Common Dynamics in Volatility: a Composite vMEM Approach

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Abstract

Measures of financial volatility exhibit clustering and persistence and can be jointly modeled as the element by element product of a vector of conditionally autoregressive scale factors and a multivariate i.i.d. innovation process (vector Multiplicative Error Model – vMEM). Since similar profiles are shared across measures, a restricted vMEM decomposes the conditional expected volatility into the sum of a common (persistent) component and a vector of measure specific components. With data on absolute returns, realized kernel volatility and daily range for five stock market indices, we show that indeed such a common component exists with the desired properties. The transitory components happen to have different features across indices.

Keywords: Volatility, (vector) Multiplicative Error Models, Long/Short Run Decomposition, GARCH, GMM.

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1 Introduction

In financial time series analysis a centerpiece is dedicated to volatility measurement and modeling/forecasting. The Multiplicative Error Model (MEM), introduced by Engle (2002) as a generalization of the ACD model (Engle and Russell (1998)), moves from the consideration that measures of volatility and other proxies of financial activity (e.g. durations, volumes, number of shares) share similar features: they are all non–negative and show persistence and clustering patterns like the ones shown by absolute or squared returns. As such, the the dynamic structure of an MEM mirrors the GARCH specification of the conditional variance of returns: the conditional expectation of the variable of interest is modeled as a linear combination of the lagged value(s) of the observed variable and of past conditional expectation(s); the variable itself is modeled as the product of such a conditional expectation and an i.i.d. innovation process with unit mean. In other words, a MEM for squared returns is a GARCH for zero mean returns. Early applications of such an approach are the CARR model for the daily range by Chou (2005), and a model for duration, volume and volatility (Manganelli (2005)).

The dynamic interdependence among several indicators of volatility (absolute returns, daily range and realized volatility) was performed in Engle and Gallo (2006) as a stack of univariate MEMs. Extending this analysis, the idea that a gain in estimation efficiency and a more reliable interpretation of the significance of the links can be obtained with a vector MEM (or vMEM) approach is pursued by Cipollini et al. (2007) and Cipollini et al. (2009). Formally, such an extension consists of jointly modeling two or more non–negative processes as the element by element product of a vector of conditionally autoregressive scale factors and a multivariate i.i.d. innovation process with a unit vector mean and a full covariance matrix.

From the extensive literature in the field of realized volatility, several measures were derived as consistent estimators of the integrated volatility in an underlying continuous time process: from the plain vanilla (Andersen and Bollerslev (1998), extensively studied by Andersen et al. (2001) and by Andersen et al. (2003)), to the refinements suggested by, to quote a few, Barndorff-Nielsen and Shephard (2002), Barndorff-Nielsen and Shephard (2004) (bipower realized variance), by Aït-Sahalia et al. (2005) (two-scale realized variance), by Bandi and Russel (2008) (to take microstructure noise into account), and by Barndorff-Nielsen et al. (2008) (realized kernels). Next to these, the range (Parkinson (1980)) has gained attention in its simplicity and satisfactory performance in forecasting (Brownlees and Gallo (2010)).

This paper addresses the need to insert an explicit latent factor which drives all indicators of volatility in a joint framework, a restriction a traditional vMEM would not impose (as noted by Hansen et al. (2011)). We motivate our new model, labeled Composite vMEM, starting from the matching patterns exhibited by several indicators of volatility, translating the idea of similar high persistence or long–term evolution in each series into a common component. Relative to the vMEM, the essential feature of the new model is the structure of the conditional mean, that is decomposed into the sum of a common (persistent) component plus a vector of idiosyncratic individual components. The main advantage is an enhanced interpretation of the dynamics: scenario analysis can be built on different assumptions relative to the evolution of the two components; relatively parsimonious formulations can capture fairly rich dynamic patterns, retaining, as with any vMEM, the ability of multistep forecasts; moreover, the common component can possibly be made dependent on lagged predetermined variables, such as credit spreads or some systemic measure of risk (cf. Brownlees and Engle (2011)).

The model, whose reduced form we trace to a more complex vMEM formulation with constrained
parameters, has connections with the Composite–MEM considered in Brownlees et al. (2011), whose characteristics have proved to adequately capture the dynamics of realized volatilities of single assets. The seminal idea, however, should be traced back to the univariate GARCH model proposed in Engle and Lee (1999), in which the dynamics of the conditional variance is decomposed into two additive components, one labeled as permanent in view of its higher persistence, and the other one as transitory.

Given the semiparametric specification of the model, inferences are obtained via GMM (Generalized Method of Moments) based on the conditional mean and variance expressions following from the model definition. Such expressions are very similar to the ones detailed in the general vMEM in Cipollini et al. (2009), so that the estimation procedures are substantially the same.

We estimate our model on three indicators of volatility (absolute returns, realized volatility and daily range) extracting the common persistent component from data on some stock indices. The results show that the common dynamics is well supported by the data and captures a high degree of the total behavior of the series; the transitory components are more erratic in pattern across indicators; finally, the degree of accuracy in estimating conditional expectations is the highest for realized volatility, followed by the daily range and by absolute returns.

