Turbulence dynamics in the wavelet representation

By C. Meneveau

The phenomenon of small-scale intermittency is shown to motivate the decomposition of the velocity field into modes that exhibit both localization in wavenumber and physical space. We review some basic properties of such a decomposition, called the wavelet transform. The wavelet-transformed Navier-Stokes equations are derived, and we define a new quantity $\Pi(r, \vec{x}, t)$, which is the flux of kinetic energy to scales smaller than r at position $\vec{x}$ (at time t). Then, the main goals of this research are summarized.

1. Introduction

One of the most important features of a turbulent flow is the transfer of kinetic energy from large to small scales of motion. For isotropic and homogeneous turbulence, the three-dimensional energy spectrum $E(k, t)$ obeys

$$\frac{\partial E(k,t)}{\partial t} = T(k,t) - 2\nu k^2 E(k,t), \tag{1}$$

where $T(k, t)$ is the net transfer of energy through wavenumbers of magnitude k. The total spectral flux of energy through wavenumber k to all smaller scales is given by

$$\Pi(k,t) = \int_k^\infty T(k',t)dk'. \tag{2}$$

Usually the mechanism of energy transfer is visualized by simplified models such as the successive break-down of 'eddies', or as the creation of small scales by the stretching and folding of vortical elements. One then argues that through scales of motion of size k^{-1}, there is a net flux of kinetic energy to smaller scales, which is equal to the time average of $\Pi(k, t)$. Notice that $\Pi(k, t)$ does not depend on position because of the Fourier representation used to obtain Eq. (2). If one now wishes to reconcile this definition of a 'flux' of energy to smaller scales with the phenomenological picture of breakdown of eddies, one needs to tacitly make the assumption that its average value is indeed physically representative of the underlying physics in any regions of space. In some loose sense, this then corresponds to the theory of Kolmogorov (1941), which neglects the phenomenon of intermittency. Of course, it has been known for a long time that the rate of dissipation $\varepsilon(x, t)$ is distributed very intermittently (Batchelor and Townsend

1949), a behavior which increases with the Reynolds number of the flow. Also, its moments increase with Reynolds number according to power-laws in the inertial range of turbulence. Among others, this permits a self-consistent statistical and geometrical representation of ε in terms of multifractals (Kolmogorov 1962, Novikov 1971, Mandelbrot 1974, Frisch and Parisi 1985, Meneveau and Sreenivasan 1987a, 1987b, 1989). The observation of power-law behavior of spatial moments of the dissipation can be modelled again rather naturally within the framework of breakdown of eddies, but now assuming that the flux of energy to smaller scales exhibits spatial fluctuations. These fluctuations accumulate as the scales of motion become smaller, and can lead to very intermittent distributions of the dissipation displaying power-law behavior. This suggests the need for defining a flux of kinetic energy to smaller scales which, as opposed to Eq. 2, should retain some degree of spatial locality.

In a very interesting paper, Kraichnan (1974) proposed to decompose the velocity field into band-limited contributions according to

$$u_i^m(\vec{x},t) = (2\pi)^{-d} \int_{|k|=2^m}^{2^{m+1}} \hat{u}_i(\vec{k},t) e^{i\vec{k}\cdot\vec{x}} d^3k, \tag{3}$$

where

$$\hat{u}_i(\vec{k},t) = \int_{-\infty}^{\infty} u_i(\vec{x},t) e^{-i\vec{k}\cdot\vec{x}} d^3x. \tag{4}$$

The equation of motion of $u_i^m(\vec{x},t)$ can be deduced from the Navier-Stokes equations, and multiplying the result by $u_i^m(\vec{x},t)$ gives the evolution equation of $[u_i^m(\vec{x},t)]^2$ which can be interpreted as the kinetic energy occurring in a wavenumber band around 2^m, at position $\vec{x}$. The result is

$$(\frac{\partial}{\partial t} - \nu\nabla^2)[u_i^m(\vec{x},t)]^2 = T^m(\vec{x},t), \tag{4}$$

where

$$T^m(\vec{x},t) = \frac{-i}{2(2\pi)^d} u_i^m(\vec{x},t) \int_{|k|=2^m}^{2^{m+1}} P_{ijk}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \int_{\vec{q}} \hat{u}_j(\vec{q}) \hat{u}_k(\vec{k}-\vec{q}) d^3q\, d^3k. \tag{5}$$

