Organized motions underlying turbulent shear flows

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1. Introduction

The objective of this project is to determine the nature and significance of the organized motions underlying turbulent shear flows. There is considerable experimental evidence for the existence of such motions. In particular, one consistently observes longitudinal streaks with a spacing of about 100 in wall units in the near-wall region of wall-bounded shear flows. Recently, an analysis based on the *direct resonance* mechanism has predicted the appearance of streaks with precisely such a spacing. Also, the *minimum channel* simulations of Jimenez and Moin have given a strong dynamical significance to that spanwise length scale. They have shown that turbulent-like flows can not be maintained when the spanwise wavelength of the motion is constrained to be less than about that critical number.

A critical review of the direct resonance ideas and the non-linear theory of Benney and Gustavsson is presented first. It is shown how this leads to the later *mean flow-first harmonic* theory of Benney. Finally, we note that a different type of analysis has led to the prediction of streaks with a similar spacing. This latter approach consists of looking for *optimum fields* and directly provides deep insights into why a particular structure or a particular scale should be preferred. Extension of past work is proposed.

2. The Direct resonance concept

The full velocity field is separated into a mean $\bar{u}(y)$ and a perturbation. The equation for the mean is obtained by averaging the incompressible Navier-Stokes equations over x, z, t:

$$\frac{1}{R}\frac{d^2}{dy^2}\bar{u} = \frac{\partial}{\partial x}\bar{P} + \frac{d}{dy}\overline{uv} \tag{1}$$

The equations for the perturbations are then derived by subtracting the averaged equations from full Navier-Stokes. Eliminating the pressure and using continuity leads to a set of equations for the remaining 2 degrees of freedom, which correspond to the vertical velocity v and the vertical vorticity η. One finds:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} - \frac{1}{R}\nabla^2\right)\nabla^2 v - \bar{u}''\frac{\partial}{\partial x}v = NL_v \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} - \frac{1}{R}\nabla^2\right)\eta + \bar{u}'\frac{\partial}{\partial z}v = NL_\eta \tag{3}$$

where the non-linear terms, NL_v, NL_η, are given in Jang, et al. (1986). The boundary conditions are $v = \frac{\partial}{\partial y}v = \eta = 0$ at the walls ($y = \pm 1$ say), together with some periodicity conditions in the x and z directions.

In the linear case, one looks for normal mode solutions of the form:

$$v = \hat{v}(y)e^{i(\alpha x+\beta z-\omega t)} \tag{4}$$

$$\eta = \hat{\eta}(y)e^{i(\alpha x+\beta z-\omega t)} \tag{5}$$

The equation for the vertical velocity v, which decouples from the vertical vorticity, is known as the Orr-Sommerfeld (OS) equation. It constitutes an eigenvalue problem. The vertical vorticity is then obtained as the solution of a forced ODE. This can be done by expanding $\hat{\eta}$ in a series based on the eigenmodes of the homogeneous problem (also known as the *Squire* equation). Each term in the expansion is proportional to $(\omega - \omega_i)^{-1}$, where ω is an eigenvalue of the OS equation and ω_i an eigenvalue of the Squire equation associated to an eigenmode $\eta_i(y)$. Of course, this procedure breaks down if any eigenvalue of the Squire equation is identical to the OS eigenvalue. This situation corresponds to a *direct resonance.* The free modes of the vertical vorticity equation are always damped (that equation is simply advection-diffusion with zero boundary conditions and no forcing). In consequence, a direct resonance can only occur for damped modes. But even in the case of near-resonance, there is the possibility that the vertical vorticity attains high amplitudes before the final viscous decay.