The paper is structured as follows. In Section 2 we present the basic vMEM to establish notation and the backdrop against which we introduce the restricted dynamics. In Section 3 we introduce the Composite vMEM, discussing its properties and its relationship with the general vMEM. In Section 4 we discuss how to make inferences on the parameters of the model via GMM. In Section 5 we present the application of the model. Section 6 contains some concluding remarks.

2 The Vector MEM

Let \( \{x_t\} \) a discrete time process with components defined on \([0, +\infty)^K\). In the vMEM (vector Multiplicative Error Model) (Cipollini et al. (2007), Cipollini et al. (2009)), \( x_t \) is structured as\(^1\)

\[
x_t = \mu_t \odot \varepsilon_t
\]

where, conditionally on the information \( \mathcal{F}_{t-1} \), \( \mu_t \) is deterministic,

\[
\mu_t = \mu(\theta, \mathcal{F}_{t-1})
\]

and \( \varepsilon_t \) is stochastic, with pdf (probability density function) defined over a \([0, +\infty)^K\) support and such that

\[
\varepsilon_t | \mathcal{F}_{t-1} \sim D^+(1, \Sigma).
\]

Note that the previous assumptions on \( \mu_t \) and \( \varepsilon_t \) give

\[
E(x_t | \mathcal{F}_{t-1}) = \mu_t
\]

\[
V(x_t | \mathcal{F}_{t-1}) = \mu_t \mu_t' \odot \Sigma = \text{diag}(\mu_t) \Sigma \text{diag}(\mu_t),
\]

where the latter is a positive definite matrix by construction.

\(^1\)We adopt the following conventions: if \( x \) is a vector or a matrix and \( a \) is a scalar, then the expressions \( x \geq 0 \) and \( x^a \) are meant element by element; if \( x_1, \ldots, x_K \) are \((m, n)\) matrices then \( (x_1; \ldots; x_K) \) means the \((mK, n)\) matrix obtained stacking the matrices \( x_i \)'s columnwise.
More practically, $\mu_t$ can be specified as
\[\mu_t = \omega + \beta_1 \mu_{t-1} + \alpha_1 x_{t-1} + \gamma_1 x_{t-1}^{(-)},\]  
where $\omega$ is $(K, 1)$, $\alpha_1$, $\gamma_1$ and $\beta_1$ are $(K, K)$ (further lags can be added, but we do not consider them here). The term $\beta_1 \mu_{t-1}$ represents an inertial component, whereas $\alpha_1 x_{t-1} + \gamma_1 x_{t-1}^{(-)}$ stands for the contribution of the more recent observation. In particular, the vector $x_{t}^{(-)}$ aims at capturing asymmetric effects associated with the sign of an observed variable and is usually structured as $x_{t,j}^{(-)} = x_{t,j} I_{t,j}^{(-)}$, where $I_{t,j}^{(-)}$ denotes the indicator of a negative value of the signed variable. For instance, when different volatility indicators of the same asset are considered, then $I_{t,j}^{(-)} = I(r_{t} < 0)$, where $r_{t}$ indicates the asset return. As a further example, in a market volatility spillover framework each market has its own indicator, so that $I_{t,j}^{(-)} = I(r_{t,j} < 0)$, where $r_{t,j}$ is the return of the $j$-th market. In what follows, we assume that all components of $I_{t,j}^{(-)}$ have conditional median zero and are conditionally uncorrelated with $x_{t}$.

An useful reinterpretation of the model is had when $x_{t}$ is assumed mean-stationary; in fact,
\[\omega = \left[ I_K - \left( \beta_1 + \alpha_1 + \frac{\gamma_1}{2} \right) \right] \mu,\]
where $\mu = E(\mu_t) = E(x_t)$. This allows to rewrite (4) as
\[\mu_t = \mu + \xi_t \]  
(5)
\[\xi_t = \beta_1 \xi_{t-1} + \alpha_1 x_{t-1}^{(\xi)} + \gamma_1 x_{t-1}^{(\xi)} \]  
(6)
\[x_t^{(\xi)} = x_t - \mu \quad x_t^{(-)} = x_t^{(-)} - \mu/2. \]  
(7)
From a practical point of view, the representation (5)-(6)-(7) constitutes a trivial reparameterization of (4), with $\mu$ replacing $\omega$. However, it has the merit of representing the dynamics of the process being driven by a zero mean, stationary component, $\xi_t$, that moves around the unconditional average level $\mu$. Depending on the context, further meaningful components, similar to $\xi_t$, can be added and/or a time–varying rather than a fixed level $\mu$, can be taken into account. An example is provided in Section 3.

