Development of renormalization group analysis of turbulence

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1. Introduction

The renormalization group (RG) procedure for nonlinear, dissipative systems is by now quite standard (Ma, 1976). The successes of its application to the problem of hydrodynamic turbulence are also becoming well-known (Forster, Nelson and Stephen, 1977, Fournier and Frisch, 1983, Yakhot and Orszag, 1986). Much progress has been made towards an understanding of what is, and what is not, accessible via RG analysis. In summary, the RG method isolates self-similar behavior and provides a systematic procedure to describe scale-invariant dynamics in terms of large scale variables only. The parameterization of the small scales in a self-consistent manner has important implications for sub-grid modeling. The limiting forms of such parameterizations are often universal, i.e. independent of the numerical coefficients in the original model.

Recognizing its limitations, the renormalization group technique is a powerful tool. RG methods will predict characteristics of the dynamics of a model that are approximately scale-invariant. Applied to the Navier Stokes equations, RG provides an expression for the eddy-damping of the large scales by the small scales. Other scale-dependent dynamics, such as sweeping, are not addressed (Chen and Kraichnan, 1989).

Skepticism has surrounded the RG predictions for turbulence because the detailed mathematics involved is not yet well understood. The method is justified mostly by its success: universal scaling laws derived using RG methods are quite accurate. Observed scaling laws are reproduced for a diverse set of problems, from population dynamics (Feigenbaum, 1979), to turbulence, to nonlinear spin dynamics near a ferromagnetic critical point (Wilson, 1974).

The deduction of experimentally known scaling laws gives credibility to the RG method. The merit of any theory, however, must be based on its predictive power. To date, the most important predictions from RG analysis of turbulence have been low Reynolds number corrections to traditional high Reynolds number models. For examples, RG formulas provide smooth transition between the Smagorinsky eddy viscosity and the molecular viscosity, and deduced modifications to the traditional $\kappa - \varepsilon$ model (Yakhot and Orszag, 1986, hereafter referred to as I). In the latter case, however, ambiguities remain with respect to procedure and interpretation. This is not surprising given the pioneering nature of the mathematics.

The RG model for homogeneous, isotropic turbulence is developed in Section 2. The steps of the RG procedure for nonlinear equations are reviewed. The meaning and consequences of the ϵ-expansion are addressed in Section 3 using the work of Fournier and Frisch (1978, 1983). Their results are given in terms of the expansion parameter ϵ. Inertial range statistics and scaling laws are recovered for the case of $\epsilon = 4$ (I). Section 4 gives some results of the theory for homogeneous, weakly anisotropic turbulence ($\epsilon = 4$ and no mean flow). Extension of the theory to include a weak mean flow is discussed in Section 5. In Section 6, errors in the Yakhot-Orszag RG $\kappa - \varepsilon$ equations are corrected. Consistency between direct numerical simulation data for channel flow, the standard $\kappa - \varepsilon$ model and the RG-based model requires a reinterpretation of the contributions to the ε-equation. Finally, Section 7 proposes application of the RG method to a sequence of model equations that converges to the Navier Stokes equations. The solutions of these particular model equations are known to have self-similar solutions.

2. The RG procedure

The renormalization group symmetry transformation consists of two steps (Ma, 1976). First, course graining is achieved by averaging over small scales. Second, space is rescaled. New independent variables are defined in the original domain by the rescaling. In most cases, the dependent variables are also rescaled.

It is not clear how to course-grain a nonlinear system in which the large scales are coupled to the small scales. This is, of course, the closure problem of turbulence. The RG technique was developed for the equations of nonlinear spin dynamics, the time-dependent Ginzburg-Landau equations. It is based on expansion about an equilibrium basic state whose Gaussian statistics are known from the theory of statistical mechanics. Although this procedure is sensible for near-equilibrium dynamics, it is not obviously relevant to turbulence. Nevertheless, the basic state in the RG analysis of turbulence is also assumed Gaussian. The meaning of the expansion will be explored using the work of Fournier and Frisch (1978, 1983) in Section 3.

The RG transformation of a nonlinear system is illustrated with homogeneous, isotropic turbulence driven by a Gaussian random force. The model equations in Fourier space are

$$\hat{v}_i[\hat{\mathbf{k}}] = G^o[\hat{\mathbf{k}}]\hat{f}_i[\hat{\mathbf{k}}] - \frac{i\lambda_o}{2}P_{imn}[\mathbf{k}]\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{q=0}^{\Lambda_o}\frac{d\mathbf{q}}{(2\pi)^d}\hat{v}_m[\hat{\mathbf{q}}]\hat{v}_n[\hat{\mathbf{k}}-\hat{\mathbf{q}}] \qquad (1)$$

$$\left\langle \hat{f}_i[\hat{\mathbf{k}}]\hat{f}_j[\hat{\mathbf{k}}']\right\rangle = (2\pi)^{d+1}2D_o\frac{F[k]}{2\pi k^2}P_{ij}[\mathbf{k}]\delta[\hat{\mathbf{k}}+\hat{\mathbf{k}}'] \qquad (2)$$

where $\hat{v}_i$ and $\hat{f}_i$ are the i^{th}-components of the Fourier amplitudes of the velocity and force vectors, $\hat{\mathbf{k}} = [\mathbf{k}, \omega]$ is a four-vector, and $G^o[\hat{\mathbf{k}}] \equiv (-i\omega + \nu_o k^2)^{-1}$

with $k \equiv |\mathbf{k}|$ and ν_o the kinematic viscosity. The tensor $P_{imn}[\mathbf{k}]$ results from elimination of the pressure using the continuity condition $k_i \hat{v}_i[\hat{\mathbf{k}}] = 0$: $P_{imn}[\mathbf{k}] \equiv k_m P_{in}[\mathbf{k}] + k_n P_{im}[\mathbf{k}]$ with projection operator $P_{ij}[\mathbf{k}] \equiv \delta_{ij} - k_i k_j / k^2$, where δ_{ij} is the Kronecker delta function. The cutoff Λ_o is the wavenumber above which viscosity wipes out all motion, $\lambda_o \doteq 1$ is an ordering parameter and d is the number of dimensions. The brackets $<>$ denote an ensemble average. The force, and thus the zeroth-order velocity (in λ_o), is homogeneous and isotropic, defined by the scalar correlation function $F[k]$.