Here $P_{ijk}(\vec{k})$ is the usual divergence-free projection operator. In analogy to Eq. (2), Kraichnan (1974) then defined a flux of kinetic energy to smaller scales as

$$\Pi^m(\vec{x},t) = \sum_{n=m}^{\infty} T^n(\vec{x},t), \tag{6}$$

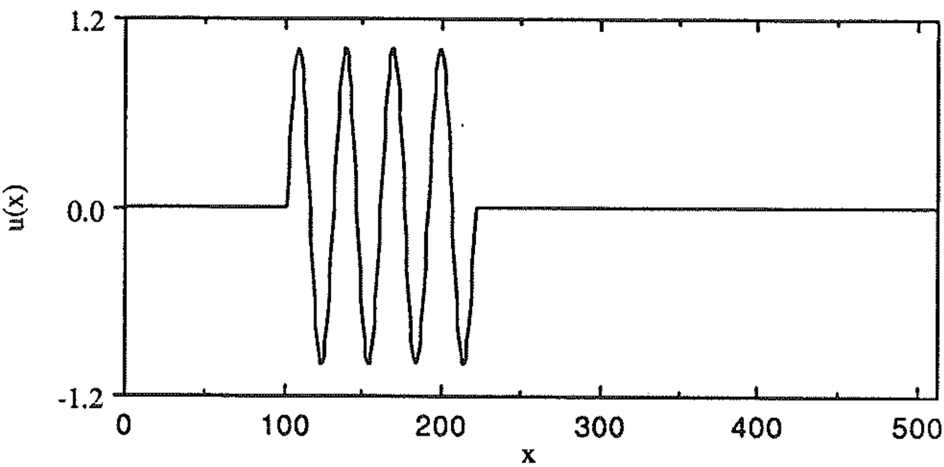

FIGURE 1 (A). Signal $u(x)$ displaying oscillations of a single scale in a confined spatial region.

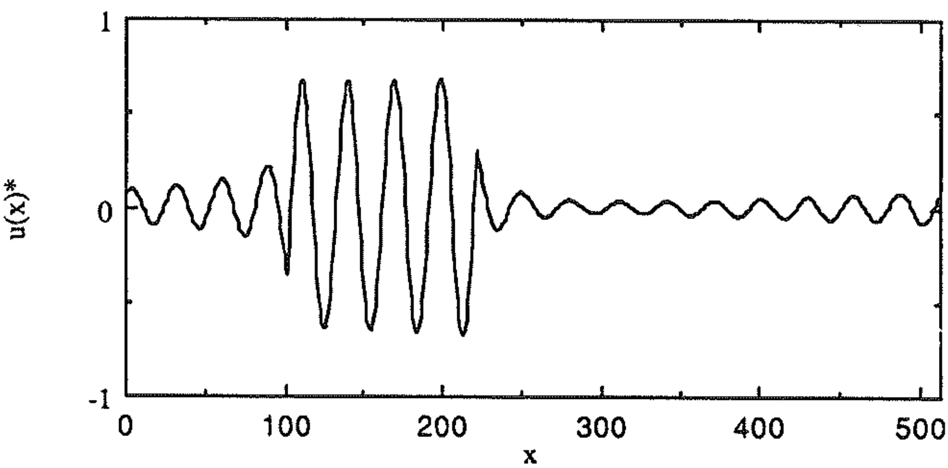

FIGURE 1 (B). High-pass filtered version of the signal $u(x)$. The filtering here consists of cutting off all discrete Fourier modes of scales larger than 30.

which is now a position-dependent quantity because of the band-pass filtering.

However, filtering using Fourier modes can be dangerous in the following sense. Take for instance the signal of Fig. (1a), where an oscillation of wavelength $\lambda = 30$ is confined to a certain region of space. This could be thought of as an extreme case of intermittency, where at a given scale λ all activity is confined to a subregion of space only. If we now high-pass filter the signal up to scales equal to ~ 30, we get the signal of Fig. (1b). It is apparent that the elimination of modes at scales larger than 30, some of which were needed to cancel the oscillations outside the domain of activity, has resulted in spreading the 'activity' everywhere. This is because of the non-local nature of the Fourier modes.