A further useful representation of (4) is based on the innovations
\[v_t = x_t - \mu_t \quad v_t^{(-)} = x_t^{(-)} - \mu_t/2, \]  
(8)
that allow to rewrite (6) as
\[\xi_t = \beta_1^* \xi_{t-1} + \alpha_1 v_{t-1} + \gamma_1 v_{t-1}^{(-)} \]  
(9)
where $\beta_1^* = \beta_1 + \alpha_1 + \gamma_1/2$. The essential difference between (5)-(6)-(7) and (5)-(9)-(8) lies in the different properties of the driving factors of $\mu_t$, namely $x_t^{(\xi)}$, $x_t^{(\xi)}$ and $v_t$, $v_t^{(-)}$: $E\left(x_t^{(\xi)}\right) = E\left(x_t^{(\xi)}\right) = 0$ for the former as opposed to $E\left(v_t|\mathcal{F}_{t-1}\right) = E\left(v_t^{(-)}|\mathcal{F}_{t-1}\right) = 0$ for the latter.

3 A Composite Vector MEM

3.1 Model Specification

Non-negative financial time series tend to show frequently very similar patterns over the sample of observation, conveying the idea of a single underlying driving force. In vMEM terms, this can be
formalized as a common dynamical component. Consequently, we propose a new formulation by changing the structure of $\mu_t$ into (5)-(6)-(7) as follows:

- the fixed $\mu$ is replaced by a time-varying component driven by a scalar common factor, $\eta_t$;
- $\xi_t$ is (ideally) structured as a vector of specific elements, i.e. each $\xi_{t,j}$ depends from its own past values only.

The implicit assumption is that the common component is able to capture adequately the main part of the cross-dependence.

More explicitly, $\mu_t$ is defined by

$$
\mu_t = \mu + \psi \eta_t + \xi_t \tag{10}
$$

$$
\xi_t = \beta_1^{(\xi)} \xi_{t-1} + \alpha_1^{(\xi)} x_{t-1} + \gamma_1^{(\xi)} x_{t-1}^{(-)}
$$

$$
\eta_t = \beta_1^{(\eta)} \eta_{t-1} + \alpha_1^{(\eta)} x_{t-1} + \gamma_1^{(\eta)} x_{t-1}^{(-)}
$$

where

$$
x_t^{(\xi)} = x_t - (\mu + \psi \eta_t)
$$

$$
x_t^{(-)} = x_t^{(-)} - (\mu + \psi \eta_t) / 2
$$

$$
x_t^{(\eta)} = x_t - (\mu + \xi_t)
$$

$$
x_t^{(-)} = x_t^{(-)} - (\mu + \xi_t) / 2
$$

$$
\eta_0 = \xi_{0,j} = x_{0,j}^{(\xi)} = x_{0,j} = x_{0,j}^{(\xi)} = 0 \quad j = 1, \ldots, K.
$$

In the base formulation, $\beta_1^{(\xi)}$, $\alpha_1^{(\xi)}$ and $\gamma_1^{(\xi)}$ are diagonal matrices, but richer interdependency structures can be considered in some applications, in particular for what concerns $\alpha_1^{(\xi)}$.

As in Section 2, an equivalent formulation of the model can be obtained by resorting to the innovations (8), instead of (11)-(12), as driving forces of the dynamics of $\mu_t$, namely

$$
\xi_t = \beta_1^{(\xi)*} \xi_{t-1} + \alpha_1^{(\xi)} v_{t-1} + \gamma_1^{(\xi)} v_{t-1}^{(-)} \tag{13}
$$

$$
\eta_t = \beta_1^{(\eta)*} \eta_{t-1} + \alpha_1^{(\eta)} v_{t-1} + \gamma_1^{(\eta)} v_{t-1}^{(-)} \tag{14}
$$

where

$$
\beta_1^{(\xi)*} = \beta_1^{(\xi)} + \alpha_1^{(\xi)} + \gamma_1^{(\xi)}/2
$$

$$
\beta_1^{(\eta)*} = \beta_1^{(\eta)} + \left(\alpha_1^{(\eta)} + \gamma_1^{(\eta)}/2\right) \psi.
$$

In such a case, the contribution led from the more recent observation, $v_{t-1}$, is the same in the two equations, so that the corresponding coefficients can be directly compared. Moreover such a representation allows a simpler derivation of the properties of the model.

A further interesting interpretation of the common component formulation can be retrieved by fusing the two addends $\mu$ and $\psi \eta_t$ into a unique time varying level $\chi_t = \mu + \psi \eta_t$. In such a case

$$
\mu_t = \chi_t + \xi_t, \tag{15}
$$

where $\chi_t$ evolves as

$$
\chi_t = \omega^{(x)} + \beta_1^{(x)*} \chi_t + \alpha_1^{(x)} v_{t-1} + \gamma_1^{(x)} v_{t-1}^{(-)} \tag{16}
$$

with $\chi_0 = \mu$. In practice, (16) looks alike the equation for $\xi_t$, but with a number of differences: the level $\omega^{(x)} = (1 - \beta_1^{(x)*})$ is non-zero; the coefficient of the inertial component, $\beta_1^{(x)*} = \beta_1^{(x)*}$, is a scalar; the coefficients of the innovations and of the ‘asymmetric’ innovations, $\alpha_1^{(x)} = \psi \alpha_1^{(\eta)}$ and $\gamma_1^{(x)} = \psi \gamma_1^{(\eta)}$ respectively, are ($K$, $K$) matrices but with unit rank.
3.2 A Reduced Form Representation