Course graining is achieved with the following steps:

1. Define $\hat{v}_i^< \equiv \hat{v}_i[0 \le k < k_c]$ and $\hat{v}_i^> \equiv \hat{v}_i[k_c \le k \le \Lambda_o]$ (with analogous definitions for $\hat{f}_i^<$ and $\hat{f}_i^>$) where k_c is the low wavenumber cutoff of the band to be eliminated.

2. In the nonlinear term let $\hat{v}_m[\hat{\mathbf{q}}]\hat{v}_n[\hat{\mathbf{k}} - \hat{\mathbf{q}}] = \hat{v}_m^<[\hat{\mathbf{q}}]\hat{v}_n^<[\hat{\mathbf{k}} - \hat{\mathbf{q}}] + 2\hat{v}_n^<[\hat{\mathbf{k}} - \hat{\mathbf{q}}]\hat{v}_m^>[\hat{\mathbf{q}}] + \hat{v}_m^>[\hat{\mathbf{q}}]\hat{v}_n^>[\hat{\mathbf{k}} - \hat{\mathbf{q}}]$.

3. Iteratively substitute for $\hat{v}_i^>$ in the equation for $\hat{v}_i^<$. Iterate a number of times equal to the order of the nonlinearity, i.e. keep terms to order λ_o^2.

4. Ensemble average over $\hat{f}_i^>$ and evaluate all four-dimensional $^>$-integrals. These are integrals whose integrand has wavenumber defined in the interval $[k_c, \Lambda_o]$. All $^>$-integrals are calculated to lowest order in the distant interaction limit. This is the limit in which $^<$-wavenumbers are small compared to $^>$-wavenumbers.

Steps 1-4 eliminate the wavenumber band $k_c \le k \le \Lambda_o$.

In addition to terms obtained by replacing $\hat{v}_i$ by $\hat{v}_i^<$ in the original equations, correction terms are generated. They are

a. force renormalization terms. These terms are zeroth-order in $\hat{v}_i^<$ and redefine the force correlation.

b. viscosity renormalization terms. These are linear in $\hat{v}_i^<$ and define an eddy viscosity, $\nu_T = \nu_o + \delta\nu$.

c. vertex renormalization terms. These are second-order in $\hat{v}_i^<$ and redefine the vertex, $\lambda_T = \lambda_o + \delta\lambda$. These terms must vanish in the infrared limit $k \to 0$ by Galilean invariance (Forster et. al., 1977).

d. higher order terms in $\hat{v}_i^<$.

To focus on scale-invariant behavior inherent in the original equations, one justifies neglect of new terms. Then one proceeds to the second half of the RG transformation, the rescaling. In this case the scale-invariant behavior is the balance between forcing and eddy damping and the new terms are higher order in $\hat{v}_i^<$.

One iterates the two-part RG symmetry transformation until the equations converge to a 'fixed point'. At a fixed point, the parameters in the model no longer change; the equations are invariant under the RG transformation and describe self-similar physics. The scaling laws at a fixed point are often independent of the initial parameter values and capture 'universal' physics contained the original model.

3. The ϵ-expansion

Fournier et. al., (1983) examined the general class of force-correlation functions $F[k] = 2\pi k^{3-\epsilon}$ for $\epsilon > 0$. (The parameter ϵ here simply defines $F[k]$ and is not the dissipation rate, traditionally denoted by the same symbol. In this paper we denote the dissipation rate of the turbulent field by ε to avoid confusion.) They found for the eddy viscosity, after elimination of the wavenumber band $k_c \leq k \leq \Lambda_o$,

$$\nu_T[k_c] = \nu_o(1 + 3\frac{\sigma_1[\epsilon]D_o}{\nu_o^3}(\frac{k_c^{-\epsilon} - \Lambda_o^{-\epsilon}}{\epsilon}))^{1/3}. \tag{3}$$

where $\sigma_1[\epsilon] = (d^2 - d - \epsilon)/(4d(d+2)\pi^2)$.

At the fixed point, which is found in the limit $\Lambda_o >> k_c$, $k_c \rightarrow k \rightarrow 0$, the following asymptotic relations hold:

$$\nu_T[k] \sim (\frac{3\sigma_1[\epsilon]}{\epsilon})^{1/3} D_o^{1/3} k^{-\epsilon/3} \tag{4}$$

$$E[k] \sim (\frac{3\sigma_1[\epsilon]}{\epsilon})^{1/3} D_o^{2/3} k^{1-2\epsilon/3} \tag{5}$$

$$\bar{\lambda} \equiv \frac{\lambda_o D_o^{1/2}}{(\nu_T^{3/2} k_c^{\epsilon/2})} \sim (3\sigma_1[\epsilon])^{-1/2} \epsilon^{1/2} \tag{6}$$

where $\bar{\lambda}$ is the non-dimensionalized expansion parameter (Reynolds number). Relations (4)-(6) are universal in the sense that they do not depend on ν_o.