This motivates the study of bases that retain locality both in wavenumber and position space. Their use in describing turbulence dynamics is the primary goal of this research, with special emphasis on the spatial characteristics of the transfer of energy to smaller scales and the implications on intermittency. The formalism will then be applied to numerical data bases of turbulent flows.

The theory and applications of the so-called wavelet bases, which are local in wavenumber and position space, has recently generated much interest (for a detailed account, see Daubechies 1988). Wavelets are currently used for speech and image processing (Mallat 1989, Kronland-Martinet et al. 1987), and can be used to describe affine coherent states in quantum mechanics (Paul 1985). The work of Siggia (1977) and Nakano (1988) attempt to describe turbulence using wavepackets, which display several similitudes with wavelets. Explicitly, the potential use of wavelets in turbulence has been pointed out in the context of coherent structures (Farge and Rabreau 1988) as well as in studies of its fractal nature (Argoul et al. 1989), even though their claim that it has proven the Richardson cascade based on single hot-wire measurements appears to be premature.

Section 2 defines the (continuous) wavelet transform of a signal, and reviews several of its properties. Section 3 defines the flux of kinetic energy to smaller scales using the wavelet representation, and also derives the wavelet-transformed Navier-Stokes equations. Section 4 contains some practical considerations related to the implementation of the discrete version of the wavelet transform, and its generalization to three dimensions. Section 5 summarizes the future objectives of the present research.

2. The wavelet transform

Given a signal $u(x)$, its wavelet transform is defined as

$$W(r,x)\{u\} = C_g^{-\frac{1}{2}} r^{-\frac{1}{2}} \int_{-\infty}^{\infty} g\left(\frac{x'-x}{r}\right) u(x')dx', \tag{7}$$

where $g(s)$ is a function called wavelet, satisfying the admissibility condition

$$C_g = \int |\omega|^{-1} |\hat{g}(\omega)|^2 d\omega < \infty. \tag{8}$$

Here $\hat{g}(\omega)$ is the Fourier transform of $g(s)$. $g(s)$ is of zero mean, will have some oscillations and will usually be real. A typical example is the mexican hat $g(s) = (1-s^2)e^{-s^2/2}$, which can approximately be viewed (Coifman 1989) as the difference between two exponentials of different sizes centered around $s = 0$. Therefore, $W(r,x)$ can be regarded as the relative contribution of scales r to the signal at position x. If $g(s)$ obeys the above conditions, the wavelet transform can be inverted (Grossmann and Morlet 1984). The inversion formula for the wavelet transform reads

$$u(x) = C_g^{-\frac{1}{2}} \int_0^{\infty} \int_{-\infty}^{\infty} r^{-\frac{5}{2}} g(\frac{x-x'}{r}) W(r,x')\{u\} dx' dr. \tag{9}$$

$W(r,x)$ can also be obtained from $\hat{u}(k)$, the Fourier transform of u(x) according to

$$W(r,x)\{u\} = C_g^{-\frac{1}{2}} (2\pi)^{-1} r^{\frac{1}{2}} \int_{-\infty}^{\infty} \hat{g}(rk)^* \hat{u}(k) e^{ixk} dk, \tag{10}$$

where $\hat{g}(\omega)$ is the Fourier transform of $g(s)$. The total energy of the signal is given by

$$\int u(x')^2 dx' = C_g^{-1} \int_0^{\infty} \int_{-\infty}^{\infty} r^{-2} [W(r,x)\{u\}]^2 dr dx. \tag{11}$$

One can also compute $\hat{u}(k)$ from $W(r,x)\{u\}$ using

$$\hat{u}(k) = C_g^{-\frac{1}{2}} (2\pi)^{-1} \int_0^{\infty} \int_{-\infty}^{\infty} r^{-\frac{3}{2}} \hat{g}(rk) e^{ixk} W(r,x)\{u\} dx dr. \tag{12}$$

The wavelet transform commutes with differentiation in the spatial variable, namely

$$\frac{\partial}{\partial x} W(r,x)\{u\} = W(r,x)\{\frac{\partial}{\partial x'} u(x')\} \tag{13}$$

For vector functions $\vec{u}(x)$ with components $u_i(x)$, the transform is a vector $\vec{W}(r,x)$ whose components are the transforms of the components of $\vec{u}(x)$.