The general solution of the time dependent problem for a vertical vorticity of the form $\eta(y,t)e^{i(\alpha x+\beta z)}$ is given by:

$$\eta(y,t) = \beta\lambda_0 \; \frac{e^{-i\omega t} - e^{-i\omega_0 t}}{\omega - \omega_0} \; \eta_0(y) + R(y,t) \tag{6}$$

where

$$\lambda_0 = \frac{\int \bar{u}' \hat{v} \; \eta_0^* \; dy}{\int \eta_0 \eta_0^* dy} \tag{7}$$

The function $R(y,t)$ contains terms of the same form for other eigenmodes η_i plus homogeneous solutions so as to satisfy the initial conditions. We are interested in the amplitude of the forced response:

$$A(t) = \beta\lambda_0 \; \frac{e^{-i\omega t} - e^{-i\omega_0 t}}{\omega - \omega_0} \tag{8}$$

For a direct resonance one gets $A(t) = \beta\lambda_0 \; te^{-i\omega t}$. After a time $t_* = |Im(\omega)|^{-1}$, the amplitude reaches its maximum value given by :

$$max|A| = \frac{\beta|\lambda_0|}{e\;|Im(\omega)|} \tag{9}$$

where the vertical bars denote an absolute value or the norm of a complex number and $Im(\omega)$ is the imaginary part of ω. For near-resonance, a good estimate of the maximum amplitude is given by

$$max|A| \simeq \frac{\beta|\lambda_0|}{e\ max\{|Im(\omega)|, |Im(\omega_0)|\} + |Re(\omega) - Re(\omega_0)|/2} \tag{10}$$

After reaching this maximum, the amplitude decays viscously. The most amplified modes are those for which the phase velocities are nearly equal and *both* damping rates are small.

The interest in this mechanism is that if the gain in amplitude is significant non-linear effects will start to play a role. The direct resonance mechanism might then bridge the gap between linear theory and the observed 3-D non-linear instability in shear flows. This is especially relevant to Couette flow for which there is no known 2-D instability while direct resonances are present for any wavenumber (and correspond to modes moving with the average velocity).

2.1. Non-linear effects

Whether direct resonance is an important mechanism or not depends on the nature of the non-linear interactions which can be triggered. Non-linearity must act quickly enough to prevent the linear viscous decay. One must realize that only the vertical vorticity is amplified by the direct resonance and this limits the possible non-linear effects. The non-linear implications of direct resonances have been investigated by Benney and Gustavsson (1981). The situation is quite different depending on whether one has a single 3-D wave or several.

For a single wave, the vertical velocity remains decoupled from the vertical vorticity. The only non-linear terms in the vertical vorticity equation have the form of an interaction between the vertical velocity and vorticity, but there is no self-interaction of the vertical vorticity. This imposes strong limitations on the non-linear effects. Benney and Gustavsson conclude that if ϵ is a measure of the amplitude of the vertical velocity perturbation, the time scale for the non-linear interactions is ϵ^{-2}, exactly as in classical weakly non-linear analyses of OS waves. This time scale must be shorter than the viscous and phase decorrelation time scales for the finite amplitude effects to act (i.e. one needs $|Im(\omega)|, |Im(\omega_0)|, |Re(\omega) - Re(\omega_0)| < \epsilon^2$, a strong restriction). However, their deduction ignores the interaction between the vertical velocity and vorticity through the mean flow. That interaction occurs on a time scale of $O(\epsilon^{-1})$, much faster than $O(\epsilon^{-2})$. That process is well illustrated by the following exact viscous non-linear solution in an unbounded domain. Consider the flow: $u = \omega_y(t)z - \omega_z(t)y$, $v = -1/2\ \omega_x(t)z$, $w = 1/2\ \omega_x(t)y$, which is a linear flow with time-dependent vorticity. The u-velocity is given by the superposition of two Couette flows. The v and w velocities are that of a rigid body rotation around the x-axis at a rate $1/2\omega_x$. The full vorticity equation, $D_t\vec{\omega} = \vec{\omega}.\vec{\nabla}\vec{v} + \nu\nabla^2\vec{\omega}$, becomes:

$$\begin{aligned} \dot{\omega}_x &= 0 \\ \dot{\omega}_y &= -\frac{1}{2}\omega_x\omega_z \\ \dot{\omega}_z &= \frac{1}{2}\omega_x\omega_y \end{aligned} \tag{11}$$

which implies that $\omega_x(t) = \omega_x(0)$ stays constant. Then if initially the flow is a Couette flow, $\omega_z(0) = -a$, $\omega_y(0) = 0$, on which a small downstream vortex is introduced, $\omega_x(0) = 2\epsilon$, the solution is:

$$\begin{aligned} \omega_y(t) &= a\ \sin(\omega_x(0)t/2) \\ \omega_z(t) &= -\ a\ \cos(\omega_x(0)t/2) \end{aligned}$$

For small times, one has:

$$\begin{aligned} \omega_y(t) &\sim a\ (\epsilon t - (\epsilon t)^3/6) \\ \omega_z(t) &\sim -\ a\ (1 - (\epsilon t)^2/2) \end{aligned}$$

The time scale is indeed of order ϵ^{-1}. In a domain bounded by two infinite horizontal planes the rigid rotation given by ω_x would be replaced by a periodic array of downstream rolls. These downstream rolls would decay on a slow viscous time scale due to the presence of the walls. The initial Couette flow would be maintained by viscous action at the same walls. As a result, one would observe a very "turbulent-looking" mean profile together with some associated streaks.

If there is a direct resonance for (α, β), there is also one for $(\alpha, -\beta)$. It is then necessary to consider the evolution when both waves are present. When several modes are present simultaneously, there is the possibility of a non-linear feedback on the vertical velocity. In that case, Benney and Gustavsson reason that the time scale for the non-linear processes is $\epsilon^{-1/2}$, which is very fast. On that time scale the vertical vorticity and the associated horizontal motions reach an amplitude of order $\epsilon^{1/2}$. Benney and Gustavsson rescale the equations assuming that the horizontal motions are of order $\epsilon^{1/2}$ while the vertical vorticity is of order ϵ. At lowest order, the resulting system consists of the homogeneous vorticity equation and a non-homogeneous equation for the vertical velocity. Although the derivation of these scalings is not available in their paper, one suspects that they proceeded as follows. Starting with a vertical velocity $\epsilon\, v(\alpha, \pm\beta)$, the vertical vorticity is "directly forced" and behaves initially as $\epsilon t\, \eta(\alpha, \pm\beta)$. The non-linear distortions are at least of order $\epsilon^2 t^2$. These distortions might interact with $\eta(\alpha, \pm\beta)$ to induce a feedback on $v(\alpha, \pm\beta)$ of order $\epsilon^3 t^4$, which introduces an $\epsilon^3 t^5$ modification of $\eta(\alpha, \pm\beta)$. Schematically, one gets:

$$v(\alpha, \pm\beta) = \epsilon\ (1 + \epsilon^2 t^4 + \ldots)e^{i(\alpha x \pm \beta z - \omega t)}$$

$$\eta(\alpha, \pm\beta) = \epsilon t(1 + \epsilon^2 t^4 + \ldots)e^{i(\alpha x \pm \beta z - \omega t)}$$

However, due to the necessary requirement of small damping for the direct resonance to lead to significant amplification, one can expect more resonances to appear. In the worst case, the vertical vorticity could resonantly force a $v(0, 2\beta)$ mode (downstream roll with half spanwise wavelength), which then induces a $\eta(0, 2\beta)$ mode (streaks). The non-linear feedback on $v(\alpha, \pm\beta)$ could be as high as order $\epsilon^3 t^6$. This cascade of interactions is represented in the following diagram.