If $\epsilon < 0$, the fixed point energy spectrum (5) results from force-correlation function $F[k] = 2\pi k^{3-2\epsilon/3}$. The case $\epsilon = -2/3$ was considered by Forster et. al., (1977), and reproduces $E[k] \propto k^2$ for low wavenumbers. This is the power law predicted by Saffman (1967) for homogeneous, isotropic turbulence. For $\epsilon < 0$, the dynamics are not universal at the fixed point.

The point $\epsilon = 0$ is called a crossover point: for $\epsilon < 0$, higher-order terms in $\hat{v}_i^<$ decay exponentially as k_c is decreased and statistics are essentially Gaussian; for $\epsilon > 0$, higher-order terms in $\hat{v}_i^<$ become important and statistics are no longer Gaussian (Fournier et. al., 1978, Kraichnan, 1987, 1989). Neglect of the higher-order terms is rigorously justified only for $\epsilon < 0$. For all $\epsilon > 0$, the higher-order terms in $\hat{v}_i^<$ are marginal (neither grow nor decay exponentially) as k_c is decreased.

For ϵ positive and near zero, the expansion in powers of $\bar{\lambda}$ is likely, but not guaranteed, to be asymptotic by relation (6). Unfortunately, ϵ near zero corresponds to an energy spectrum near $E[k] \sim k$, which is not often observed in nature.

Despite the mathematical uncertainties associated with positive values of ϵ away from zero, Yakhot et. al., (I), applied the RG procedure to the forced Navier Stokes equations (1) and (2) with $\epsilon = 4$. This case models the physically relevant spectrum $E[k] \sim k^{-5/3}$. Their results are exactly equations (3)-(6) with $\epsilon = 4$ everywhere except in the coefficient σ_1. The value $\epsilon = 0$ is used to evaluate σ_1. By relating the parameter D_o to the flow-averaged dissipation rate $\bar{\varepsilon}$, they found the universal scaling law $E[k] = 1.617\bar{\varepsilon}^{2/3}k^{-5/3}$ (Leslie, 1973, I, Dannevik, Yakhot, and Orszag, 1987).

The prediction for Kolmogorov's constant 1.617 is very close to the observed values, which are in the range 1.4-1.6. It is found using $\sigma_1[0] = 1/(10\pi^2)$. If $\sigma_1[4] = 1/(30\pi^2)$ is used, the RG value of Kolmogorov's constant is 1.11. It is not apparent why the coefficient at the fixed point should be evaluated at $\epsilon = 0$ instead of $\epsilon = 4$. Indeed, the general procedure advocated in I is to evaluate all coefficients at the fixed point using $\epsilon = 0$. This procedure is supported by most of the RG results for turbulence. As another example, the Obukhov-Corrsin constant derived using $\epsilon = 0$ is 1.16, while the value derived using $\epsilon = 4$ is 0.41. However, we show in Section 6 that evaluating coefficients in the RG equation for the dissipation rate at $\epsilon = 0$ leads to results that are inconsistent with direct numerical simulations and the traditional model. Paper I does not explain why amplitudes of a k^1-spectrum are used for the theory of a $k^{-5/3}$-spectrum.

If we are only interested in scale-invariant physics, the RG-expansion is likely to reflect its essential features, regardless of the value of ϵ. The difference between ϵ from its crossover value gives a rough idea of the importance of the dynamics that are being neglected and the departure from Gaussian statistics. For high Reynolds number turbulence, with a well-developed $k^{-5/3}$-spectrum, the eddy viscosity (3) with $\epsilon = 4$ may capture the eddy-damping effect of small scales even though all other effects are ignored in the RG analysis. Numerical tests will be decisive (Karniadakis, Yakhot, Rakib, Orszag and Yakhot, 1989).

To summarize, the RG-expansion probably provides an accurate description of self-similar physics. The difference between the expansion parameter and its crossover value is a measure of the importance of other dynamics and of non-Gaussianity. It is not clear if and/or why amplitudes should always be evaluated at the crossover value of the renormalized expansion parameter.

4. Weakly anisotropic turbulence

A model for weakly anisotropic turbulence can be developed by extending the force correlation to depend linearly on the anisotropy tensor b_{ij},

$$< \hat{f}_i[\hat{\mathbf{k}}]\hat{f}_j[\hat{\mathbf{k}}'] >= (2\pi)^{d+1}\delta[\hat{\mathbf{k}} + \hat{\mathbf{k}}']D_o \frac{F[k]}{2\pi k^2}\{2P_{ij}[\mathbf{k}]+$$

$$+\psi\{b_{ij} - \frac{(b_{in}k_n k_j + b_{jn}k_n k_i)}{k^2} + \frac{b_{nm}k_n k_m k_i k_j}{k^4}+$$

$$+\alpha_o(\frac{b_{nm}k_n k_m \delta_{ij}}{k^2} - \frac{b_{nm}k_n k_m k_i k_j}{k^4})\}\} \quad (7)$$

where α_o and ψ are constants. The anisotropy tensor is defined as $b_{ij} \equiv (< v_i[\mathbf{x},t]v_j[\mathbf{x},t] > -(1/3)\kappa\delta_{ij})/\kappa$ where $\kappa \equiv (1/2) < v_i[\mathbf{x},t]v_i[\mathbf{x},t] >$. Relation (7) is the most general weakly anisotropic correlation (i.e. linear in b_{ij}) that satisfies continuity and the required symmetry conditions (Reynolds, 1987). Note that b_{ij} is a matrix of constants because the flow is assumed homogeneous.