For functions defined in higher dimensions, it is recommendable to use decomposable wavelets. In three dimensions we use

$$g(\vec{s}) = g(s_1, s_2, s_3) = g_1(s_1) g_2(s_2) g_3(s_3). \tag{14}$$

One can then prove the following useful relations:

$$\vec{\nabla}_{\vec{x}} \cdot \vec{W}(r,\vec{x})\{\vec{u}(\vec{x}')\} = W(r,\vec{x})\{\vec{\nabla}_{\vec{x}'} \cdot \vec{u}(\vec{x}')]\} \tag{15}$$

and

$$\vec{\nabla}_{\vec{x}} \vec{W}(r,\vec{x})\{\vec{u}(\vec{x}')\} = W(r,\vec{x})\{\vec{\nabla}_{\vec{x}'} \vec{u}(\vec{x}')\}. \tag{16}$$

3. Wavelet representation of turbulence dynamics

Let us define $W_i(r,\vec{x},t)$ as the wavelet transform of the velocity field $u_i(\vec{x},t)$. (From here on we simplify the notation by using $W_i(r,\vec{x},t)$ instead of $W_i(r,\vec{x},t)\{u_i\}$). Because of Eq. (15), the incompressibility condition reads

$$\vec{\nabla}_{\vec{x}} \cdot \vec{W}(r,\vec{x},t) = 0. \tag{17}$$

Multiplying the Fourier-transformed Navier-Stokes equations by $C_g^{-\frac{1}{2}} r^{\frac{d}{2}} (2\pi)^{-d} \times \hat{g}(r\vec{k})^* e^{i\vec{x}\cdot\vec{k}}$, integrating over wavenumber space and using Eq. (12) gives

$$(\frac{\partial}{\partial t} - \nu\nabla^2)[W_i(r,\vec{x},t)] =$$

$$\int_{r'}\int_{r''}\int_{\vec{x}'}\int_{\vec{x}''} W_j(r',\vec{x}',t) W_k(r'',\vec{x}'',t) I_{ijk}(r,\vec{x};r',r'',\vec{x}',\vec{x}'') dr' dr'' d^3x' d^3x'', \tag{18}$$

where

$$I_{ijk}(r,\vec{x};r',r'',\vec{x}',\vec{x}'') =$$

$$-\frac{ir^{\frac{d}{2}}}{2C_g^{\frac{1}{2}}(2\pi)^{3d}(r'r'')^{\frac{1+d}{2}}} \int_{\vec{k}} \hat{g}(r\vec{k})^* P_{ijk}(\vec{k}) e^{i\vec{k}\cdot(\vec{x}+\vec{x}'')}$$

$$\times \int_{\vec{q}} \hat{g}(r'\vec{q})\hat{g}(r''(\vec{k}-\vec{q})) e^{i\vec{q}\cdot(\vec{x}'-\vec{x}'')} d^3q\, d^3k. \tag{19}$$

This illustrates that there are now interactions of the $W_i(r,\vec{x},t)$ occurring at different positions as well as different scales. These non-local and inter-scale interactions are dictated by the properties of $I_{ijk}(r,\vec{x};r',r'',\vec{x}',\vec{x}'')$. Additionally, one can, of course, apply the wavelet transform in time.

Of more immediate interest is to define a quantity analogous to Eq. (6) in the wavelet representation. For this we start with Eq. (19) and multiply by $W_i(r,\vec{x},t)$ and express the right-hand-side as a function of the velocity field. We obtain

$$(\frac{\partial}{\partial t} - \nu\nabla^2)[W_i(r,\vec{x},t)]^2 = T(r,\vec{x},t), \tag{20}$$

where

$$T(r,\vec{x},t) =$$

$$-\frac{ir^d}{2(2\pi)^{2d}}[\int_{\vec{p}} \hat{g}(r\vec{p})^* \hat{u}_i(\vec{p}) e^{i\vec{p}\cdot\vec{x}} d^3p] \int_{\vec{k}} P_{ijk}(\vec{k}) \hat{g}(r\vec{k})^* e^{i\vec{k}\cdot\vec{x}}$$

$$\times [\int_{\vec{q}} \hat{u}_j(\vec{q}) \hat{u}_k(\vec{k} - \vec{q}) d^3 q] d^3 k. \tag{21}$$