A better understanding of the specification introduced can be gained by deriving its reduced form, according to Lütkepohl (2005, Section 11.6). By considering a general polynomial formulation of the model, the stacked vector of the two dynamical components, \( \xi_t^+ = (\eta_t; \xi_t) \), evolves according to

\[
[I_{K+1} - \beta^+(L)] \xi_t^+ = \alpha^+(L) v_t + \gamma^+(L) v_t^(-)
\]

where

\[
\beta^+(L) = \text{diag} (\beta^{(n)}(L), \beta^{(\xi)}(L)) \quad \alpha^+(L) = (\alpha^{(n)}(L); \alpha^{(\xi)}(L)) \quad \gamma^+(L) = (\gamma^{(n)}(L); \gamma^{(\xi)}(L)).
\]

The conditional mean is then a reduced form of the process \( \xi_t^+ \), obtained as

\[
\mu_t - \mu = F \xi_t^+
\]

with \( F = (\psi, I_K) \). Hence,

\[
|I_{K+1} - \beta^+(L)| (\mu_t - \mu) = F |I_{K+1} - \beta^+(L)|^{adj} \left[ \alpha^+(L) v_t + \gamma^+(L) v_t^(-) \right]
\]

or

\[
(1 - \beta^{(n)}(L)) |I_K - \beta^{(\xi)}(L)| \mu_t = (1 - \beta^{(n)}(1)) |I_K - \beta^{(\xi)}(1)| \mu + |I_K - \beta^{(\xi)}(L)| \psi \left[ \alpha^{(n)}(L') v_t + \gamma^{(n)}(L') v_t^(-) \right] + (1 - \beta^{(n)}(L)) |I_K - \beta^{(\xi)}(L)|^{adj} \left[ \alpha^{(\xi)}(L) v_t + \gamma^{(\xi)}(L) v_t^(-) \right]
\]  

(17)

This representation has a number of implications.

The constant of the reduced form, namely

\[
(1 - \beta^{(n)}(1)) |I_K - \beta^{(\xi)}(1)| \mu
\]

depends essentially on \( \mu \). Note that the inclusion of a constant term (say \( \omega^{(n)} \)) into the dynamics of \( \eta_t \) would cause an overparameterization, because of the further addend \( |I_K - \beta^{(\xi)}(1)| \psi \omega^{(n)} \) into (18).

The \( \psi \) parameter acts in the model as a rescaling factor of the contribution of the common component \( \eta_t \) when entering into the conditional mean \( \mu_t \). In the reduced form, it simply multiplies the \( \alpha^{(n)} \) and \( \gamma^{(n)} \) coefficients. This implies that a reciprocal rescaling of \( \psi \) and all such coefficients (namely, \( \psi / k, k\alpha^{(n)}_l \) and \( k\gamma^{(n)}_l \) \( \forall l \), where \( k > 0 \)) leaves unchanged their impact on \( \mu_t \). We suggest to normalize \( \psi \) as

\[
\psi' 1 = K.
\]

(19)

If all elements of \( \psi \) are non-negative, equation (19) implies that \( \eta_t \) is scaled to an average level; as a comparison, \( \psi' 1 = 1 \) would scale the common component to an aggregated level. A \( \psi_j > 1 \ (< 1) \) implies that the contribution of \( \eta_t \) is amplified (dumped) when plugged into \( \mu_{t,j} \).

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2In what follows we will adopt the following conventions: if \( x_1, \ldots, x_K \) are matrices with the same number of columns (rows), then \( (x_1; \ldots; x_K) ((x_1, \ldots, x_K)) \) indicates the matrix obtained stacking the matrices \( x_i \)’s columnwise (rowwise).
The diagonal structure of the matrices appearing into the $\beta^{(\xi)}(L)$ polynomial leads to simplifications of both the determinant and the adjoint matrix of $I_K - \beta^{(\xi)}(L)$. In particular, staking the diagonal elements of $\beta^{(\xi)}(L)$ into the vector $\beta^{(\xi,v)}(L) = \left( \beta^{(\xi)}_{1,1}(L); \ldots; \beta^{(\xi)}_{K,K}(L) \right)$, after some algebra we obtain

\[
(1 - \beta^{(\eta)}(L)) \left( \mathbb{1} - \beta^{(\xi,v)}(L) \right) \odot \mu_t = (1 - \beta^{(\eta)}(1)) \left( \mathbb{1} - \beta^{(\xi,v)}(1) \right) \odot \mu \\
+ \left[ \left( \mathbb{1} - \beta^{(\xi,v)}(L) \right) \odot \psi \alpha^{(\eta)}(L)' + (1 - \beta^{(\eta)}(L)) \alpha^{(\xi)}(L) \right] v_t \\
+ \left[ \left( \mathbb{1} - \beta^{(\xi,v)}(L) \right) \odot \psi \gamma^{(\eta)}(L)' + (1 - \beta^{(\eta)}(L)) \gamma^{(\xi)}(L) \right] v_t^{(-)} ,
\]