In anticipation of an anisotropic eddy viscosity, let

$$G^o[\hat{\mathbf{k}}] = (-i\omega + \nu_o k^2 + \beta_o k_m k_n b_{mn})^{-1} \quad (8)$$

in the forced Navier Stokes equations (1), where $\beta_o = 0$. The model given by (1), (7) and (8) has no mean flow.

The RG steps 1-4 result in renormalized equations with correction terms a-d (Section 2) where the eddy damping is now defined by $\nu_T = \nu_o + \delta\nu$ and $\beta_T = \beta_o + \delta\beta$. In addition, a fifth type of term is generated which couples the equation for $\hat{v}_i[\hat{\mathbf{k}}]$ to the equation for $\hat{v}_j[\hat{\mathbf{k}}]$:

e. linear coupling terms. In the equation for $\hat{v}_i[\hat{\mathbf{k}}]$, these are linear in $\hat{v}_j[\hat{\mathbf{k}}]$ and have the form $(k^2 b_{ij} - k_i k_m b_{mj})\hat{v}_j^<[\hat{\mathbf{k}}] \equiv M_{ij}\hat{v}_j^<[\hat{\mathbf{k}}]$.

The linear coupling terms show that the small scales can force $\hat{v}_i$ through interaction with $\hat{v}_j$.

One can suppress this forcing by choosing α such that it vanishes at each iteration of the RG scale elimination. The choice

$$\alpha^{(n)} = 1 + \psi\frac{7\beta^{(n)}}{2\nu^{(n)}} \quad (9)$$

makes the coefficient of the linear coupling terms zero. The superscript n is the iteration number.

With constraint (9), $F[k] = 2\pi k^{3-\epsilon}$ and $\epsilon = 4$ ($E[k] \sim k^{-5/3}$), one finds

$$\nu_T^{(n+1)} = \nu_T^{(n)} + \sigma_1[\epsilon]\frac{D_o}{(\nu_T^{(n)})^2}\frac{1}{(\Lambda[nr])^4}\frac{(e^{4r}-1)}{4} \quad (10)$$

$$\beta_T^{(n+1)} = \beta_T^{(n)} + \sigma_2[\epsilon]\frac{D_o}{(\nu_T^{(n)})^2}(\frac{\beta_T^{(n)}}{\nu_T^{(n)}} + \psi)\frac{1}{(\Lambda[nr])^4}\frac{(e^{4r}-1)}{4} \quad (11)$$

where $\sigma_1[0] = 1/(10\pi^2)$ as above, $\sigma_2[0] = 1/(40\pi^2)$ and $\Lambda[nr] = \Lambda_o e^{nr} = k_c$. The cutoff k_c is now the last eliminated wavenumber.

The differential equations appropriate for repeated elimination of infinitesimal bands ($r \to 0$) are

$$\frac{d\nu_T[\eta]}{d\eta} = \sigma_1 \nu_T[\eta](\bar{\lambda}[\eta])^2 \tag{12}$$

$$\frac{d\beta_T[\eta]}{d\eta} = \sigma_2(\beta_T[\eta] + \psi\nu_T[\eta])(\bar{\lambda}[\eta])^2 \tag{13}$$

where $\eta \equiv nr$ and $\bar{\lambda}$ is the renormalized expansion parameter given by (6). The solution of (12) subject to $\nu_T[0] = \nu_o$ is equation (3) with $k_c = \Lambda_o e^{-\eta}$; the solution of (13) subject to condition $\beta_T[0] = 0$ is

$$\beta_T[k_c] = \frac{(\sigma_2/\sigma_1)\psi}{(1-\sigma_2/\sigma_1)}(\nu[k_c] - \nu_o^{(1-\sigma_2/\sigma_1)}\nu_T^{\sigma_2/\sigma_1}[k_c]). \tag{14}$$

As in the isotropic theory given in Sections 2 and 3, the fixed point is at $\eta \to \infty$, which corresponds to $k_c \to k \to 0$. According to the theory of Yakhot et. al., one evaluates the coefficients σ_1 and σ_2 at the crossover $\epsilon = 0$. Then $\nu_T[k]$, $E[k]$ and $\bar{\lambda}$ are given by (4)-(6), $\beta_T[k] \sim (1/3)\psi\nu_T[k]$ and

$$\alpha \sim \frac{13}{6}. \tag{15}$$

The parameter α defines the turbulent states that are energetically possible in the model. As α increases, the function space of realizable states decreases (Shih et. al., *to be submitted to J. Fluid Mech.*). The value of α for the Reynolds-stress model of Launder, Reece and Rodi, (1975), is $\alpha_{LRR} = .527$; the value for the Reynolds stress model that matches Rapid Distortion Theory (RDT) is $\alpha_{RDT} = 3/2$. Both models have small regions of realizability around the isotropic turbulence point.

If we suppress the linear coupling terms in the RG analysis, α increases from 1 to 13/6 as more and more scales are eliminated from the problem. In view of the large values of α necessary to prevent linear coupling, turbulence models based on RG theory which includes this coupling seem more promising. The extension is relatively simple: the model equations become

$$\hat{v}_i[\hat{\mathbf{k}}] = G^o[\hat{\mathbf{k}}]\hat{f}_i[\hat{\mathbf{k}}] + \Theta_o M_{ij}[\mathbf{k}]\hat{v}_j[\hat{\mathbf{k}}] - \frac{i\lambda_o}{2}P_{imn}[\mathbf{k}]\int \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}}\hat{v}_m[\hat{\mathbf{q}}]\hat{v}_n[\hat{\mathbf{k}} - \hat{\mathbf{q}}] \tag{16}$$

where M_{ij} is defined above and $\Theta_o = 0$. Two special cases were treated by Rubinstein and Barton (1987). This is a possible area of further research. The equations of a passive scalar are amenable to similar analysis.

