Linkages across Sovereign Debt Markets*

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Abstract

This paper studies linkages across sovereign debt markets when countries choose to default and renegotiate their debt. Countries are linked to one another by borrowing and renegotiating from common lenders who have concave payoffs. Countries are strategic large players and understand that their choices for loans, defaults, and renegotiations affect debt prices and recoveries as well as future choices for other countries. Defaults and renegotiations failures in one country lower lenders’ payoffs which increase the cost of funds for other countries and lead to more defaults and renegotiation failures. The simultaneity in defaults induces correlation in interest rates across countries. In the model, renegotiations with one country can have positive spillovers to other countries by reducing their risk of default. The model can rationalize some of the recent events in Europe.

*The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. E-mails: arellano.cristina@gmail.com; yanbai06@gmail.com
1 Introduction

Sovereign debt crises tend to happen in bunches. During the 1980s almost all Latin American countries defaulted and subsequently renegotiated their sovereign debt. During the recent European debt crises, questions about the debt sustainability arose for multiple countries including Greece, Ireland, Italy, Portugal, and Spain. During these crises, interest rates increase simultaneously for multiple countries. Figure 1 illustrates the co-movement in country interest rates in Europe since 2009. Recently discussions about renegotiations and bailouts to Greece are justified as a way to prevent further contagion in the region. Despite sovereign debt crises happening in tandem, theoretical work on sovereign default has often been restricted to study countries in isolation.

This paper studies linkages across sovereign debt markets when countries choose to default and renegotiate their debt. Countries are linked to one another by borrowing and renegotiating from common lenders who have concave payoffs. Countries are strategic large players and understand that their choices for loans, defaults, and renegotiations affect debt prices and recoveries as well as future choices for other countries. Defaults and renegotiations failures in one country lower lenders’ payoffs which increase the cost of funds for other countries and lead to more defaults and renegotiation failures. The simultaneity in defaults induces correlation in interest rates across countries. The model also provides a theory where renegotiations with one country can have positive spillovers to other countries by reducing the risk of default.

The model economy consists of two symmetric countries who borrow from a continuum of competitive lenders. The borrowing countries can default on their debt. Default entails costs in terms of access to financial markets and direct output costs. Countries in default choose to renegotiate the debt and pay the recovery. After renegotiation, sanctions are lifted for defaulters and they regain access to financial markets. Lenders are symmetric and their kernel depends on the choices of loans, defaults and renegotiations of the borrowing countries. The recovery of the defaulted debt is determined through Nash bargaining during renegotiation.
Countries are linked through one another because the prices of debt and the recovery are determined jointly and depend on the choices of default, borrowing, and renegotiation of all countries. Importantly, borrowing countries understand that their choices impact all debt prices and recoveries and consider these effects when optimizing. The price of debt reflects the risk-adjusted compensation for the loss lenders face in case of default and incorporate three main elements: the lenders’ cost of funds, the risk-adjusted default probability, and the risk-adjusted recovery rate. When one country decides to default (or to not renegotiate) the price of debt for the other country worsens because the lender’s marginal valuation rises which increases the cost of funds as well as the risk-adjusted default and recovery rate.

Renegotiations also respond to other countries choices and states. Recovery is determined such that the marginal cost for the borrower from paying the recovery equals the marginal benefit of the lender from receiving the recovery. When other countries default (or fail to renegotiate), the marginal benefit for the lender increases which raises the recovery. Higher recovery lowers the likelihood of renegotiation. However, renegotiations can also deter other countries from defaulting. Hence, when other countries are close to default, recoveries are actually lower because the marginal benefit to the lender also includes preventing the costs associated with a second default.

We solve the model numerically and parameterize to Europe for data since 2002. Specifically we target the concavity of lenders’ payoffs, the discount factor of countries, and the bargaining parameter such that the model reproduces the level and volatility of capital ratios of German and French banks, and the average spread and recovery in Greece. To focus on our mechanisms, we study the case of uncorrelated income shocks across countries.

Our model predicts that country interest rates co-move as in the data because defaults happen together. We find that the conditional spread of the home country is about 5% higher when the foreign country has spreads above its median relative to below its median. Interesting when the foreign country defaults spreads are 3% lower than when the foreign country has spreads above its median. Our model also predicts that recoveries in the home country are linked to the credit conditions of
the foreign country. Recovery rates are about 10% lower at home when the foreign
country has spreads above its median relative to below its median. When the foreign
country defaults, the recovery rates jump about 70 percentage points. Such increase
in recovery rates makes renegotiation about 20% less likely at home when the foreign
is in default.

The model in this paper builds on the work of Aguiar and Gopinath (2006)
and Arellano (2008), who model equilibrium default with incomplete markets, as in
the seminal paper on sovereign debt by Eaton and Gersovitz (1981). These papers
analyze the case of risk neutral lenders, abstract from recovery, and focus on default
experiences of single countries. Borri and Verdelhan (2009), Presno and Puozo (2011)
and Lizarazo (2010a) study the case of risk averse lenders. They show that risk
aversion allows the model to generate spreads larger than default probabilities, which
is a feature of the data. Borri and Verdelhan also show empirically that a common
factor drives a substantial portion of the variation observed. Lizarazo (2010b) studies
contagion in a model similar to ours where multiple borrowers trade with a risk averse
lenders. Her model can generate co-movement in spreads across borrowing countries
however she abstracts from any debt renegotiation and strategic interactions. Yue
(2010), D’Erasmo (2011), and Benjamin and Wright (2009) study debt renegotiation
in a model with risk neutral lenders. They find that debt renegotiation allows the
model to match better the default frequencies and the debt to output ratios.