that in the 1-lag formulation (namely, 1 one lag for the three components $\mu_t, v_t$ and $v_t^{(-)}$) simplifies as

\[
(1 - \beta^{(\eta)}_1 L) \left( \mathbb{1} - \beta^{(\xi,v)}_1 L \right) \odot \mu_t = \left( 1 - \beta^{(\eta)}_1 \right) \left( \mathbb{1} - \beta^{(\xi,v)}_1 \right) \odot \mu \\
+ \left[ \left( \psi \alpha^{(\eta)}_1 + \alpha^{(\xi)}_1 \right) - \left( \beta^{(\xi,v)}_1 \odot \psi \alpha^{(\eta)}_1 + \beta^{(\eta)}_1 \alpha^{(\xi)}_1 \right) L \right] L v_t \\
+ \left[ \left( \psi \gamma^{(\eta)}_1 + \gamma^{(\xi)}_1 \right) - \left( \beta^{(\xi,v)}_1 \odot \psi \gamma^{(\eta)}_1 + \beta^{(\eta)}_1 \gamma^{(\xi)}_1 \right) L \right] L v_t^{(-)} ,
\]

where $\beta^{(\xi,v)}_1 = \left( \beta^{(\xi)}_{1,1}; \ldots; \beta^{(\xi)}_{K,K} \right)$. Equation (20) indicates that the 1-lag formulation of the Composite vMEM with a diagonal $\beta^{(\xi)}_1$ matrix has a reduced form expression similar to a 2-lag vMEM with restrictions on the parameters.

### 3.3 Related Models

The Composite vMEM introduced above has similarities with other univariate models. In fact, for $K = 1$ (15), (13) and (16) become

\[
\begin{align*}
\mu_t & = \chi_t + \xi_t \\
\xi_t & = \beta^{(\xi)}_1 \xi_{t-1} + \alpha^{(\xi)}_1 v_{t-1} + \gamma^{(\xi)}_1 v_{t-1}^{(-)} \\
\chi_t & = \omega^{(\chi)} + \beta^{(\chi)}_1 \chi_{t-1} + \alpha^{(\chi)}_1 v_{t-1} + \gamma^{(\chi)}_1 v_{t-1}^{(-)},
\end{align*}
\]

whose structure is identical to the $\mu_t$ formulation of the Composite-MEM introduced in Brownlees et al. (2011). We can thus interpret the Composite vMEM as a vector extension of such a model. We remark that the univariate case requires the identification condition $\beta^{(\chi)}_1 = \beta^{(\eta)}_1 > 0$, since otherwise the corresponding $\chi$ and $\eta$ parameters could be exchanged without modifying the reduced form representation. This allows to interpret $\chi_t$ and $\xi_t$ as permanent and transitory components, respectively, according to Engle and Lee (1999). This constraint is however not needed for $K > 1$: the parameter exchange mentioned before is not possible since the same scalar component $\eta_t$ (although rescaled by $\psi$ and the shifted by $\mu$) drives the level of all components of $x_t$.

By the same token, as discussed in Brownlees et al. (2011) the Composite vMEM and the Composite-MEM have close similarities with the component GARCH of Engle and Lee (1999). In that paper, the conditional variance of $r_t$ given $\mathcal{F}_{t-1}$ evolves according to

\[
\begin{align*}
h_t & = \phi (r_{t-1}^2 - q_{t-1}^2) + \beta (h_{t-1} - q_{t-1}) \\
q_t & = \omega + \rho q_{t-1}^2 + \phi (r_{t-1}^2 - h_{t-1}) + \delta (r_{t-1}^2 - h_{t-1}/2)
\end{align*}
\]
where \( q_0 = h_0 \) and \( r_t^{2(-)} = r_t^2 I(r_t < 0) \). In practice, within this model the time-varying component \( q_t \) replaces the usual fixed unconditional variance of a weakly stationary GARCH. With \( E(r_t|\mathcal{F}_{t-1}) = 0 \) and \( x_t = r_t^2 \) the parallel with a Composite MEM is established.

4 Model Inference

In this section we describe how to obtain inferences on the Composite vMEM by means of Generalized Method of Moments (GMM). The formulation of the model preserves the mean–variance relations (2) and (3) as functions of \( \mu_t \) and \( \Sigma \). Such relations are typical of the vMEM and, by consequence, inferences can be derived without the need to specify the conditional distribution of the multiplicative error term as detailed in Cipollini et al. (2009). In what follows we summarize the main elements of the approach. The interested reader can be referred to the cited work for more details on GMM estimation of vMEMs and to Newey and McFadden (1994) and Wooldridge (1994) for the general aspects of GMM inference.