The flux of kinetic energy to all smaller scales can then be defined as

$$\Pi(r, \vec{x}, t) = \int_0^r T(r', \vec{x}, t) dr'. \tag{22}$$

Therefore, given the Fourier transform of the velocity field, the quantity $\Pi(r, \vec{x}, t)$ can be computed. Other quantities whose spatial distribution is of interest is the dissipation term

$$\varepsilon(r, \vec{x}, t) = \frac{\nu}{2} [\frac{\partial W_i}{\partial x_j} + \frac{\partial W_j}{\partial x_i}]^2. \tag{23}$$

Assuming constant mean shear, the production term is

$$P(r, \vec{x}, t) = W_i(r, \vec{x}, t) W_j(r, \vec{x}, t) S_{ij}, \tag{24}$$

where S_{ij} is the mean rate of strain.

4. Wavelet bases and discretization

There are many possible choices for the wavelet $g(s)$. The simplest is the Haar function $g(s) = 2^{\frac{1}{2}}$ for $0 \leq s \leq \frac{1}{2}$ and $g(s) = -2^{\frac{1}{2}}$ for $\frac{1}{2} < s \leq 1$. Another is the mexican hat mentioned in section 2. In terms of the discretization of the transform, assume that one has a signal on a discrete grid consisting of N points. One possibility is to space r logarithmically and 'slide' the spatial variable over all N points of the signal. In such a case one obtains of the order of $N log N$ values of the transform. This is what has generally been used in qualitative studies, such as by Kronland-Martinet et al. (1987) and Argoul et al. (1989). The fact that the transform consists of more points than the original signal comes from the non-orthogonality of the wavelet functions in such a case.

Intuitively, for larger values of r one could use a coarser spatial grid than for smaller values of r. This observation has led (see Mallat 1989) to the definition of basis functions of the form

$$g^{m,i}(x) = g(\frac{x - i b_0 a_0^m}{a_0^m}) = g(a_0^{-m} x - i b_0), \tag{25}$$

where a_0 and b_0 are dilation and translation steps. Notice that now the translation depends on the dilation, both being logarithmically spaced. Choices for a_0 and b_0 are not completely arbitrary (Daubechies 1988); here we will use the simplest case $a_0 = 2$ and $b_0 = 1$. Notice that the Haar basis with such a choice of a_0 and b_0 constitutes an orthonormal system, because

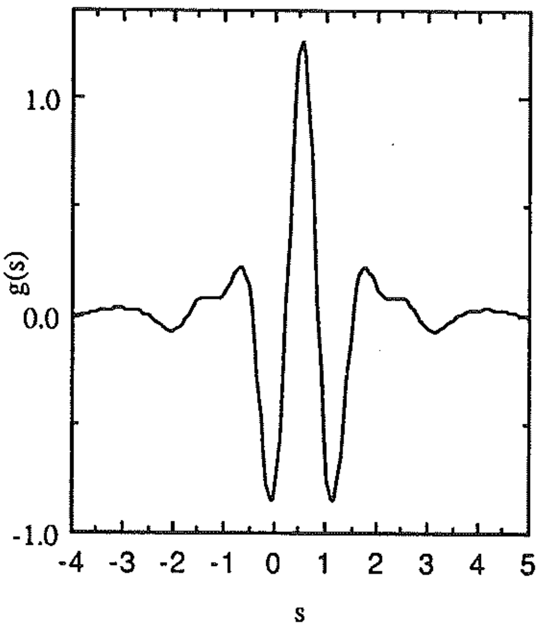

FIGURE 2. Lemarie-Battle wavelet with exponential decay in physical space. For a method of constructing such a wavelet, see Mallat(1989).

$$\int g^{m,i}(x) g^{n,j}(x) dx = \delta_{mn}\delta_{ij}. \tag{26}$$

The discrete wavelet coefficients of a continuous function u(x) are defined as

$$W^{m,i} = \int g^{m,i}(x) u(x) dx, \tag{27}$$

and the (discrete) reconstruction formula is the wavelet series expansion of $u(x)$

$$u(x) = C_g^{-\frac{1}{2}} \sum_m \sum_i 2^{-\frac{m}{2}} W^{m,i} g^{m,i}(x). \tag{28}$$