2 Model

Consider an economy where two symmetric countries borrow from a continuum of
foreign lenders. Debt contracts are unenforceable and countries can choose to default
on their debt whenever they want. Countries that default get a bad credit standing,
are excluded from borrowing, and suffer a direct output cost. Countries in default
can renegotiate their debt. During renegotiation the defaulting country and the
lender bargain over the recovery. After renegotiation is complete, countries regain
good credit standing.
We consider an economy where each borrowing country $i$ for $i = \{1, 2\}$ receives a stochastic endowment $y_i$ each period. Let $y = \{y_1, y_2\}$ be the vector of endowments for each country in a period. These shocks follow a Markov process with transition matrix $\pi_y(y', y)$. We assume lenders face no additional shocks.

The timing of events in this economy is as follows. Each country starts each period with a level of debt $b_i$ and a credit standing $h_i$. Countries with good credit standing have $h_i = 0$ and decide whether to default or repay their debts with the indicator function $d_i$. If they repay they set $d_i = 0$, maintain their good standing for next period and choose new debt choices $b'_i$. If they default $d_i = 1$, they don’t pay their debt and start next period with bad credit standing $h_i = 1$. Countries with bad credit standing decide whether to renegotiate or not with indicator function $z_i$. If country $i$ renegotiates $z_i = 1$ and if it doesn’t renegotiate $z_i = 0$. If they renegotiate, then they bargain with the lenders over the recovery $\phi_i$ to be paid. Countries that renegotiate start the next period with good credit standing and zero debt. If they don’t renegotiate, then the maintain their bad credit standing. The endogenous aggregate states consist of the vector of debt holdings $b = \{b_1, b_2\}$ and their credit standing $h = \{h_1, h_2\}$. The economy wide state $s$ incorporates the endogenous and exogenous states: $s = \{b, h, y\}$.

### 2.1 Borrowing Countries

The representative household in each borrowing country $i$ receives utility from consumption $c_{it}$ and has preferences given by

$$E \sum_{t=0}^{\infty} \beta^t u(c_{it}),$$

where $0 < \beta < 1$ is the time discount factor and $u(\cdot)$ is increasing and concave.

The government of the borrowing country is benevolent and its objective is to maximize the utility of households. While in good credit standing, the government trades one period discount bonds with foreign lenders. The government also decides
whether to repay or default on its debt. While in default, the government is in bad credit standing and it decides whether to renegotiate or not. If the government renegotiations then it bargain with the lenders over the recovery to be repaid. The government rebates back to households all the proceedings from its credit operations in a lump sum fashion.

The price for bonds is a function \( q_i(o_i, o_{-i}, s) \) that depends on the choices of country \( i \) to default and new loans \( o_i = \{d_i, b'_i\} \), the choices of the other country, \( o_{-i} \), which include default, new loans, and renegotiation, \( o_{-i} = \{d_{-i}, b'_{-i}, z_{-i}\} \) and all aggregate states \( s \). The bond price compensates the lender for the risk adjusted loss in case of default and it depends on the choices of country 1 and 2 and the aggregate states, \( (o_1, o_2, s) \), because the lender’s kernel, as well as default, renegotiation, and recovery depend on all these variables. Below we specify how the bond price functions are determined.

When the government is in good credit standing \( h_i = 0 \) and chooses to repay its debts \( d_i = 0 \), the resource constraint for borrowing country \( i \) is the following

\[
c_i = y_i - b_i + q_i(o_i, o_{-i}, s)b'_i
\]

If the government with \( h_i = 0 \) defaults by setting \( d_i = 1 \), the government doesn’t pay its outstanding debt \( b_i \), it is excluded from trading international bonds, and it incurs output costs \( y_{it}^{def} \). Consumption equal output during these periods.

\[
c_i = y_{it}^{def}.
\]

Following Arellano (2008) we assume that borrowers lose a fraction \( \lambda \) of output if output is above a threshold: \( y_{it}^{def} = \begin{cases} y_t & \text{if } y_t \leq (1 - \lambda)\bar{y} \\
(1 - \lambda)\bar{y} & \text{if } y_t > (1 - \lambda)\bar{y} \end{cases} \), where \( \bar{y} \) is the mean level of output.

Default changes the credit standing of the country to \( h_i = 1 \). Every period a government with \( h_i = 1 \) chooses to renegotiate its debts or not. The indicator function \( z_i = 0 \) if it doesn’t renegotiate and \( z_i = 1 \) if it renegotiates. In periods when the government doesn’t renegotiate, consumption equals output \( c_{it} = y_{it}^{def} \).
If the government renegotiates, then it bargains with the lenders over the recovery \( \phi_i(o_{-i}, s) \). The recovery is the amount that the government pays back lenders to regain its good credit standing. The recovery depends on the other country’s choices \( o'_{-i} \) as well as the aggregate state \( s \) because the bargaining outcome depend on these. When \( z_i = 1 \) the resource constraint for the economy is

\[
c_{it} = y_{it} - \phi_i(o_{-i}, s) \tag{4}
\]

After renegotiating, the borrower country starts next period with zero debt due \( b_0' = 0 \) and good credit standing \( h_i = 0 \).

We represent the borrowing country’s problem as a recursive dynamic programing problem. Let \( v_i(s) \) be the value function of the borrowing country \( i \) that has good credit standing \( h_i = 0 \). Borrowing country \( i \) decides whether to default or not after endowment shocks are realized

\[
v_i(s) = \max_{d_i = \{0, 1\}} \{d_i v_i^{nd}(s) + (1 - d_i) v_i^d(s)\} \tag{5}
\]

where \( v_i^{nd}(s) \) is the value to the country conditional on not defaulting and \( v_i^d(s) \) is the value of default. \( d_i(s) = 1 \) if the country chooses default and zero otherwise.

If the country repays the debt \( h'_0 = 0 \), and the country chooses optimal consumption and savings

\[
v_i^{nd}(s) = \max_{c_i, b'_i} \{u(c_i) + \beta \sum s' \pi(s', s) v_i(s') \} \tag{6}
\]

subject to (2), taking as given the other country’s choices \( o_{-i} = \{d_{-i}, b'_{-i}, z_{-i}, c_{-i} \} \) with

\[
o_{-i} = O(s) \tag{7}
\]

The aggregate state the following period \( s' = \{b', h', y'\} \) is completely determined by both countries choices.