The model is ruled by two parameters: \( \theta \), which controls the dynamics of \( \mu_t \) and is the parameter of main interest; \( \Sigma \), the variance matrix of the error term, which represents a nuisance parameter.\(^3\)

Let us define

\[
 u_t = x_t \odot \mu_t - 1,
\]

where \( \odot \) indicates the element–by–element division and we suppressed dependency of \( u_t \) on the parameter \( \theta \), on the information \( \mathcal{F}_{t-1} \) and on the current value of the dependent variable \( x_t \).

(2) and (3) imply that \( u_t \) is a conditionally homoskedastic martingale difference, that is

\[
 E(u_t|\mathcal{F}_{t-1}) = 0 \quad (24)
\]

\[
 V(u_t|\mathcal{F}_{t-1}) = \Sigma. \quad (25)
\]

According to Wooldridge (1994, sect. 7), conditional moment restrictions like (24) can be used as a key ingredient for estimation. Any \((M, K)\) matrix \( G_t \) depending deterministically on the information \( \mathcal{F}_{t-1} \) gives

\[
 E(G_t u_t|\mathcal{F}_{t-1}) = 0 \quad \implies \quad E(G_t u_t) = 0. \quad (26)
\]

so that \( G_t \) is uncorrelated with \( u_t \). Such an instrument matrix may depend on some nuisance parameters (assumed known for the moment).

Equation (26) provides \( M \) moment conditions. If \( M = p \), we have as many equations as the dimension of \( \theta \), thus leading to the MM criterion

\[
 \bar{g} = \frac{1}{T} \sum_{t=1}^{T} g_t = 0 \quad (27)
\]

where \( g_t = G_t u_t \).

\(^3\)Because of the matrix symmetry, the unique nuisance parameters could be better gathered in \( \text{vech}(\Sigma) \), the vector obtained stacking the portion of each column up to the main diagonal included, but we want to keep the notation simple.
In general, under (26) and some regularity conditions, the GMM estimator \(\hat{\theta}_T\), obtained solving (27) for \(\theta\), is consistent (Wooldridge (1994, th. 7.1)). Furthermore, under some additional regularity conditions we have asymptotic normality of \(\hat{\theta}_T\), with asymptotic variance matrix

\[
\text{Avar}(\hat{\theta}_T) = \left( S' V^{-1} S \right)^{-1}.
\] (28)

where

\[
S = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E(\nabla_{\theta} g_t) \quad V = \lim_{T \to \infty} \frac{1}{T} V \left( \sum_{t=1}^{T} g_t \right)
\] (29)

(Wooldridge (1994, th. 7.2)), where \(\nabla_{x'y}\) indicates the matrix of partial derivatives with generic element \(\partial y_i / \partial x_j\).

Considering now the vMEM, its structure allows some considerable simplifications, stemming in particular from \(u_t\) being a martingale difference (see Wooldridge (1994)). In fact, such characteristic implies that \(g_t = G_t u_t\) also is a martingale difference (equation (26)): this leads to simplifications in the assumptions needed for the asymptotic normality, by virtue of the martingale CLT, and it is a sufficient condition for making terms \(g_t\)'s into (29) serially uncorrelated, thus giving

\[
V = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E(g_t g_t').
\] (30)

The martingale difference structure of \(u_t\) allows also a simple formulation for the efficient choice of the instrument \(G_t\), where efficient is meant producing the 'smallest' asymptotic variance among the GMM estimators coming from \(g\) functions structured as in (27), with \(g_t = G_t u_t\) and \(G_t\) an instrument. Such efficient choice is

\[
G_t^* = -E(\nabla_{\theta} u_t' | F_{t-1}) V(u_t | F_{t-1})^{-1}.
\] (31)

By computing \(E(g_t g_t')\) into (30) and \(E(\nabla_{\theta} g_t)\) into (29) we obtain \(E(g_t g_t') = -E(\nabla_{\theta} g_t) = E(G_t^* \Sigma G_t^*)\), so that

\[
V = -S = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E(G_t^* \Sigma G_t^*)
\]

and (28) specializes as

\[
\text{Avar}(\hat{\theta}_T) = -S^{-1} = V^{-1}.
\] (32)

Considering the analytical structure of \(u_t\) in a vMEM (equation (23)) and taking \(a_t = \nabla_{\theta} \mu_t' \text{diag}(\mu_t)^{-1}\) we have

\[
\nabla_{\theta} u_t' = -a_t \text{diag}(u_t + 1),
\]

so that (31) becomes

\[
G_t^* = a_t^* \Sigma^{-1}.
\]

Replacing it into \(g_t = G_t u_t\) and this, in turn, into (27), we obtain that the GMM estimator of \(\theta\) in the vector MEM solves the MM equation

\[
\frac{1}{T} \sum_{t=1}^{T} a_t \Sigma^{-1} u_t = 0
\] (33)
and has asymptotic variance matrix

\[
A\var(\hat{\theta}_T) = \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E \left[ a_t \Sigma^{-1} a_t' \right] \right]^{-1}.
\]  
(34)

It is interesting to remark that in the univariate MEM the moment condition (33) becomes

\[
\sigma^{-2} \frac{1}{T} \sum_{t=1}^{T} a_t u_t = \sigma^{-2} \frac{1}{T} \sum_{t=1}^{T} \nabla_{\theta} \mu_t \frac{x_t - \mu_t}{\mu_t^2} = 0,
\]  
(35)

that is identical to the first–order conditions of the univariate MEM when the multiplicative error term is assumed Gamma distributed (Engle and Gallo (2006)). We remark, however, as the vector version differs from the univariate case in two relevant points: the moment condition (33) cannot be derived from an explicit formulation of the pdf of the error term; the variance matrix of the error term, \( \Sigma \), cannot be removed from the equation (33).