In practice, $u(x)$ itself is discrete and the integration in Eq. (27) needs to be replaced by a sum. In the formulation to be adopted here, the discrete samples $u(x_n)$ are viewed (Mallat 1989, Daubechies 1988) as resulting from the convolution of $u(x)$ with a function $\phi_{0,n}(x)$ according to

$$u(x_n) = \int u(x) \phi_{0,n}(x) dx. \tag{29}$$

It turns out that the conditions of orthonormality of the entire wavelet basis (as well as several other considerations) are related to the properties of $\phi_{0,n}(x)$

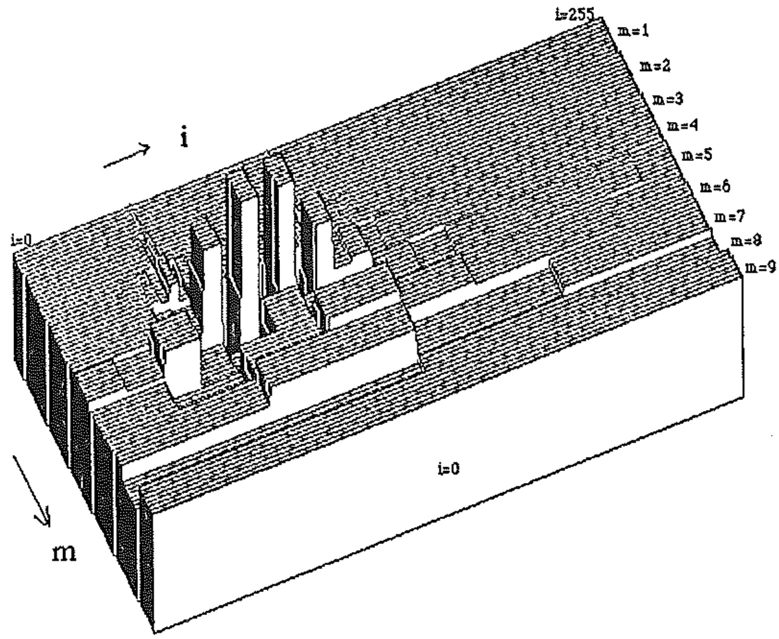

FIGURE 3. Wavelet transform of the signal of Fig. 1(a) using the Lemarie-Battle wavelet and the fast algorithm of Mallat (1989). The index m denotes the scale and runs from $m = 1$ to $m = log_2 N = 9$. The index i runs from 0 to $2^{-m}N - 1$. The spatial resolution thus decreases as m increases. The total number of values of the transform is $N - 1$, and for the decomposition to be complete, one also needs to know, say, the mean of the signal.

(Mallat 1989, Daubechies 1988). For instance, the use of such a formulation naturally leads to an algorithm to compute fast wavelet transforms (FWT).

Several issues other than orthonormality need to be taken into account when deciding which wavelets to use. One very important issue is the degree of locality. The Haar system is very well localized in space (it has compact support in $[0, 1]$), but has very poor spectral locality. This is a disadvantage, because we would like the wavelet coefficients corresponding to a certain scale r to be large only when the signal actually contains oscillations of that scale. In other words, one is interested in fast decay both in wavenumber and position space. A very convenient function complying with the conditions of discrete orthonormality was discovered by Lemarie and Battle (see Mallat 1989). This function decays as k^{-4} in wavenumber space and exponentially in physical space, and was used by Mallat (1989) for image analysis. Figure 2 shows this function. Figure 3 displays the discrete wavelet transform of the signal of Fig. 1a. Notice the spacing that becomes more coarse-grained as the dilation factor $r = 2^m$ increases. The transform peaks near $m = 5$ (corresponding to a scale $\lambda = 32$) only in the vicinity of the oscillations of the signal. Inverting the transform for scales up to 32 ($m = 0$ to 5) gives the signal of Fig. 4. Since the wavelet coefficients away