If the country defaults, it is does not pay the debt, cannot borrow, consumes
output \( y^d \) and \( h_i' = 1 \)

\[
v^d_i(s) = \{u(y^d_i) + \beta \sum_{s'} \pi(s', s)w_i(s')\}
\]  

(8)

subject to (7).

After default the country has bad credit standing \( h_i = 1 \) and maintains the level of the defaulted debt \( b_i \). Let \( w_i \) be the value function associated with being in bad credit standing. Once \( h_i = 1 \) the country decides whether to renegotiate or not

\[
w_i(s) = \max_{z_i = \{0, 1\}} \{z_i w^r_i(s) + (1 - z_i) w^{nr}_i(s)\}.
\]  

(9)

\( z_i(s) = 1 \) if the country chooses renegotiate and zero otherwise. The length of renegotiation is endogenous and equals to the time that the borrower takes to choose to renegotiate. Let \( w^r_i(s) \) be the value associated with renegotiation and \( w^{nr}_i(s) \) the value of not renegotiating the debt.

If the country renegotiates, then and the country repays the recovery \( \phi_i(o_{-i}, s) \). Recovery is a function that will be derived below. This function depends on the current states \( s \) as well as the choices of the other country \( o'_{-i} \) because the payoffs from renegotiation depend on these variables. Renegotiation allows the country to avoid the output cost as well as access to international borrowing. Following renegotiation the country starts with zero debt \( b'_i = 0 \), and with good credit standing \( h_i' = 0 \)

\[
w^r_i(s) = \{u(y_i - \phi_i(o_{-i}, s)) + \beta \sum_{s'} \pi(s', s)v_i(s')\}
\]  

(10)

subject to (7) where \( s' \) incorporates that country’s \( i \) state for debt, \( b'_i = 0 \), and credit standing \( h_i' = 0 \). If the country does not renegotiate, then it remains excluded from financial markets and consuming \( y^d_i \) with \( h_i' = 1 \)

\[
w^{nr}_i(s) = \{u(y^d_i) + \beta \sum_{s'} \pi(s', s)w_i(s')\}
\]

subject to (7) where \( s' \) incorporates that country’s \( i \) state for debt, \( b'_i = b_i \), and
credit standing $h_i' = 1$. Note that $w_i^{nr}(s) = v_i^d(s)$.

This problem delivers value functions $v_i(s)$ and $w_i(s)$ and decision rules for debt $B_i(s)$, default $D_i(s)$, and repayment $Z_i(s)$ which we label with capital letters.

### 2.2 Lenders

There is continuum of foreign lenders whose objective is to maximize the present discounted value of dividends $d_{Lt}$

$$
E \sum_{t=0}^{\infty} \delta^t g(d_{Lt}), \quad (11)
$$

where $0 < \delta < 1$ is the lender’s time discount factor and $g(\cdot)$ is an increasing and concave function. We assume that $\beta > \delta$.

Every period lenders receives a constant payoff from the net operations of other loans $r_LL$ and deposits $r_D D$ which we summarize by $y_L = r_LL - r_D D$. Lenders trade bonds with the two borrowing countries. We assume that lenders honor all financial contracts. To make it explicit that lenders are competitive, we will denote the lenders’ holding of the countries’ bonds by $\ell_i$ (of course in equilibrium $\ell_i = b_i$) and the equilibrium prices of bonds and recovery rates with $Q_i(s)$ and $\Phi_i(s)$.

Lenders choose optimal dividends $d_L$ and loans to the borrowing countries $\ell_1$, and $\ell_2$, taking as given the prices of bonds $Q_i(s)$ and the recovery rate of bonds $\Phi_i(s)$. The value function for the lender is given by

$$
v^L(\ell_1, \ell_2, s) = \max_{c_L, \{\ell_i' \text{ if } h_i=0 \text{ and } d_i(s)=0\} \ell_{i=1,2}} \{g(c_L) + \delta \sum_{s'} \pi(s', s) v^L(\ell_1', \ell_2', s')\} \quad (12)
$$

They maximize their value subject to their budget constraint that depends on the credit standing of each borrowing country and whether they default or renegotiate

$$
c_L(s) = y_L + \sum_{i=1,2} (1 - h_i) [1 - d_i(s)] (\ell_i - Q_i(s) \ell_i') + \sum_{i=1,2} h_i z_i(s) \frac{\Phi_i(s) \ell_i}{b_i}, \quad (13)
$$

the evolution of the endogenous states in states where they don’t trade with each
country

\[
\ell'_i = \begin{cases} 
\ell'_i & \text{if } (h_i = 0 \text{ and } d_i(s) = 1) \text{ or } (h_i = 1 \text{ and } z_i(s) = 0) \\
0 & \text{if } (h_i = 1 \text{ and } z_i(s) = 1) 
\end{cases} 
\text{ for } i = \{1, 2\} 
\]

(14)

and the law motion of the state

\[s' = S(s)\]

(15)

When lenders are trading with the country \(i\) the first order conditions for \(\ell'_i\) is

\[
g'(c_L)Q_i(s) = \sum_{s'} \pi(s', s)v_{\ell'_{i}}(|\ell'_1, \ell'_2, s'; (h'_i = 0))
\]

The envelope conditions for this problem depend on the state \(s\). It is useful to separate these envelope conditions in states where the borrowing country is in good or bad credit standing

\[
v_{\ell'_i}(\ell_1, \ell_2, s; (h_i = 0)) = [1 - d_i(s)]g'(c_L) + d_i(s)\delta \sum_{s'} \pi(s', s)v_{\ell'_i}(\ell'_1, \ell'_2, s'; (h'_i = 1))
\]