5. Homogeneous turbulence with a weak mean flow

A formulation of the RG theory of turbulence without an artificial external force would be appealing. One might think that providing an internal production mechanism by including a mean flow would alleviate the necessity of an external force. However, if the zeroth-order turbulent velocity field is sustained by interaction with the mean, then wavenumbers are changing in time as quickly as Fourier amplitudes. For the RG analysis to be meaningful, one requires that wavenumbers stay constant at least in the turnover time of a large eddy.

Thus we continue to assume that the turbulence is sustained by an external force. The homogeneous mean must be considered weak and corrects the zeroth-order solution given by the balance of external forcing and viscous diffusion. Our model equations for the fluctuations $\hat{v}_i'$ are

$$\hat{v}_i'[\hat{\mathbf{k}}] = G^o[\hat{\mathbf{k}}]\hat{f}_i[\hat{\mathbf{k}}] + N_{ij}[\mathbf{k}]\hat{v}_j'[\hat{\mathbf{k}}] - \frac{i\lambda_o}{2}P_{imn}[\mathbf{k}]\int \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}}\hat{v}_m'[\hat{\mathbf{q}}]\hat{v}_n'[\hat{\mathbf{k}}-\hat{\mathbf{q}}] \quad (17)$$

$$N_{ij}[\mathbf{k}] = \Pi_{mn}k_m\frac{\partial}{\partial k_n}\delta_{ij} - \Pi_{ij} + 2\frac{k_m k_i}{k^2}\Pi_{mj} \quad (18)$$

where $\hat{v}_i = U_i + \hat{v}_i'$, $U_i \equiv < \hat{v}_i > = \Pi_{ij}x_j$ and Π_{ij} is constant (Leslie, 1973).

For simplicity, one may first consider isotropic, homogeneous forcing given by the correlation (2). The RG procedure is carried out as in Section 2, by repeated substitution of $(\hat{v}_i')^>$ in the equation for $(\hat{v}_i')^<$. For consistent asymptotics, terms of order $\Pi_{ij}^2\lambda_o$ and $\Pi_{ij}\lambda_o^2$ must be retained, while terms of order λ_o^3 may be dropped. One anticipates interesting changes in the RG eddy viscosity.

The RG analysis for homogeneous shear should reproduce the universal scalar spectrum of the Reynolds stress in the inertial range. For shear in the x_2 direction, $U_i = U\delta_{i1}x_2$, RG should predict $4\pi k^2 < \hat{v}_1'[\hat{\mathbf{k}}]\hat{v}_2'[\hat{\mathbf{k}}] > \propto k^{-7/3}$.

One immediately notices the similarity between the model equations with a mean flow (17) and the renormalized model equations due to anisotropic external forcing (16). This similarity can be exploited to reduce the amount of work in the problem with a weak mean flow. It is a straightforward extension to flow with a homogeneous mean driven by a weakly anisotropic external forcing.

6. The RG $\kappa - \varepsilon$ model

The most important RG contribution to turbulence modeling has thus far been low Reynolds number corrections to previously established high Reynolds number equations. The RG corrections are derived, unlike their ad-hoc predecessors.

Unfortunately, there remain unresolved issues in the high Reynolds number RG $\kappa - \varepsilon$ model. These should be reconciled before study of the low Reynolds number corrections. For better understanding of the RG $\kappa - \varepsilon$ equations, the traditional model is reviewed. The Yakhot-Orszag RG model is then discussed. We give a corrections to, and a reinterpretation of, the results cited in I.

6.1. The traditional $\kappa - \varepsilon$ model

The dissipation rate of fluctuations in homogeneous turbulence is $\varepsilon \equiv \nu_o < (\nabla_j v'_i)^2 >$ where $v'_i[\mathbf{x}, t]$ are the zero-averaged fluctuations from the mean. As in Section 5, $v_i = U_i + v'_i$, $U_i \equiv < v_i > = \Pi_{ij} x_j$ where Π_{ij} is constant. The time rate of change of ε is

$$\frac{\partial \varepsilon}{\partial t} = -\overbrace{2\nu_o^2 < (\nabla_j \nabla_m v'_i)^2 >}^{1} - \overbrace{2\nu_o < (\nabla_j v'_i)(\nabla_m v'_i) > \nabla_j U_m}^{2} -$$

$$-\overbrace{2\nu_o < (\nabla_j v'_i)(\nabla_j v'_m) > \nabla_m U_i}^{3} - \overbrace{2\nu_o < (\nabla_j v'_i)(\nabla_j v'_m)(\nabla_m v'_i) >}^{4}. \quad (19)$$

In the standard, high Reynolds number model of equation (19), the total dissipation of ε is represented by the combination of the dissipation term 1 and the turbulent transport term 4,

$$-2\nu_o^2 < (\nabla_j \nabla_m v'_i)^2 > -2\nu_o < (\nabla_j v'_i)(\nabla_j v'_m)(\nabla_m v'_i) > \sim -C_\varepsilon^2 \frac{\varepsilon^2}{\kappa} \quad (20)$$

where C_ε^2 is an adjustable constant. A typical value of C_ε^2 is 1.8. The total production is traditionally modeled by the sum of the two remaining terms, 2 and 3,

$$-2\nu_o < (\nabla_j v'_i)(\nabla_m v'_i) > \nabla_j U_m - 2\nu_o < (\nabla_j v'_i)(\nabla_j v'_m) > \nabla_m U_i$$

$$\sim -C_\varepsilon^1 \frac{\varepsilon}{\kappa} \nabla_j U_i < v'_i v'_j > . \quad (21)$$

The constant C_ε^1 is also adjustable. A typical value for C_ε^1 is 1.4.

