0, T)(1 - R) \approx \frac{\sum_{s=1}^{T} (1 - P(A_i \cup A_j))^{s-1}(P(A_i) - (1 - S)P(A_i \cap A_j))}{\sum_{s=1}^{T} \delta(0, s-1)(1 - P(A_i \cup A_j))^{s-1}} \]

and then approximate the right hand side around \( P(A_i) = 0, P(A_j) = 0, P(A_i \cap A_j) = 0 \). The result is:

\[ \frac{z_{ij}}{\delta(0, T)(1 - R)} \approx \frac{T}{\sum_{s=1}^{T} \delta(0, s-1)} (P(A_j) - (1 - S)P(A_i \cap A_j)) \]

It is important to check how good is the approximation for a realistic range of parameters. For several different points in time (every 50 days) between 1/1/2007 and 3/31/2009, I compare the correct spread and the approximated spread, computed using the US yield curve at that time, considering:

- different values of \( P(A_j) \): between 0 and the maximum probability implied by bond data under no liquidity assumptions (\( \max_j \{h_j(0)\} \)).
- different values of \( P(A_i \cap A_j) \): between 0 and \( P(A_j) \)
- different values of \( R \) and \( S \): between 0.1 and 0.4.

The average error (approximated spread-true spread) in the monthly spread is 0.3 basis points, while the median error is 0.2bp. The standard deviation is 0.3bp. This indicates a tendency to
overestimate spreads by about 0.5% on average in the simulations, with a standard deviation of 0.4%. The maximum error in absolute value in the parameter range selected was 1% of the CDS spread (about 2bp). Therefore, the approximation is extremely good.

**Proof of Proposition 3**

Start from a symmetric LP problem

$$\max c'p$$

$$s.t. Ap \leq b$$

Suppose that $p^*$ is a solution to the problem. Given the definition of symmetry presented in the text, it is clear that $p^*_J$ is also a solution to the problem: $c'p^* = c'p^*_J$ and similarly hold for every row of the constraints, for every $J$.

Now, construct $p^{**}$ as follows:

$$p^{**} = \frac{1}{2N} \sum J p^*_J$$

where the first $J$ correspond to no permutation, and $J$ cycles across all permutations of indices $A_1, ..., A_N$.

Note that it is also possible to construct $p^{**}$ in the following way, considering the binary representation introduced in Proposition 2. Every $b_i$ vector has $O_i$ ones and $N - O_i$ zeros. Call $H_i$ the set of all vectors of size $N$ that have $O_i$ ones and $N - O_i$ zeroes in different positions. Call $b_{ih}$ the vector corresponding to element $h$ from $H_i$. Then, for every $i$, construct $p^{**}$ as:

$$p^{**}_i = \left( \frac{O_i}{N} \right)^{-1} \sum_{h \in H_i} b_{ih}$$

From the first construction, it is clear why $p^{**}$ is a solution to the maximization problem, being just an average of solutions. Plus, $p^{**}$ is symmetric, which proves the statement of the Proposition.

An example with $N = 3$. As explained in Proposition 2, we can construct the probability system $p^*$ as follows:

$$p^*_1 = Pr\{\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3\}$$
Suppose \( p^* \) solves the maximization problem, and construct \( p^{**} \) as:

\[
\begin{align*}
\ p^{**}_1 &= p^*_1 \\
\ p^{**}_2 &= p^{**}_3 = p^{**}_5 = \frac{p^*_2 + p^*_3 + p^*_5}{3} \\
\ p^{**}_4 &= p^{**}_6 = p^{**}_7 = \frac{p^*_4 + p^*_6 + p^*_7}{3} \\
\ p^{**}_8 &= p^*_8
\end{align*}
\]

\( p^{**} \) solves the maximization problem and is symmetric.

**Proof of Proposition 4**

Given Proposition 3, the following theorem holds, which can be derived as a special case of the bounds presented in Boros and Prekopa (1989), in section 8.

Here, \( S_1 = Nq_1 \) and \( S_2 = \frac{N(N-1)}{2}q_2 \). The bounds are nonlinear functions of \( S_1 \) and \( S_2 \), and take the following form, for \( 3 \leq r \leq N - 1 \).