(16)

\[
v_{\ell'_i}(\ell_1, \ell_2, s; (h_i = 1)) = z_i(s)g'(c_L)\frac{\Phi_i(s)}{b_i} + [1 - z_i(s)]\delta \sum_{s'} \pi(s', s)v_{\ell'_i}(\ell'_1, \ell'_2, s'; (h'_i = 1))
\]

(17)

Combining the first order condition and envelope conditions we get that the bond price \(Q_i(s)\) satisfies

\[
g'(d_L)Q_i(s) = \delta \sum_{s'} \pi(s', s) \left\{ g'(d'_L)[1 - d_i(s')] + d_i(s')\delta \sum_{s''} \pi(s'', s')v_{\ell'_{i}}(\ell''_1, \ell''_2, s'; (h''_i = 1)) \right\}
\]

(18)

where \(d_L\) is defined by (13) and \(v_{\ell'_{i}}(\ell''_1, \ell''_2, s'; (h''_i = 1))\) is defined recursively by (17).

The price of bonds can be written in a more intuitive manner by defining lenders’
pricing kernel \( m(s', s) \) as the marginal rate of substitution for the representative lender across periods

\[
m(s', s) = \frac{\delta \pi(s', s) g'(s')}{g'(s)}. \tag{19}
\]

and defining the present value of recovery \( \zeta_i(s) \) as \( \zeta_i(s) = \delta \sum_{s'} \pi(s', s) \frac{g'(s')}{g(s)} v^L_{\ell_1}(\ell_1, \ell_2, s; (h_i = 1)) \).

The price of bonds \( Q_i(s) \) can now be compactly written as

\[
Q_i(s) = \sum_{s'} \left[ m(s', s)(1 - d_i(s')) + d_i(s')m(s', s)\zeta_i(s') \right] = \sum_{s'} \left[ m(s', s)(1 - d_i(s')(1 - \zeta_i(s'))) \right] \tag{20}
\]

where the present value of recovery is defined recursively by

\[
\zeta_i(s) = \sum_{s'} \left[ m(s', s)z_i(s') \frac{\Phi_i(s')}{b_i} + (1 - z_i(s'))\zeta_i(s') \right] \tag{21}
\]

The bond price (20) and the value of recovery (21) are easily interpretable. The bond price contains two elements: the payoff in non-default states \( d_i(s') = 0 \) and in default states \( d_i(s') = 1 \). The lender discounts cash flows by the pricing kernel \( m(s', s) \) and hence states are weighted by \( m(s', s) \). For every unit of loan \( \ell_i \), the lender gets 1 unit in the non-default states and the value of recovery \( \zeta_i(s') \) in default states.

The recovery value is the expected payoff from defaulted debt the following period. It also contains two pieces. If the country renegotiates next period \( z_i(s') = 1 \), and the value for recovery for every unit of loan is \( \frac{\Phi_i(s')}{b_i} \). If the country doesn’t renegotiate, \( z_i(s') = 0 \) and the present value of recovery is given by the discounted value of future recovery given by \( \zeta_i(s') \). These future recovery values are weighted by the pricing kernel \( m(s', s) \) which implies that recovery values are weighted more heavily for states \( s' \) that feature a higher pricing kernel.

The bond price compensates the lender for any covariation between its kernel and the bond payoffs. If default happens in states when \( m(s', s) \) is high, the price contains
a positive risk premia for the default event. Moreover, if the value of recovery is low when \( m(s', s) \) is high, the price also contains positive risk premia for the covariation of recovery.

### 2.3 Renegotiation protocol

When the defaulter country chooses to renegotiate by setting \( z_i(s) = 1 \), it bargains immediately with a committee of all lenders on the recovery that will be repaid. The renegotiation process follows Nash bargaining. Consider a candidate recovery value \( \hat{\phi}_i \). The payoff for lenders from renegotiating and receiving recovery \( \hat{\phi}_i \) equals the value of the representative lender evaluated at the aggregate debt values, \( v^L(b_1, b_2, s; \hat{\phi}_i) \). The payoff for the borrower from renegotiation is \( w^r_i(s; \hat{\phi}_i) \) for this candidate value of recovery \( \hat{\phi}_i \).

If the two parties do not reach an agreement, the defaulting country is in permanent financial autarky and \( y_i = y_i^d \) and gets a threat value equal to

\[
v_i^{aut}(y_i) = \left\{ u(y_i^d) + \beta \sum_{y', \sigma'} \pi_y(y'_i | y_i) v_i^{aut}(y') \right\}
\]

In case of no agreement, all lenders receive zero of the debt and will be permanently in financial autarky with the defaulter country. Lenders will still have access to financial trading with the other non-defaulting country. Let \( v^{L,1}(b_1, b_2, s) \) be the value to all lenders from trading with only one borrowing country, which is specified below.

The recovery \( \phi_i \) maximizes the weighted surplus for the defaulter country and the lenders. The bargaining power for the borrower is \( \theta \) and that for lenders is \( (1 - \theta) \). Recovery \( \phi_i \) solves

\[
\phi_i = \max_{\phi \in [0,1]} \left\{ \left[ w^r_i(s; \phi_i) - v_i^{aut}(y_i) \right]^{\theta} \left[ v^L(b_1, b_2, s; \phi_i) - v^{L,1}(s) \right]^{1-\theta} \right\}
\] (22)

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subject to both parties receiving a non-negative surplus from the renegotiation

\[ w^r(s; \phi) - v^{aut}(y_i) \geq 0 \]
\[ v^L(s; \phi) - v^{L,1}(s) \geq 0 \]

and law of motion \( s' = S(s) \).