$$\begin{aligned} \epsilon\, v(\alpha, \pm\beta) &\longrightarrow \epsilon t\, \eta(\alpha, \pm\beta) \\ \epsilon^2 t^2\, \eta\eta^* &\longrightarrow \epsilon^2 t^3\, v(0, 2\beta) \\ \epsilon^2 t^3\, v(0, 2\beta) &\longrightarrow \epsilon^2 t^4\, \eta(0, 2\beta) \\ \epsilon^3 t^5\, \eta(\alpha, \pm\beta)\eta(0, 2\beta) &\longrightarrow \epsilon^3 t^6\, v(\alpha, \pm\beta) \end{aligned}$$

The first and third interactions are linear and correspond to "near direct resonances". The second interaction was observed by Jang, Benney and Gran, it is further discussed below. The fourth interaction has not yet been explicitly established. If this scenario takes place, the correct expansion would rather be:

$$v(\alpha, \pm\beta) = \epsilon\, (1 + \epsilon^2 t^6 + \ldots)e^{i(\alpha x \pm \beta z - \omega t)}$$

$$\eta(\alpha, \pm\beta) = \epsilon t(1 + \epsilon^2 t^6 + \ldots)e^{i(\alpha x \pm \beta z - \omega t)}$$

implying a non-linear time scale of order $\epsilon^{-1/3}$.

3. Applications

Gustavsson has looked for and found direct resonances for laminar Couette, plane, and pipe Poiseuille flows. No exact resonances were found for laminar boundary layer profiles; however, Jang, Benney and Gran (1986) found one for a *turbulent* boundary layer profile. The use of the theory for a turbulent profile is more delicate to justify, as in that case finite perturbations must exist to maintain the turbulent mean. Yet, considering the linear perturbation equations around a turbulent mean can be seen as an effort to determine a "proper eigenmodal decomposition" of the fluctuating field. Kim has located several near-resonances in the case of a turbulent channel flow profile. We are now confronted with a selection problem. Which of these near resonances, if any, is the relevant one?

The first, and only, resonance found by Jang, Benney and Gran corresponds to a wavenumber intriguingly close from the peak of experimentally measured power spectral distributions. In addition, the vertical velocity motion induced by the non-linear interaction of the vertical vorticity with itself corresponds to a downstream roll with a spacing of 90 in wall units. This is a very interesting nonlinear process which gives a mechanism to generate streamwise vorticity from vertical vorticity. Longitudinal streaks are then introduced by the interaction of this streamwise vortex with the mean profile. Physically, the large horizontal motions coming from the large vertical vorticity induced by the direct resonance create downstream and spanwise vorticity ($\frac{\partial}{\partial y}w$ and $\frac{\partial}{\partial y}u$) as a consequence of the no-slip condition at the walls. These vorticity components are stretched ($\frac{\partial}{\partial y}w\frac{\partial}{\partial x}u$) and rotated ($-\frac{\partial}{\partial x}w\frac{\partial}{\partial y}u$), respectively, as can be deduced from the equation for the x-vorticity:

$$\frac{D}{Dt}\omega_x = \omega_x\frac{\partial}{\partial x}u + \omega_y\frac{\partial}{\partial y}u + \omega_z\frac{\partial}{\partial z}u + \nu\nabla^2\omega_x$$

neglecting $\frac{\partial}{\partial z}v$ and $\frac{\partial}{\partial x}v$ in the expressions for ω_x and ω_z, one finds:

$$\frac{D}{Dt}\omega_x \simeq \frac{\partial}{\partial y}w\frac{\partial}{\partial x}u - \frac{\partial}{\partial x}w\frac{\partial}{\partial y}u + \nu\nabla^2\omega_x$$

Mathematically this process translates into the non-linear forcing of a $v(0,2\beta)$ vertical velocity mode. This mechanism is particularly relevant to the studies of John Kim (1983). One emphasizes that according to the mechanism explained above, the downstream vorticity is generated from the vertical vorticity rather than from a "splatting" effect (Kim, 1983).