**General lower bound**

Call

\[
\begin{align*}
\ A_1 &= (r - 1)N - (r + N - 2)S_1 + 2S_2 \\
\ A_2 &= -(r - 2)S_1 + 2S_2
\end{align*}
\]
• If $A_2 \leq 0$

\[ p_r \geq 0 \]

• If $A_1 \geq 0$ and $A_2 \geq 0$

\[ p_r \geq \frac{-(r-2)S_1 + 2S_2}{(N+1-r)N} \]

• If $A_1 < 0$

\[ p_r \geq \frac{(r-1)(r-2i-2)+2iS_1-2S_2}{(i-r+2)(i-r+1)} \]

with

\[ i = \left\lfloor \frac{2S_2-(r-2)S_1}{S_1-(r-1)} \right\rfloor \]

where $[x]$ is the largest integer smaller than $x$.

Note that in general either $A_1 < 0$ or $A_2 < 0$ in the region that satisfies the necessary consistency relations between $S_1$ and $S_2$.

Call

\[ B_1 = rN - (r+N-1)S_1 + 2S_2 \]
\[ B_2 = -(r-1)S_1 + 2S_2 \]

• If $B_1 < 0$

\[ p_r \leq 1 \]

• If $B_1 \geq 0$ and $B_2 \geq 0$ then

\[ p_r \leq \frac{(r+N-1)S_1 - 2S_2}{rN} \]

• If $B_2 < 0$

\[ p_r \leq \frac{i(i+1)-2iS_1+2S_2}{(r-i-1)(r-i)} \]

where

\[ i = \left\lfloor \frac{(r-1)S_1 - 2S_2}{r-S_1} \right\rfloor \]
Lower bound in the symmetric case

Call
\[ A_1 = (r - 1)N - (r + N - 2)Nq_1 + N(N - 1)q_2 \]
\[ A_2 = -(r - 2)Nq_1 + N(N - 1)q_2 \]
and remember that \( q_2 \leq q_1 \). Then,
\[ A_2 \leq 0 \iff q_2 \leq \frac{(r - 2)}{(N - 1)} q_1 \]
\[ A_1 \leq 0 \iff q_2 \leq \frac{(r + N - 2)q_1 - r + 1}{N - 1} = \frac{(r - 2)q_1}{N - 1} + \frac{N_q_1 - r + 1}{N - 1} \]

Remember the following relations between \( S_1 \) and \( S_2 \) (or respectively \( q_1 \) and \( q_2 \)):
\[ q_1 \geq q_2 \]

and (Bonferroni Inequality):
\[ S_1 - S_2 \leq 1 \]
so that:
\[ \frac{N(N - 1)}{2} q_2 \geq Nq_1 - 1 \]
or:
\[ \frac{2}{N - 1} \left( q_1 - \frac{1}{N} \right) \leq q_2 \leq q_1 \]

In addition, note that the Bonferroni Inequality is not sharp, so that not all \( q_2 \) that satisfy that condition are actually compatible with the existence of an underlying probability system. The correct lower bound for \( q_2 \) given \( q_1 \) is given by the requirement that the lower bound for \( r = 1 \) (the probability of the union of all events) is less or equal to one. Call this number \( \underline{q_2} \).

The bounds then can be written as follows.

CASE 1: If \( q_1 < \frac{r - 1}{N} \)

- If \( \underline{q_2} \leq q_2 \leq \frac{(r - 2)}{(N - 1)} q_1 \)
  \[ p_r \geq 0 \]
- If \( \frac{(r - 2)}{(N - 1)} q_1 \leq q_2 \leq q_1 \)
  \[ p_r \geq \frac{-(r - 2)Nq_1 + N(N - 1)q_2}{(N + 1 - r)N} \]
CASE 2: If \( q_1 > \frac{r-1}{N} \)

- If \( q_2 < q_2 \leq \frac{(r-2)q_1}{N-1} + \frac{Nq_1-r+1}{N-1} \)

\[
i = \left\lfloor \frac{N(N-1)q_2 - (r-2)Nq_1}{Nq_1 - (r-1)} \right\rfloor
\]

\[
p_r \geq \frac{(r-1)(r-2i-2) + 2iNq_1 - N(N-1)q_2}{(i-r+2)(i-r+1)}
\]

- If \( q_2 \geq \frac{(r-2)}{(N-1)}q_1 + \frac{Nq_1-r+1}{N-1} \)

\[
p_r \geq \frac{-(r-2)Nq_1 + N(N-1)q_2}{(N+1-r)N}
\]

Note that in this case, to the left of \( \frac{(r-2)}{(N-1)}q_1 \) we would have \( A_2 \leq 0 \). However, this number has to be less than \( q_2 \): otherwise, there would be a discontinuity in the solution to the minimization problem, since the bound is decreasing to the right of \( \frac{(r-2)}{(N-1)}q_1 \) and positive, and would be 0 to the left. But we know that the bound is a continuous function of the constraints \( q_1 \) and \( q_2 \).