The recovery \( \phi_i \) is such that the marginal cost to the borrower equals the marginal benefit to lender as follows

\[
\frac{\theta u_c(y_i - \phi_i)}{w^r(s; \phi_i) - v^{aut}(y)} = \frac{(1 - \theta)u_c(c_L; \phi_i)}{v^L(s; \phi_i; q_1) - v^{L,1}(b_{-i}, y_{-i})}.
\] (23)

In considering the threat point for renegotiating the debt, we assume that lenders will trade only with the non-defaulting country from then on. The value to the lenders of trading only with one country is similar to the problem above except that it only trades with country \(-i\).

\[ v^{L,1}(b_{-i}, s) = \max_{c_L, b'_{-i}} \{ u(c_L) + \delta \sum_{s'} \pi(s', s)v^{L,1}(s') \} \]

subject to its budget constraint

\[ c = y_L + (1 - h_{-i}) [1 - d_{-i}(s)] \left( b_{-i} - Q_{-i}(s)b'_{-i} \right) + h_{-i}z_{-i}(s)\Phi_{-i}(s)b_{-i} \]

and the law of motion \( s' = S(s) \).

### 2.4 Functions for Bond Prices and Recovery

In our model, competitive lenders trade bonds with the two borrowing countries, who are big players. Borrowing countries internalize the effects their choices of default, borrowing and renegotiation \( \{d_1, b'_1, z_1, d_2, b'_2, z_2\} \) have on bond prices and engage in Cournot competition with one another.

Consider first the case when both countries are in good credit standing, \( h_1 = h_2 = 0 \) and they are choosing default and new loans. Countries understand that
for every choice $o = \{d_1, b'_1, d_2, b'_2\}$, bond prices $\{q_1, q_2\}$ have to satisfy the demand system determined by lenders’ first order conditions:

$$
q_1 = \sum_{s'} m(s'_o, s; q_1, q_2, o) \left[ 1 - d_1(s'_o)(1 - \zeta_1(s'_o)) \right] \\
q_2 = \sum_{s'} m(s'_o, s; q_1, q_2, o) \left[ 1 - d_2(s'_o)(1 - \zeta_2(s'_o)) \right]
$$

(24)

where the state tomorrow $s'_o$ depends countries’ choices $o = \{d_1, b'_1, d_2, b'_2\}$, i.e. the states for debt $b'$ and credit standing $h'$ in $s'$ depend on today’s choice of debt and default, and where the lender’s kernel $m(s'_o, s; q_1, q_2, o)$ is itself a function of the equilibrium prices, countries choices, and current and future states. (XX Be more explicit about prices when $d = 1$)

To make explicit that borrowing countries internalize the demand system, it is informative to expand lender’s kernel $m(s'_o, s; q_1, q_2, o)$ by considering a candidate choice $\hat{o} = \{\hat{d}_1, \hat{b}'_1, \hat{d}_2, \hat{b}'_2\}$ which induce a state tomorrow $s'_{\hat{o}}$. The lenders kernel equals to the marginal utility of dividends tomorrow relative to today. Dividends today and tomorrow are given by

$$
d_L(s, q_1, q_2, o) = y_L + \sum_{i=1,2} \left[ 1 - \hat{d}_i \right] \left( b_i - q_i \hat{b}'_i \right),
$$

$$
d''_L(s'_o) = y_L + \sum_{i=1,2} (1 - \hat{h}'_i) \left[ 1 - D_i(s'_o) \right] \left( \hat{b}'_i - Q_i(s'_o)B''_i(s'_o) \right) + \sum_{i=1,2} \hat{h}_i Z_i(s'_o) \Phi_i(s'_o)
$$

Borrowing countries understand that their choices of debt and default directly affect $d_L$ and that $d_L$ depends on prices $\{q_1, q_2\}$. Borrowing countries also understand that $d''_L$ is directly affected by their choice $\hat{b}'_i$ and $\hat{h}_i$ (which is mapped from $\hat{d}_i$). Moreover, borrowing countries take future functions for borrowing $B''_i(s'_o)$, renegotiation $Z_i(s'_o)$, default $D_i(s'_o)$, prices $Q_i(s'_o)$, and recovery $\Phi_i(s'_o)$ as given but they understand the their choices today affect the state tomorrow and hence the associated
values for these variables.

We now define the bond price functions $q_1(o'_1, o'_2, s)$ and $q_2(o'_2, o'_1, s)$.

**Definition 1** When $h_1 = h_2 = 0$, the bond price functions $q_1(o'_1, o'_2, s)$ and $q_2(o'_2, o'_1, s)$ solve (24).

Consider now the case when country $i$ is in good credit standing and country $-i$ is in bad credit standing. Here country $i$ is choosing to default and new loans and country $-i$ is choosing whether to renegotiate or not. Similarly in this case, for every choice $o = \{d_i, b'_i, z_{-i}\}$ the bond price and recovery $\{q_i, \phi_{-i}\}$ solve

\[
q_i = \sum_s m(s'_o, s; q_i, \phi_{-i}, o)(1 - d_i(s'_o))(1 - \zeta_i(s'_o)) 
\]

(25)

\[
\frac{\theta u'(y_i - \phi_{-i})}{[w^r(s; \phi_{-i}) - v^{\text{aut}}(y)]} = \frac{(1 - \theta)g'c_L(s, q_i, \phi_{-i}, o)}{[v^L(b_i, b_{-i}, s; q_i, \phi_{-i}, o) - v^{L,1}(b_i, y_i)]}
\]

where the lender’s consumption and values are evaluated for every choice $o$ and corresponding price and recovery, $q_i, \phi_{-i}$.