From the nature of the non-linear interactions, the streamwise vortex always has twice the spanwise wavenumber of the 3-D vertical vorticity which generated it. Thus *double* pairs of counter-rotating vortices should be observed if this process is relevant. The "minimum channel" simulations of Jimenez and Moin, show that "turbulence" can be maintained with only *one* pair of counter-rotating streamwise vortices. This would imply that the mechanism for their generation can not come from the non-linear interaction of the vertical vorticity with itself as proposed by Jang, Benney and Gran. More cautiously, there must be another mechanism for their creation.

4. Mean flow-first harmonic model

The appearance of the new resonances discussed above imposes some significant modifications to the non-linear theory of Benney and Gustavsson. It is necessary to reformulate the problem in order to account for the intrinsic spanwise modulation of the mean flow. Steps in that direction have been taken by Benney and Chow (1989). These authors have formulated a *mean flow-first harmonic* theory where the mean varies in both the vertical and spanwise direction and the perturbation is composed of only one downstream fourier mode. No extensive analysis of the solutions of these equations have yet been made. This self-contained theory is still in a primitive state. It seems that some careful numerical simulations could test the validity of this approach. This mean flow-first harmonic theory is in some sense based on an idea of triad resonances between modes of the form $(\alpha, \pm\beta)$ and $(0, 2\beta)$, but it could also describe interaction between $(\alpha, \pm 2\beta)$, $(0, \pm 2\beta)$ and $(\alpha, 0)$. In other words, in the case of a mean + a spanwise mode $(0, 2\beta)$, one might find both *fundamental* and *subharmonic* instabilities (just as for a mean + a 2-D Tollmien-Schlichting $(\alpha, 0)$ wave). The fundamental instability could be relevant to the minimum channel simulations. Such instabilities of a spanwise periodic basic state might be the other side of the 3-D instability of a downstream periodic basic flow. The question is which basic state should be studied? Laminar Couette flow modified by its slowest decaying downstream roll eigenmode is a good candidate. The same basic state could be chosen for channel flow, or, alternatively, a state generated from the computed turbulent profiles could be used. This state would be obtained by averaging the full-field over the downstream x direction and time.

In the mean flow-first harmonic theory, one hopes that the waves developing on a spanwise varying basic state are such that their non-linear interactions maintain the mean and especially the downstream rolls. It seems more likely to the present author that the downstream structures would rather be formed by a 3-D instability of the developing wave (i.e. the 3-D instability of a mean + a downstream mode $(\alpha, 0)$; the elliptical instability).

5. Optimum fields

A related investigation is to determine *optimum* perturbation fields maintaining the mean and being chosen so as to maximize various mean moments (e.g. production) under some critical constraints derived from Navier-Stokes. In this approach as in the mean field-first harmonic theory, the mean flow equations are exact while approximations are made on the fluctuation equations. Busse (1978), for instance, showed that the field which maximizes the averaged Reynolds stresses, while maintaining the mean and satisfying the boundary conditions, the incompressibility constraint and an energy constraint, corresponds to a downstream roll-streak structure with a spacing of about 50 in wall units. Without a doubt, a numerical investigation including additional constraints will improve this value. The advantages of this approach is that it is mathematically rigorous and gives some definite physical insights such as what are the important constraints on the real motions, and why a particular structure is observed. It is an excellent way of getting the organized motions in a turbulent field. Once a solution is found, it can then serve as the basis for a new expansion or analysis. Busse's solution, for instance, could serve as the basic state in the 3-D stability calculations referred to above.

The following question could be quite relevant to the minimum channel simulations. Given the computed turbulent velocity profile, what is the most "efficient" way of maintaining it? By "efficient" we mean, for example, that the ratio of the total average turbulent energy production to the total average kinetic energy of the fluctuations is maximized. Of course, one could look for other optima (such as max average Reynolds stresses over rms fluctuations). Mathematically, the problem is that of determining the maximum of:

$$\frac{-<\overline{uv}\frac{d}{dy}\bar{u}>}{<u^2+v^2+w^2>} \tag{12}$$

where the brackets $<.>$ stand for an average over all variables x, y, z, t. The overbar, as before, is an average over x, z, t. We request that this optimum fluctuating field maintains the mean, that is :

$$\frac{1}{R}\frac{d^2}{dy^2}\bar{u}=\frac{\partial}{\partial x}\bar{P}+\frac{d}{dy}\overline{uv} \tag{13}$$