**Upper bound in the symmetric case**

Call

\[
B_1 = rN - (r + N - 1)Nq_1 + N(N-1)q_2
\]

\[
B_2 = -(r - 1)Nq_1 + N(N-2)q_2
\]

The key points are whether

\[
\frac{(r + N - 1)}{N-1}q_1 - \frac{r}{N-1} > \frac{(r-1)}{(N-2)}q_1
\]

or

\[
\frac{(N - 2)(r + N - 1) - (r - 1)(N - 1)}{(N-2)}q_1 > r
\]

\[
q_1 > \frac{N - 2}{N(N-2) - r + 1}r
\]
Note that in this case we have the requirement that $q_2$ is not so low that the probability of the intersection given the bounds comes out less than 0. I.e. we need that $q_2$ is at least $\hat{q}_2$ where $\hat{q}_2$ is such that the upper bound with $r = N$ is greater than 0.

Then we have:

CASE 1: If $q_1 > \frac{N-2}{N(N-2)-r+1} r$

- If $\hat{q}_2 \leq q_2 < \frac{(r+N-1)}{N-1}q_1 - \frac{r}{N-1}$, $p_r \leq 1$

- If $q_2 \geq \frac{(r+N-1)}{N-1}q_1 - \frac{r}{N-1}$

          $p_r \leq \frac{(r+N-1)Nq_1 - N(N-1)q_2}{rN}$

CASE 2: If $q_1 < \frac{N-2}{N(N-2)-r+1} r$

- If $\hat{q}_2 \leq q_2 < \frac{(r-1)}{(N-2)} q_1$

          $p_r \leq \frac{i(i + 1) - 2iNq_1 + N(N-1)q_2}{(r - i - 1)(r - i)}$

          where

          $i = \left[ \frac{(r-1)Nq_1 - 2N(N-1)q_2}{r - \hat{N}q_1} \right]$    

- If $q_2 \geq \frac{(r-1)}{(N-2)} q_1$

          $p_r \leq \frac{(r+N-1)Nq_1 - N(N-1)q_2}{rN}$
## Tables

### Table 1

<table>
<thead>
<tr>
<th>Bank of America</th>
<th>Citigroup</th>
<th>Goldman Sachs</th>
<th>Lehman Brothers</th>
<th>JP Morgan</th>
<th>Merrill Lynch</th>
<th>Morgan Stanley</th>
<th>Wachovia</th>
<th>Abn Amro</th>
<th>Bnp Paribas</th>
<th>Barclays</th>
<th>Credit Suisse</th>
<th>Deutsche Bank</th>
<th>HSBC</th>
<th>UBS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg CDS spread</td>
<td>0.0043</td>
<td>0.0058</td>
<td>0.0069</td>
<td>0.0067</td>
<td>0.0050</td>
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<td>0.0058</td>
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<tr>
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<td>0.0007</td>
<td>0.0019</td>
<td>0.0019</td>
<td>0.0011</td>
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<td>0.0010</td>
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<td>0.0008</td>
<td>0.0010</td>
<td>0.0005</td>
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<tr>
<td>Max spread</td>
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<td>Std basis</td>
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<td>Max basis</td>
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</table>

Note: table reports descriptive characteristics on the CDS spread and the yield spread for the 15 institutions in the sample. The yield spread is computed as the linearly interpolated yield for a 5-year maturity bond in excess of the corresponding Treasury rate.

### Table 2

<table>
<thead>
<tr>
<th>Bank of America</th>
<th>Citigroup</th>
<th>Goldman Sachs</th>
<th>Lehman Brothers</th>
<th>JP Morgan</th>
<th>Merrill Lynch</th>
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<th>Barclays</th>
<th>Credit Suisse</th>
<th>Deutsche Bank</th>
<th>HSBC</th>
<th>UBS</th>
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</thead>
<tbody>
<tr>
<td>Avg valid bonds</td>
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<td>16.9</td>
<td>13.7</td>
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Note: first column reports average number of bonds for each institution that are used for the estimation of marginal default probabilities. Columns 2-7 break this number down by year.
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Figure 8: Part of the network in the high systemic risk scenario as of 08/19/2008
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