**Definition 2** When $h_i = 0$ and $h_{-i} = 1$, the bond price and recovery functions $q_i(o'_i, o'_{-i}, s)$ and $\phi_{-i}(o'_{-i}, o'_i, s)$ solve (25)

Finally, when both countries are in bad credit standing, $h_1 = h_2 = 1$ they are choosing to renegotiate or not and for every choice $o = \{z_i, z_{-i}\}$ recovery values $\{\phi_i, \phi_{-i}\}$ solve

\[
\frac{\theta u_c(y_i - \phi_{-i})}{[w^r(s; \phi_{-i}) - v^{\text{aut}}(y)]} = \frac{(1 - \theta)g'c_L(\phi_i, \phi_{-i}, o)}{[v^L(s; \phi_i, \phi_{-i}, o) - v^{L,1}(b_i, y_i)]} 
\]

(26)

\[
\frac{\theta u_c(y_i - \phi_{-i})}{[w^r(s; \phi_{-i}) - v^{\text{aut}}(y)]} = \frac{(1 - \theta)u_c(c_L; \phi_i, \phi_{-i})}{[v^L(b_i, b_{-i}, s; \phi_i, \phi_{-i}, o) - v^{L,1}(b_i, y_i)]}
\]

**Definition 3** When $h_1 = h_2 = 1$, the recovery functions $\phi_1(o'_1, o'_2, s)$ and $\phi_2(o'_2, o'_1, s)$ solve (26)
2.5 Equilibrium

We focus on recursive Markov equilibria in which all decision rules are functions only of the state variable \( s = \{b_1, b_2, h_1, h_2, y_1, y_2\} \). A recursive equilibrium for this economy consists on (i) the policy functions \( O_i(s) \) for every borrowing country \( i \) which include policies for debt choices \( B'_i(s) \), default and renegotiation decisions \( D_i(s) \) and \( Z_i(s) \), and for consumption \( C_i(s) \), the borrowing countries’ value functions \( v_i(s), v_i^{nd}(s), v_i^d(s), w_i(s), w_i^{nr}(s) \), and \( w_i^r(s) \), (ii) the lender policy functions for dividends \( d_L(s) \), debt choices \( L'_1(s) \), and \( L'_2(s) \), and value functions \( v^L(\ell_1, \ell_2, s) \) and \( v^{L,1}(\ell_i, s) \), (iii) the recovery functions \( \phi_i(o, s) \) for each country, (iv) the bond price functions for every country \( q_i(o', s) \), (v) the equilibrium price of debt \( Q_i(s) \) and recovery rate \( \Phi_i(s) \), and (vi) the evolution of the aggregate state \( S(s) \), such that:

1. Taking as given the bond price function \( q_i(o, s) \), the recovery function \( \phi_i(o, s) \), and the policy functions for the other country \( O_{-i}(s) \), the policy and value functions functions for every country \( i \), \( O_i(s) = [B'_i(s), D_i(s), Z_i(s)], C_i(s), v_i(s), v_i^{nd}(s), v_i^d(s), w_i(s), w_i^{nr}(s), w_i^r(s) \) satisfy its optimization problems.

2. Taking as given the bond price \( Q_i(s) \), the recovery \( \Phi_i(s) \), and the evolution of the aggregate states \( S'(s) \), the policy functions and value functions for the lenders \( L'_1(s), L'_2(s), c_L(s), v^L(s) \) and \( v^{L,1}(s) \) satisfy their optimization problem.

3. The bond price functions \( q_1(o, s) \) and \( q_2(o, s) \) and recovery functions \( \phi_1(o, s) \) and \( \phi_2(o, s) \) satisfy equations (24), (25), and (26).

4. The price of debt \( Q_i(s) \) clears the bond market for every \( i \)

\[ q_i(o, s) = Q_i(s) \]  

(27)

5. The recovery rate \( \Phi_i(s) \)

6. The goods market clears

\[ c_1 + c_2 + c_L = y_1 + y_2 + y_L \]  

(28)
7. The law of motion for the evolution countries’ choices (7) and the aggregate endogenous states (15) are consistent with the individual decision rules and shocks.

2.6 Strategic decisions

We analyze the strategic interactions resulting from the borrowing countries’ problems. For illustration, we assume that the bond prices functions $q_1$ and $q_2$ as well as the value function $v$ are differentiable. Consider the case both countries are in good credit standing. Countries know that their choices for borrowing and default affect their current price and their neighbors’. We can write the optimal first order condition for optimal borrowing for country 1 as

$$u'(c_1) \left[ q_1(o,s) + \frac{\partial q_1(o,s)}{\partial b_1} b_1 \right] = -\beta \sum_{s'} \pi(s',s) v_{1,b_1}(s')$$

(29)

The left hand side of this expression is the marginal benefit for borrowing an additional unit of debt. As in standard default models, the marginal cost of debt is not only the debt price but also the derivative of the price with respect to their borrowing choices. In our model however, as in standard oligopoly type models, the prices of two countries and are connected through the lender’s demand system (24) and the countries understand this. To understand how the price changes with the choice of the debt, it is useful to write the bond price as a product between the risk-neutral expected repayment times expected kernel

$$q_i = r_i^Q(s_o', y) E[m(s_o', s, q_1, q_2, o) | y]$$ for $i = 1, 2$

where expected repayment under the risk neutral measure is

$$r_i^Q(s_o', y) = \sum_{s'} \frac{m(s_o', s, q_1, q_2, o)}{E[m(s_o, s, q_1, q_2, o)]} [1 - d_1(s_o') (1 - \zeta_1(s_o'))].$$

The demand system implies that the ratio of the prices equal the ratio of the risk neutral repayment, which implies that country 1, for example sees its price determination as
The derivative of the price with respect to debt can be found by differentiating the above expression

\[
\frac{\partial q_1(o, s)}{\partial b'_1} = \frac{Em\frac{dr_1^Q(s'_o)}{b'_1} + \left[ Em\nu_1 + q_1 Emq_2 \left( \frac{r_2^Q(s'_o, y)}{r_1^Q(s'_o, y)} \right) \nu'_1 \right] r_1^Q(s'_o, y)}{1 - \left[ r_1^Q(s'_o, y)Emq_1 + r_2^Q(s'_o, y)Emq_2 \right]}
\]

where we have suppressed the arguments inside the kernel for short-hand. As the above expression shows, the derivative of the price with respect to debt is a fairly complicated object. The price of debt changes with debt because debt reduces risk neutral repayment \( dr_1^Q(s'_o)/b'_1 \), affects the kernel \( Em\nu_1 \) directly and changes the \( q_2 \) through the demand system \( \left( \frac{r_2^Q(s'_o, y)}{r_1^Q(s'_o, y)} \right) \nu'_1 \) which affects the kernel \( Emq_2 \). The price \( q_1 \) also affects the kernel directly and indirectly through its impact on \( q_2 \). Borrowing countries weight all these forces when choosing their optimal borrowing.