The optimum field should satisfy the incompressibility constraint: $\vec{\nabla}.\vec{u}=0$, and the boundary conditions. Finally, we impose that it also satisfies the energy constraint that, for a statistically steady state, the turbulent energy production is equal to the dissipation rate. This reads:

$$-<\overline{uv}\frac{d}{dy}\bar{u}>=\frac{1}{R}<(\vec{\nabla}u)^2+(\vec{\nabla}v)^2+(\vec{\nabla}w)^2> \tag{14}$$

As the turbulent profile and the Reynolds number are imposed, this last constraint implies that we are maximizing the functional:

$$\frac{<(\vec{\nabla}u)^2+(\vec{\nabla}v)^2+(\vec{\nabla}w)^2>}{<u^2+v^2+w^2>} \tag{15}$$

For this type of problem, one knows from Busse's work that the optimum field corresponds to x-independent structures, thus we are really maximizing the spanwise wavenumber. The question has thus become: what is the smallest spanwise wavelength which could maintain the turbulent mean? The equations for the optimum fluctuating field are obtained from variational calculus, after some manipulations they read:

$$[\lambda-\frac{1}{R}(\frac{d^2}{dy^2}-\beta^2)]\,(\frac{d^2}{dy^2}-\beta^2)\;\hat{v}(y)=\beta^2\lambda_2(y)\hat{u}(y) \tag{16}$$

$$[\lambda-\frac{1}{R}(\frac{d^2}{dy^2}-\beta^2)]\;\hat{u}(y)=-\lambda_2(y)\hat{v}(y) \tag{17}$$

with the boundary conditions: $\hat{v}=d\hat{v}/dy=\hat{u}=0$ at $y=\pm1$. The *Lagrange multipliers* λ and $\lambda_2(y)$ are determined from the constraints that $<(\vec{\nabla}\vec{u})^2>$ and $\overline{uv}$ have fixed values. This is a fairly simple numerical problem. One will note the strong similarity between this system and the OS and vertical vorticity equations. The most important difference is that we now have a production term for the downstream roll (v). This term models some optimum process maintaining the rolls.

5.1. Improvements

The most efficient way of subtracting energy from the mean flow corresponds to x-independent structures. Conservation of energy was insured but no further constraint was imposed on how this energy should be spread among the 2 degrees of freedom v and u, i.e. among downstream rolls and streaks. However, we know that there are strong constraints on such a process. Indeed, if one introduces downstream rolls into the flow they are very efficient at taking energy out of the mean, but all that energy goes into the streaks. These streaks are themselves precisely determined by the mean profile and the rolls.

One way of improving the results is to proceed as in the mean field-first harmonic theory and recognize that the mean should have an intrinsic spanwise variation ($u(y,z)$), with associated downstream rolls (v, w motions). The problem is then to determine what optimum fluctuations could maintain such a mean. The fluctuations will now be x-dependent. This problem will predict a mean profile, streak spacing *and* an optimum x-scale.

A simpler alternative has been proposed by Malkus (1967). Instead of considering a different mean, the idea is to impose more constraints on the fluctuations. Malkus' suggestion is to include the equation for the total streamwise enstrophy. This should give insight into the mechanism of production of streamwise vorticity. Yet another way is to split *a priori* the fluctuating field into its 2 degrees of freedom and impose energetic constraints for both of them simultaneously. In this way, the repartition of the turbulent energy production among the 2 degrees of freedom would be imposed from Navier-Stokes, instead of being freely determined by the variational problem.

Dr. Leslie Smith has been working on related topics, and these optimum fields projects will be realized with her collaboration.

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