Concerning the latter point, as detailed in Cipollini et al. (2009) the moment function in the left-hand side of (33) is, however, \( \Sigma \)-insensitive, in the sense that inconsistency in estimating \( \Sigma \) in the first step does not lead to inconsistency in estimating \( \theta \). This implies that (34) is the asymptotic variance matrix of \( \hat{\theta}_T \) even when \( \Sigma \) is estimated, instead of being known as assumed until now, and can be consistently estimated by

\[
\hat{A}\var(\hat{\theta}_T) = \left[ \frac{1}{T} \sum_{t=1}^{T} \hat{a}_t \hat{\Sigma}_T^{-1} \hat{a}_t' \right]^{-1}
\]  
(36)

where \( \hat{a}_t \) is computed on the basis of \( \hat{\theta}_T \) and \( \hat{\Sigma}_T \) is a consistent estimator of \( \Sigma \). As suggested by equation (25), a ‘natural’ estimator for the nuisance parameter is

\[
\hat{\Sigma}_T = \frac{1}{T} \sum_{t=1}^{T} u_t u_t'
\]  
(37)

where \( u_t \) indicates here the residual (23) computed by using the current values of \( \hat{\theta}_T \).

5 Empirical Application: Common Dynamics in Volatility

We adopt the Composite vMEM for modeling the joint dynamics of absolute returns (\(|ret|\)), realized kernel volatility (\(rkv\)) and high-low range (\(hl\)) for several indices, covering the period February 2001 – February 2009 (\( T = 2009 \) observations).\(^4\) Realized kernel volatilities are computed from tick by tick data according to Barndorff-Nielsen et al. (2008) and taken from the Oxford Man Institute (OMI) Realized Library, that uses data from Reuters DataScope Tick History\(^5\). Returns and daily ranges are computed using the daily highs and lows downloaded from Datastream. All measures are expressed in annualized percentage terms.

The aim of the analysis is to illustrate the separate contribution to the overall dynamics of the common and measure–specific transitory components, investigating their relative importance across the three

\(^4\)Tickers and names: DJINDUS (Dow Jones Industrials), S&PCOMP (S&P 500), S&PMIDC (S&P 400 Midcap), NASA100 (Nasdaq 100), FTSE100 (FTSE 100).

\(^5\)http://realized.oxford-man.ox.ac.uk/
indicators. In order to gain some insight into the changes provided by the Common Component, the Composite vMEM is compared against a Diagonal vMEM, i.e. a vMEM in which all $\alpha_1^{(\xi)}$, $\gamma_1^{(\xi)}$, $\beta_1^{(\xi)}$ coefficient matrices (expressed in the Composite notation) are diagonal. All models are estimated by joint GMM.

The time series patterns, shown in the top panel of the figures (1 to 5, highlight the strong similar pattern of the long term evolution of the three indicators with an initial period of (relatively) high volatility, followed by a long period of low volatility (between 2004 and the first half of 2007), in turn followed by a progressive increase up to the burst at the end of 2008. Around this long term pattern, $rkv$ appears considerably less noisy than $|ret|$, with $hl$ in an intermediate position.

The tables of parameter estimates (Tables 1 to 5) show a high persistence of the common component throughout, with the $\beta_1^{(\eta)^*}$ estimates within the range 0.98 - 0.99, substantially in line with the univariate analyses in Brownlees et al. (2011) and Engle and Lee (1999). The main contribution to this component is provided by the $rkv$ innovations ($\alpha_2^{(\eta)}$ is highly significant), and this seems consistent with the smoother evolution of this indicator relative to the other ones. The impact of the $|ret|$ and $hl$ innovations on the common component is by far less important in size and, at any rate, $\alpha_{3,1}^{(\eta)}$ is significant, while $\alpha_{3,1}^{(\eta)}$ is not, except for DJINDUS. Judging on the size of the common component’s loadings, $\eta_t$ is usually lowered when transferring into $rkv$ ($\psi_2$ significantly smaller than 1 for all indices), while it is amplified when entering into $|ret|$ and $hl$ ($\psi_1, \psi_3$ significantly higher than 1).

The estimated parameter results also point to the fact that the persistent component captures an important part of the persistence of the specific series: while the Diagonal vMEMs have a fairly high $\beta_1^{(\xi)^*}$’s coefficients, moving to the Composite model we see a decrease in their value (and, generally, sharply so). By the same token, the $\alpha_1^{(\xi)}$ coefficients tend to decrease in value and in significance for all indicators when moving from the Diagonal to the Composite vMEM, while the asymmetric parameters tend to be about the same, and continue to be highly significant.