The right hand side of expression in (29) is the marginal cost from having debt the following period. After using the envelope theorem, the marginal value is

\[
-v_{1,b_1}(s) = (1 - d_1(s))u'(c) \left[ 1 - \frac{\partial q_1}{\partial b'_1}b'_1 \right] + (1 - d_1(s))(1 - d_2(s)) \left[ u'(c) \frac{\partial q_1}{\partial B'_2} \frac{\partial B'_2}{\partial b_1} + \beta Ev_{1,\nu'_2}(s) \frac{\partial B'_2}{\partial b_1} \right]
\]

In our model, having an extra unit of debt is costly not only because of the cost of paying it in the states in which the borrower does not default, but also because debt affects directly the price of new debt today. Having an extra unit of debt also affects the other country’s choices of debt and default which in turn affect the price of debt today through the derivative \( \frac{\partial q_1}{\partial B'_2} \frac{\partial B'_2}{\partial b_1} \). Finally, having more debt today also affects all future choices of debt and default of the other country which is encoded in the derivative \( v_{1,\nu'_2}(s) \frac{\partial B'_2}{\partial b_1} \). These future choices of the other country matter because
they affect the future debt prices of this country.

3 Quantitative Analysis

We solve the model numerically and analyze the linkages across the two borrowing countries in country interest rates, defaults and renegotiations. The model predicts that borrowing rates and default probabilities are higher for one of the borrowing countries when the other country has a high risk of default. The model also predicts recoveries are lower and renegotiations faster for one of the borrowing countries when the other country has a high risk of default.

3.1 Parameterization

The utility function for the borrowing countries is $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. We set the risk aversion coefficients $\sigma$ to 2, which is a common value used in real business cycle studies. The utility of lenders is exponential given by $g(d_L) = -e^{-d_L}$. The length of a period is one quarter. The stochastic process for output for the borrowing countries is independent from one another and follow a log-normal AR(1) process, $\log(y_{t+1}) = \log(y_t) + \varepsilon_{t+1}$ with $E[\varepsilon^2] = \eta^2$. We discretize the shocks into a nine-state Markov chain using a quadrature-based procedure (Tauchen and Hussey, 1991). We use annual series of linearly detrended GDP for Greece for 1960–2011 taken from the World Development Indicators to calibrate the volatility and persistence of output. Note that the parameter controlling the lenders’ constant resources $y_L$ does not affect any result because $g(d_L)$ is CARA.

We calibrate four parameters: the lender and borrowers’ discount rates $\delta$ and $\beta$, the default cost $\lambda$, and the borrower’s bargaining parameter $\theta$, to match three moments: an average risk free rate of 1.2%, an average spread of 6%, and average recovery of 20%. Table 1 summarizes the parameter values.
Table 1: Parameters Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Countries’ risk aversion</td>
<td>$\sigma = 2$</td>
<td>Standard value</td>
</tr>
<tr>
<td>Stochastic structure for shocks</td>
<td>$\rho_y = 0.93, \eta_y = 0.02$</td>
<td>Greece output</td>
</tr>
<tr>
<td>Calibrated parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Output cost after default</td>
<td>$\lambda = 0.01$</td>
<td>Spread 6%</td>
</tr>
<tr>
<td>Borrowers’ discount factor</td>
<td>$\beta = 0.90$</td>
<td>Risk free rate 1.2%</td>
</tr>
<tr>
<td>Lender’s discount factor</td>
<td>$\delta = 0.95$</td>
<td>Recovery rate 20%</td>
</tr>
<tr>
<td>Bargaining power</td>
<td>$\theta = 0.58$</td>
<td></td>
</tr>
</tbody>
</table>

3.2 Results

We simulate the model and report statistics summarizing debt markets for one of the borrowing countries. Due to symmetry, statistics for the second country are equal.

Borrowing countries do not interact directly with one another or have common shocks. Linkages across borrowing countries are encoded in the bond price and recovery schedules $q(o, s)$ and $\phi(o, s)$ which themselves depend on other states and choices of the borrowing countries.

Table (2) reports statistics default probabilities, spreads, recovery rates, length of renegotiation and risk free rates for country 1, home, conditional on the debt market conditions for country 2, foreign.

The risk free rate is defined as the inverse of the lender’s kernel $r_f = 1/Em - 1$. Spreads are defined as the difference between the country interest rate and the risk free rate $spr_i = 1/q_i - r_f - 1$. Recovery rates are defined as the recovery relative to the debt in default $\phi/b_i$. Finally the length of renegotiation is defined as the average number of periods it takes countries to renegotiate after default.

Consider first the overall average statistics of the model. The model replicates the calibrated average spread 5.9%, risk free rate of 1.2% and recovery rate of about 20%. The model also predicts that the average default probability is 3.4% and renegotiations happen quickly, about 1 period after defaulting. The spread is composed of the risk-adjusted default probability, and the risk-adjusted present value of recovery. Risk premia on both default and recovery is positive in the model but account
for a modest fraction of the country interest rate. The actuarially fair spread defined with a risk neutral pricing kernel equals 4%, showing that the majority of the spread is accounted for by actuarially fair default probabilities and recovery.

