Integrate-and-Fire Neuron

Layer 5 pyramidal cell from rat visual cortex [Izhikevich07]

A minimum current is required for spiking

Spike frequency increases linearly at high rates

Integrate-and-Fire Model

Integrate and Fire

Reset

Positive Feedback

Integrate and Na current

Reset

Neurons spike when an inward Na current overcomes an outward leak current

This model does not include spike generation—the spike is pasted on when $V_m$ reaches $V_{th}$, the threshold voltage. It only captures the neuron's behavior below threshold (inflexion point).
Membrane-Voltage Equation

Intracellular

\[ I_{\text{in}} \]

\[ C_m \]

\[ g_{lk} \]

\[ I_{lk} \]

Extracellular

\[ V_m \]

Current–Voltage Relations

Conductance:

\[ I_{lk} = g_{lk} V_m \]

Capacitance:

\[ I_C = C_m \frac{dV_m}{dt} \]

Capacitor models membrane; conductance models leak

\[ I_{\text{C}} + I_{lk} = I_{\text{in}} \]

\[ \Rightarrow C_m \frac{dV_m}{dt} + g_{lk} V_m = I_{\text{in}} \]

\[ \Rightarrow \tau_m \frac{dV_m}{dt} + V_m = V_{\infty} \]

where \( \tau_m = \frac{C_m}{g_{lk}} \) and \( V_{\infty} = \frac{I_{\text{in}}}{g_{lk}} \)

\( V_m \) reaches \( V_{\text{th}} \) only if \( V_{\infty} > V_{\text{th}} \). Thus, \( I_{\text{in}} = g_{lk} V_{\text{th}} \) is the minimum current for spiking.

Dimensionless Form

Voltage Units

Threshold

Rest

\[ V_m \quad V_{\text{inf}} \quad V_{\text{th}} \]

Dimensionless Units

Threshold

Rest

\[ x \quad x_{\text{inf}} \quad 1 \]

Voltages are expressed as multiples of the threshold voltage

Dividing both sides by \( V_{\text{th}} \) and defining \( x = V / V_{\text{th}} \) yields

\[ \tau_m \frac{dx}{dt} + x = x_{\infty} \]

where \( x_{\infty} = \frac{V_{\infty}}{V_{\text{th}}} \)

Note that \( x_{\infty} \) must exceed 1 for spiking to occur.
Determining the Time-Constant

The initial slope (left) increases linearly with the rest-level (right)

For \( x \ll x_\infty \), we have

\[
\ln[1]:= \tau_m \frac{dx}{dt} \approx x_\infty \frac{dx}{dt} = \frac{x_\infty}{\tau_m}
\]

The membrane voltage would take \( \tau_m \) seconds to reach \( x_\infty \) at its initial rate of change.

We determine \( \tau_m \) in lab by measuring the intial slope for different input currents: Data should fall on a straight-line with slope \( 1/\tau_m \).

Stability: Phase-Plot

The phase-plot (left) explains the neuron's behavior (right)

We solve the membrane-equation for the derivative and plot it versus \( x \):

\[
\frac{dx}{dt} = \frac{x_\infty - x}{\tau_m}
\]
The plot reveals that the point $x = x_\infty$ is stable: $\dot{x} > 0$ when $x < x_\infty$ and $\dot{x} < 0$ when $x > x_\infty$.

That is, the derivative's sign is such that $x$ returns to $x_\infty$ when perturbed.

### Spike Frequency

The time $T$ when $x = 1$ equals the period; frequency is $1/T$ (plotted for $\tau_m = 10 \text{ ms}$)

Given steady-state $x_\infty$, time-constant $\tau_m$, and initial condition $x(0) = 0$:

$$x(t) = x_\infty + (0 - x_\infty) e^{-t/\tau_m} = x_\infty (1 - e^{-t/\tau_m})$$

Setting $x(T) = 1$ and solving for $T$ yields:

$$T = \tau_m \ln \left( \frac{x_\infty}{x_\infty - 1} \right) \implies T \approx \frac{\tau_m}{x_\infty - 1/2} \text{ for } x_\infty \gg 1$$

### Sodium Channels

An additional (voltage-dependent) inward current $I_{Na}$ models Na channels

The sodium current is modeled as:

$$I_{Na} = \frac{1}{3} \left( \frac{V_m}{V_{th}} \right)^2 I_{1k} V_m$$
$I_{\text{Na}}$ causes $V_m$ to increase, which then causes $I_{\text{Na}}$ to increase further (positive-feedback).

**Positive-Feedback Neuron**