The simplest model of ε for inhomogeneous turbulence simply restores diffusion and advection by the mean,

$$\frac{\partial \varepsilon}{\partial t} + U_j \nabla_j \varepsilon = C_\varepsilon^1 \frac{\varepsilon}{\kappa} P_\kappa - C_\varepsilon^2 \frac{\varepsilon^2}{\kappa} + \nabla_j \chi_T \nabla_j \varepsilon \quad (22)$$

where χ_T is an eddy diffusivity and $P_\kappa = -\nabla_j U_i < v'_j v'_i >$. The Reynolds stress $< v'_i v'_j >$ is usually modeled by $- < v'_i v'_j > = \nu_T \nabla_j U_i + 2\kappa \delta_{ij}/3$.

Though the parameterizations in (22) are for high Reynolds number turbulence, their signs and general trends are supported by direct numerical simulations of turbulent channel flow (Mansour, Kim, and Moin, 1988). The simulations are necessarily at low Reynolds numbers. In section 6.3, consistency with the simulation data is used to reinterpret the RG-based ε-equation.

6.2. The Yakhot-Orszag RG ε-equation

The goal is to calculate the effect of the small scale velocity field on the large scale variations of ε. The strategy is to assume that the high wavenumber velocity field obeys forced Navier Stokes equations, for examples (1) or (18). The model worked out in I assumes that the high wavenumbers are governed by (1) with homogeneous, isotropic forcing given by (2).

The steps used in I to derive the RG ε-equation are given in CTR Manuscript 106 (Smith, 1989). Many assumptions of the procedure are not explicitly addressed by the authors of I. A large amount of second guessing is required to understand their interpretation of the results. Due to the complexity and vagueness of their method, the steps will not be presented here.

Here we simply state the results reported in I and give corrections. The corrections are to purely mechanical errors and do not address assumptions or interpretation. These more important issues are discussed in section 6.3.

The Yakhot-Orszag high Reynolds number, RG ε-equation is

$$\frac{\partial\varepsilon}{\partial t} + U_j\nabla_j\varepsilon = -1.063\frac{\varepsilon}{\kappa}P_\kappa - 1.7215\frac{\varepsilon^2}{\kappa} + \nabla_j\chi_T\nabla_j\varepsilon \tag{23}$$

where $P_\kappa = -\nabla_j U_i < v_i' v_j' >$ as above. The Reynolds stress is again modeled by $- < v_i' v_j' >= \nu_T\nabla_j U_i + 2\kappa\delta_{ij}/3$. The RG theory gives ν_T and χ_T as functions of ε and κ.

The corrected model, based on the same method, assumptions and interpretation, is

$$\frac{\partial\varepsilon}{\partial t} + U_j\nabla_j\varepsilon = 1.594\varepsilon(a_\infty + \varepsilon b_\infty) + 0.9\frac{\varepsilon^2}{\kappa} + \nabla_j\chi_T\nabla_j\varepsilon \tag{24}$$

where the coefficient of the production term is identically zero (CTR Manuscript 102, 1989). The constants a_∞ and b_∞ are defined by integrals,

$$a_\infty = -2\int_0^\infty d\eta\ \nu_T[\eta]\Lambda[\eta]$$

$$b_\infty = 0.2\int_0^\infty d\eta\ \frac{1}{\nu_T^2[\eta]\Lambda^2[\eta]} \tag{25}$$

where $\nu_T[\eta]$ is given by (3) with $k_c = \Lambda[\eta] = \Lambda_o e^{-\eta}$. The term $1.594\varepsilon(a_\infty + \varepsilon b_\infty)$ has the same scaling as the term $10.5\varepsilon^2/\kappa$, but is of the opposite sign and larger in magnitude. All coefficients in models (23) and (24) are evaluated at the crossover value of the renormalized expansion parameter, $\epsilon = 0$. Models (23) and (24) should be compared with the standard model (22).

6.3. Corrected results reinterpreted

In paper I, the starting point to derive the RG ε-equation is the equation for $\phi \equiv \nu_o(\nabla_j v_i[\mathbf{x},t])^2$. The average of ϕ is the dissipation rate in homogeneous turbulence. The exact equation for ϕ is

$$\frac{\partial \phi}{\partial t} = -v_j \nabla_j \phi + \nu_o \nabla_j \nabla_j \phi - 2\nu_o (\nabla_m \nabla_j v_i)^2 -$$

$$-2\frac{\nu_o}{\rho}(\nabla_j v_i)(\nabla_j \nabla_i P) - 2\nu_o (\nabla_j v_i)(\nabla_j v_m)(\nabla_m v_i) \tag{26}$$

The origin in (26) of the terms in model (24) suggests a reinterpretation consistent with the standard model (22) and direct numerical simulation data for turbulent channel flow. The RG analysis is actually performed on the transform of equation (26) with $\hat{v}_i = \hat{v}_i^< + \hat{v}_i^>$, $\hat{\phi} = \hat{\phi}^< + \hat{\phi}^>$ and $\hat{P} = \hat{P}^< + \hat{P}^>$. The following list gives the origin, Fourier integral and final contribution in real space in the format

*). origin in equation (27)