### Table 2: Debt Linkages

<table>
<thead>
<tr>
<th>Home</th>
<th>Overall</th>
<th>Foreign Good Credit</th>
<th>Foreign Bad Credit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average</td>
<td>$spr &gt; p50$</td>
<td>$spr \leq p50$</td>
</tr>
<tr>
<td>Spread</td>
<td>5.9</td>
<td>8.5</td>
<td>3.5</td>
</tr>
<tr>
<td>Default prob.</td>
<td>3.4</td>
<td>3.9</td>
<td>2.9</td>
</tr>
<tr>
<td>Recovery</td>
<td>19.7</td>
<td>7.1</td>
<td>23.2</td>
</tr>
<tr>
<td>Length renego</td>
<td>1.02</td>
<td>1.0</td>
<td>1.01</td>
</tr>
<tr>
<td>Cost of funds</td>
<td>1.2</td>
<td>1.3</td>
<td>1.2</td>
</tr>
</tbody>
</table>

To illustrate debt market linkages across borrowing countries, table (2) reports debt market statistics for the home country conditional on the debt market conditions for the foreign country. Consider first the case when the foreign country is in good credit standing, where we partition the states. We find that spreads are correlated: the average spread for home equals 8.5% when the spread for the foreign country is above its median while it is 3.5% when the foreign spread is below the median. The main reasons for this correlation is that when spreads in the foreign country are high, the home country defaults more often and recoveries are lower. The default rate of home equals 3.9% and the recovery rate is about 7% when the foreign spread is high. When the foreign spread is low, default probabilities are lower equals 2.9% and recovery rates are higher at 23%. Interestingly, when the foreign country experiences an actual default, spreads for the home are lower than when the foreign country has high spreads and high default risk. This is because during actual foreign defaults, default probabilities for home are actually lower than when the foreign has large spreads and also recovery rates are much larger.

The co-movement in spreads across countries can be understood from the links across countries prices and recoveries. First, when the foreign country has high spreads, it is often experiencing a capital inflow which mechanically maps into a higher risk free rate and a "tighter" home debt price schedule. A implication of the
tighter bond price schedule is that spreads at home are high too. Second, when the foreign country has high spreads, default becomes less costly for the home country because recovery rates are low. This force increases default probabilities pushing up spreads.

Now consider the case when the foreign country is in bad credit standing because of a previous default. The implications for the home country depend dramatically on whether the foreign country renegotiates its debt or not. When the foreign country renegotiate, default probabilities and spreads are low, yet recovery rates continue to be quite high. The risk free rate during times of renegotiations are quite low because the lender receives the recovery which lowers its marginal value. When the foreign country does not renegotiate though, conditions are home are the worst. Spreads and default probabilities experience sharp increases and recoveries explode.

3.3 Crises events

We now illustrate how joint debt crises manifest themselves over time. To this end, we plot time series of various debt market measures for the home country when the foreign country is in a debt crisis. We define a period of debt crisis in the foreign country the periods where the foreign country has a spread above its median. To construct the times series in Figures 2 and 3 we start with the limiting distribution over states \( s = \{b, h, y\} \) and follow each point in the distribution through its law of motion \( S'(s) \) over 3 periods. The figures are the corresponding average statistics across these 3 periods for the subset of points in the limiting distribution that at time 0 correspond to a foreign crisis.

Figure 2 plots a time series for both spreads, default incidence, output and the cost of funds in three periods. These statistics correspond to the states when both countries, i.e. have access to borrowing and can default. Period 0 is the crisis period where the foreign country has high spreads. Consider first the time series of spreads. The time series of spreads shows that the foreign country experiences a rapid increase in spreads from 7 to 12% in crisis period. The home spreads follow the foreign spreads increasing from about 6.5 to 8.5% in the crisis period. After the crisis period, if the
foreign country manages to continue to repay its debts, its spreads fall to about 5.5%. The home spreads however remain high after the crisis period.

Now consider the incidence of default. By construction, the foreign country does not default in periods -1 and 0. We are precisely considering periods prior to a foreign default. After the crisis period however, the foreign country has a high probability of default of about 7%. The home probability of default decreases about 1% into the crisis period and increases about 0.5%. Hence, the joint increase in spreads leading to the crisis is driven by the higher default incidence in period 1 of foreign and home (the spread incorporates future default probabilities).

Figure 2 also shows time series for output and the risk free interest rate. Output is pretty flat across these periods, although the output of the foreign country is below its mean because crises are associated with downturns. The risk free interest rate is fairly flat too, although in the crisis period is slightly higher about 0.03% higher. The stability of risk free rates illustrates that debt linkages across countries do not operate strongly through a change in the level of the risk free rate. [So what is it then?]

Let’s now consider what happens to the home country if it is in bad credit standing during a crisis in the foreign country. In Figure 3 plots time series of the recovery rate and the renegotiation rate at home. During the crisis period, recovery rates fall for the home country because the surplus to the lender from the renegotiating with the home country is larger if the foreign country has high spreads. The surplus is larger because a renegotiation prevents an immediate foreign default, which is costly to the lender. After the crisis period, recovery rates increase a bit, this is largely due to the fact that a large incidence of foreign defaults events raise the recovery for home. Renegotiation rates are pretty high, but after the crisis they fall due to a higher recovery.

4 Conclusion

We developed a multi-country model of sovereign default and renegotiation. Debt market conditions for borrowing countries are linked to one another because they
borrow from a common risk averse lender. In our model country interest rates are correlated and renegotiations with one country have spillover effects to other countries. The model provides a rationale for shorter renegotiations and larger haircuts with one country to prevent default in other countries. Our model provides a framework to study some of the recent events in Europe.

References


Figure 2

- **Spread**
  - Foreign
  - Home

- **Fraction of Default**
  - Home
  - Foreign

- **Output**
  - Home
  - Foreign

- **Cost of Fund**