The medium and bottom panels of Figures 1 to 5 illustrate the persistent–transitory decomposition for the three indicators. The persistent components appear fairly smooth (but not so smooth as a slow-moving component extracted via non parametric methods, cf. Engle and Rangel (2008), Brownlees and Gallo (2010), Barigozzi et al. (2010)); the transitory components contribute much less to the overall conditional expected volatility. The degree of persistence exhibited by these components varies across series with very little persistence for DJINDUS and SPCOMP, and a much higher degree of persistence in SPMIDC (which has the highest $\beta_1^{(\xi)}$ coefficients). The estimated error term oscillates around one, but very skewed for absolute returns and fairly symmetrical with occasional spikes (jumps?) for realized kernels. It provides further evidence of the conditional expectation being more accurately estimated for $rkv$ (values close to one) relative to $hl$ and, even more so, to $|ret|$.

6 Conclusions

Starting from the stylized fact of a high degree of common movements and similar persistence exhibited by time series of several volatility measures (absolute returns, realized kernels and daily range), we suggest a specification of a constrained vMEM (Cipollini et al. (2007), and Cipollini et al. (2009)), forcing its dynamics to follow the evolution of a common component (which all volatility measures contribute to) and of measure–specific components characterized by a lesser degree of persistence. The model can be estimated in a GMM framework on the basis of the conditional expectations and
conditional variances of the variables of interest.

Our empirical application suggests a certain regularity in the results obtained on data from five stock market indices: the common component is determined mainly by the realized kernel and has a high persistence coefficient as expected; the measure–specific transitory components show a more diversified pattern with a substantial absence of persistence only for some indices. As expected, the accuracy of the estimated conditional expectation, as measured by the estimated multiplicative errors, is the best for the realized kernels, followed by the daily range and much lower for absolute returns. We notice some occasional spikes in the estimated error, which probably deserve some further attention as occasional bursts in volatility.

The model suggested here is geared toward the extraction of a common component in several measures of the volatility of the same asset, so that it is natural to think of a single component in the background. The same could hold true for several measures of trading activity other than volatility (volume, number of trades, average duration, etc.). It seems more of an empirical question whether a just a single component is adequate in representing the dynamics of several volatility indicators as in a volatility spillover study, and which time series would significantly contribute to it.

References


Figure 1: Graphs of data, estimated persistent component ($\tilde{\chi}_t = \hat{\mu} + \hat{\psi}_t \hat{\eta}_t$), short term component ($\hat{\xi}_t$) and error ($\hat{\varepsilon}_t$) for the DJINDUS ticker (20010201 – 20090227).
Figure 2: Graphs of data, estimated persistent component ($\hat{\chi}_t = \hat{\mu} + \hat{\psi}\hat{\eta}_t$), short term component ($\hat{\xi}_t$) and error ($\hat{\epsilon}_t$) for the S&PCOMP ticker (20010201 – 20090227).
Figure 3: Graphs of data, estimated persistent component ($\hat{\chi}_t = \hat{\mu} + \hat{\psi} \eta_t$), short term component ($\hat{\xi}_t$) and error ($\hat{\epsilon}_t$) for the S&PMIDC ticker (20010201 – 20090227).
Figure 4: Graphs of data, estimated persistent component ($\hat{\chi}_t = \hat{\mu} + \hat{\psi}\hat{\eta}_t$), short term component ($\hat{\xi}_t$) and error ($\hat{\varepsilon}_t$) for the NASA100 ticker (20010201 – 20090227).
Figure 5: Graphs of data, estimated persistent component ($\hat{\chi}_t = \hat{\mu} + \hat{\psi}_t\hat{\eta}_t$), short term component ($\hat{\xi}_t$) and error ($\hat{\epsilon}_t$) for the FTSE100 ticker (20010201 – 20090227).
Table 1: Parameter estimates for Diagonal and Composite vMEM on the joint dynamics of ($ret$, $rkv$, $hl$) for the DJINDUS ticker (20010201 – 20090227).

<table>
<thead>
<tr>
<th>parameter</th>
<th>Diagonal vMEM</th>
<th>Composite vMEM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tr>
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<tr>
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Table 2: Parameter estimates for Diagonal and Composite vMEM on the joint dynamics of \((|ret|, rkv, hl)\) for the S&PCOMP ticker (20010201 – 20090227).

<table>
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<tr>
<th>parameter</th>
<th>Diagonal vMEM</th>
<th>Composite vMEM</th>
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Table 4: Parameter estimates for Diagonal and Composite vMEM on the joint dynamics of ($ret$, $rkv$, $hl$) for the NASA100 ticker (20010201 – 20090227).

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Table 5: Parameter estimates for Diagonal and Composite vMEM on the joint dynamics of \((|\text{ret}|, rkv, hl)\) for the FTSE100 ticker (20010201 – 20090227).

<table>
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<th>Parameter</th>
<th>Diagonal vMEM</th>
<th>Composite vMEM</th>
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