Fourier integral

$\rightarrow$ final contribution to RG model

a.) a renormalized diffusion term, generated by the $-v_j \nabla_j \phi$ term:

$$-ik_j \int \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \hat{v}_j^>[\hat{\mathbf{q}}] \hat{\phi}^>[\hat{\mathbf{k}} - \hat{\mathbf{q}}]$$

$$\rightarrow \nabla_j \chi_T \nabla_j \phi^<. \tag{27}$$

b.) a contribution from $-2\nu_o(\nabla_m \nabla_j v_i)^2$:

$$-2\nu_o \int \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}} q_m q_j (k-q)_m (k-q)_j \hat{v}_i^>[\hat{\mathbf{q}}] \hat{v}_i^>[\hat{\mathbf{k}} - \hat{\mathbf{q}}]$$

$$\rightarrow \tilde{B}_d \varepsilon (a_\infty + 3\nu_T \Lambda_f^2). \tag{28}$$

c.) contributions from $-2\nu_o(\nabla_j v_i)(\nabla_j v_m)(\nabla_m v_i)$:

$$2i\nu_o \int \frac{d\hat{\mathbf{q}} d\hat{\mathbf{r}} d\hat{\mathbf{p}}}{(2\pi)^{2d+2}} \delta[\hat{\mathbf{k}} - \hat{\mathbf{q}} - \hat{\mathbf{r}} - \hat{\mathbf{p}}] q_j \hat{v}_i^>[\hat{\mathbf{q}}] p_m \hat{v}_i^>[\hat{\mathbf{p}}] r_j \hat{v}_m^<[\hat{\mathbf{r}}] \quad +$$

$$+ \quad 2i\nu_o \int \frac{d\hat{\mathbf{q}} d\hat{\mathbf{r}} d\hat{\mathbf{p}}}{(2\pi)^{2d+2}} \delta[\hat{\mathbf{k}} - \hat{\mathbf{q}} - \hat{\mathbf{r}} - \hat{\mathbf{p}}] q_j \hat{v}_i^>[\hat{\mathbf{q}}] r_j \hat{v}_m^>[\hat{\mathbf{r}}] p_m \hat{v}_i^<[\hat{\mathbf{p}}]$$

$$\rightarrow \tilde{B}_d \varepsilon \nu_T (\nabla_j v_i^<)^2 (b_\infty + \frac{3}{2}\frac{(d^2+d+\epsilon-6)}{2d(d+2)}\frac{1}{\nu_T^2\Lambda_f^2}). \tag{29}$$

In b.) and c.), Λ_f is the integral scale. The relationships between Λ_f, ν_T, ε and κ are

$$\nu_T \Lambda_f^2 = \frac{3}{2} C_K \gamma[\epsilon] \frac{\varepsilon}{\kappa}$$

$$\nu_T = c_\nu[\epsilon]\frac{\kappa}{\varepsilon} \tag{30}$$

where $\tilde{A}_d = (d^2-d-\epsilon)/(2d(d+2))$, $\tilde{B}_d = 1.5\tilde{A}_d/.1904$, $\gamma[\epsilon] = (3\tilde{A}_d\tilde{B}_d/8)^{1/3}$, $c_\nu[\epsilon] = (4\gamma[\epsilon])/(9C_K^2[\epsilon])$ and $C_K[\epsilon]$ is the RG prediction for Kolmogorov's constant.

In the corrected Yakhot-Orszag model described by (24) and (25), the constants were evaluated at the crossover value of the renormalized expansion parameter ϵ. Here the ϵ-dependence is shown explicitly. The definition of b_∞ as a function of ϵ is

$$b_\infty = -\frac{(d^2+d+\epsilon-6)}{2d(d+2)}\int_0^\infty d\eta \, \frac{1}{\nu_T^2[\eta]\Lambda^2[\eta]}. \tag{31}$$

The expression (3) for $\nu_T[\eta]$ and $\Lambda[\eta] = \Lambda_o e^{-\eta}$ may still be used to evaluate a_∞ and b_∞. The coefficient defining a_∞ is not a function of ϵ. The renormalized diffusivity χ_T is a function of ϵ.

The division into $^>$ and $^<$ functions identifies the sub-equation in (26) that generates the renormalized ε-equation. The derivation in I associates $v_i^>$ with v_i', $v_i^<$ with U_i and $\varepsilon = \nu_o < (\nabla_j v_i^>)^2 >= \nu_T < (\nabla_j v_i^<)^2 >$. The contributing sub-equation is then

$$\frac{\partial \varepsilon}{\partial t} = -U_j \nabla_j \varepsilon + \nabla_j \chi_T \nabla_j \varepsilon - 2\nu_o^2 < (\nabla_j \nabla_m v_i')^2 > -$$

$$-2\nu_o < (\nabla_j v_i')(\nabla_m v_i') > \nabla_j U_m - 2\nu_o < (\nabla_j v_i')(\nabla_j v_m') > \nabla_m U_i. \tag{32}$$

Equation (32) is the exact equation (19) for ε without the turbulent transport term and with advection and diffusion restored.

The simulation data for channel flow indicates that the contribution c.) should be a production term. If the Reynolds stress is modeled by $- < v_i' v_j' >= \nu_T \nabla_j U_i$, then $P_\kappa = \nu_T(\nabla_j U_i)^2 = \nu_T < (\nabla_j v_i^<)^2 >= \varepsilon$. Thus, in the context of RG and traditional modeling, we may label $\nu_T < (\nabla_j v_i^<)^2 >$ either P_κ or ε, depending on the sign of its coefficient.

The sign of contribution c.) is positive, in accord with the numerical simulation data. Thus we should identify c.) as

$$1.594\varepsilon P_\kappa(b_\infty[0] - \frac{4.8}{\kappa}), \qquad (33)$$

where b_∞ is larger in magnitude than the $1/\kappa$ term and has the opposite sign.

The interpretation (33) is consistent with standard model (22) and the simulation data for channel flow, and gives

$$\frac{\partial \varepsilon}{\partial t} + U_j \nabla_j \varepsilon = 1.594\varepsilon(a_\infty[0] + \varepsilon b_\infty[0]) +$$

$$+5.7\frac{\varepsilon^2}{\kappa} - 4.8\frac{\varepsilon}{\kappa}P_\kappa + \nabla_j \chi_T \nabla_j \varepsilon. \qquad (34)$$

In equations (33) and (34) all coefficients have been evaluated at $\epsilon = 0$.

7. RG analysis of optimal equations

The Euler-Lagrange (EL) equations governing the optimization of a mean field moment, subject to constraints derived from the Navier Stokes equations, have smooth, ordered solutions. The EL solutions better approximate the ordered features of turbulent flow with each additional constraint. A particular class of EL equations that approximates the equations of shear turbulence has solutions of self-similar, downstream rolls (Busse, 1970). This scale-invariant structure suggests that RG analysis of EL equations may be fruitful.

Well chosen EL equations may adequately capture 'order within disorder' and predict the organized motions observed in real turbulent flows. For example, the size of the smallest downstream roll in the above mentioned solutions is a prediction for the spacing of the streaks near the wall in shear flows. The fact that these EL equations capture self-similar physics indicates that optimal theory and RG theory are different approaches that may sometimes isolate the same phenomena. Perhaps they are complimentary when applied to the turbulence problem.

Optimal theory has until now been restricted to semi-analytically tractable EL equations. Thus, the constraints have been limited to the boundary conditions, continuity and the integral statement of energy balance. A joint project with F. Waleffe is an upper bound formulation based on additional constraints which impose the balance of vorticity. Such a formulation requires numerical solution, but will unquestionably provide better and more accurate information about the ordered structures in turbulent shear flows. (See the CTR 1989 Annual Report by F. Waleffe.)

The optimal equations constrained by the boundary conditions, continuity and the integral statement of energy have the same linear terms as the Navier Stokes equations and different nonlinear terms. Only the nonlinear terms are affected by the addition of more constraints. The existence of nonlinear terms which represent only the ordered, self-similar physics inherent in the Navier Stokes nonlinear terms would be intriguing. The RG method is not limited by nonlinearity, however complicated. Features such as the streak spacing and the slope of the logarithmic layer should be products of RG analysis.

8. Conclusions

There remain unanswered questions about the Yakhot-Orszag theory of turbulence based on renormalization group techniques. Among them are 'What is the meaning of evaluating coefficients at the crossover value of the nondimensionalized expansion parameter?' and 'What is the correct procedure for deriving a model equation for the dissipation rate?'. Extension of the theory to weakly anisotropic flow, and to flow with a mean, may help answer these questions as well as improve eddy viscosity/diffusivity models. Finally, RG analysis of optimal equations may help isolate the ordered features of turbulent flows.

REFERENCES

BUSSE, F., 1970 *J. Fluid Mech.* **41**, 219.

CHEN, S. & KRAICHNAN, R. H. 1989 *Phys. Fluids A.* **1** 2019.

DANNEVIK, P., YAKHOT, V., & ORSZAG, S. A. 1987 *Phys. Fluids* **30**, 2021.

FEIGENBAUM, M. J. 1979 *Journal of Statistical Physics* **21**, no. 6, 669.

FORSTER, D., NELSON, D. R., & STEPHEN, M. J. 1977 *Physical Review A* **16**, no. 2, 732.

FOURNIER, J.-D. & FRISCH, U. 1977 *Physical Review A* **17**, no. 2, 747.

FOURNIER, J.-D. & FRISCH, U. 1983 *Physical Review A* **28**, no. 2, 1000.

KARNIADAKIS, G., YAKHOT, A., RAKIB, S., ORSZAG, S., & YAKHOT, V. 1989 7^{th} Symposium on Turbulent Shear Flows, Stanford University.

KRAICHNAN, R. H., 1987 *Phys. Fluids* **30**, 2400.

KRAICHNAN, R. H., 1989 *Physica D* **37**, 160.

LAUNDER, B. E., REECE, G. L., & RODI, W. 1975 *J. Fluid Mech.* **68**, 537.

LESLIE, D. C. 1973 *Developments in the Theory of Turbulence*, Clarendon Press, Oxford.

LUMLEY, J. L. 1978 *Advances in Applied Mathematics* **18**, Academic Press, New York, 123-176.

MA, S.-K. 1976 Modern Theory of Critical Phenomena, (Frontiers in Physics; 46).

MANSOUR, N. N., KIM, J., & MOIN, P 1988 *J. Fluid Mech.* **194**, 15.

REYNOLDS, W. C. 1987 Fundamentals of Turbulence for Turbulence Modeling and Simulation, Von Karman Institute lecture notes.

RUBINSTEIN, R. & BARTON, J. M. 1987 *Phys. Fluids* **30**, 2987.

SAFFMAN, P. G. 1967 *J. Fluid Mech.* **27**, 581.

SHIH, T.-S. & REYNOLDS, W. C. *to be submitted to J. Fluid Mech.*

SMITH, L. M. 1990 CTR Manuscript 106.

WILSON, K. G. & KOGUT, J. 1974 *Phys. Rep.* **12**, 75.

YAKHOT, V. & ORSZAG, S. A. 1986 *Journal of Scientific Computing* **1**, no. 1, 3.