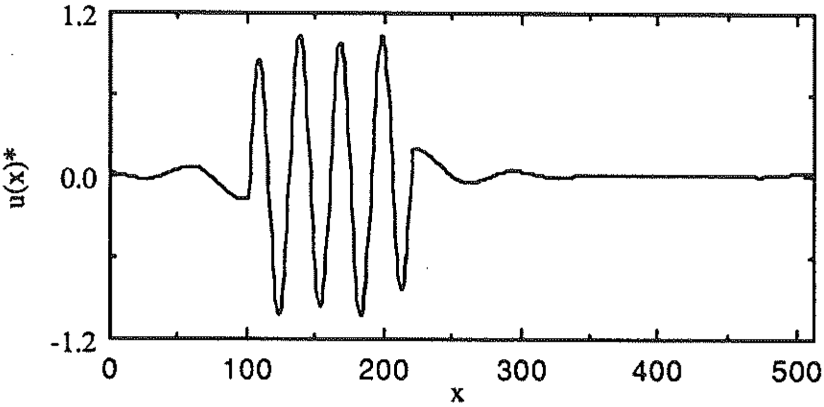

FIGURE 4. High-pass filtered version of the signal of Fig. 1(a) using scales corresponding to $m = 1$ to 5. Here we have applied the (discrete) inverse-wavelet transform algorithm of Mallat (1989).

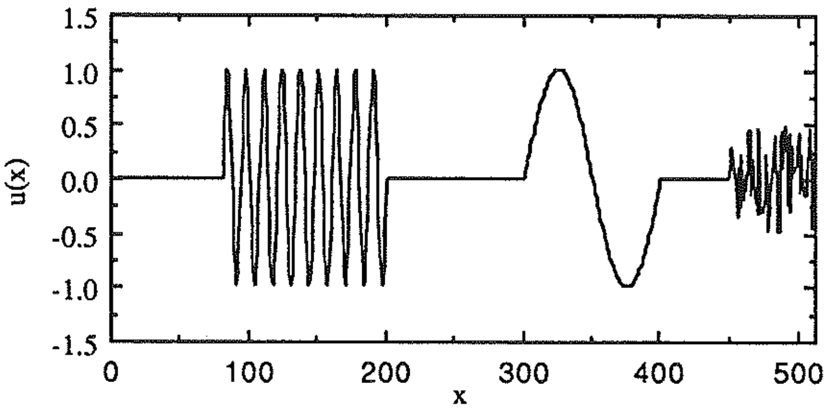

FIGURE 5. Signal displaying oscillations of different frequencies at different locations, as well as random numbers (right portion).

from the activity are very small, there is little risk in incurring the problems that occurred with the Fourier representation (see Fig. 1b).

Figure 5 shows another function consisting of oscillations of different scales located at different positions. Figure 6 is its discrete wavelet transform. Figures 7 and 8 correspond to high-pass and low-pass filtered versions of the signal. The wavelet transform is seen to separate events of different scales in a fashion which respects their location in space.

Even though $g(s)$ of the Lemarie-Battle wavelets has fast decay in space, it has non-local support (i.e. $g^{m,i}(x) \neq 0$ even at large $| x |$). If one were to

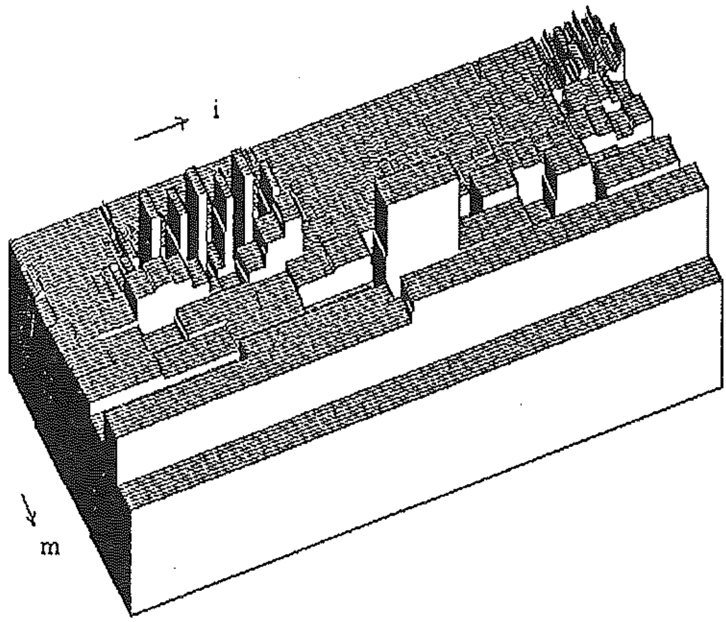

FIGURE 6. Wavelet transform of Fig. 5 using the Lemarie-Battle wavelet and the fast algorithm of Mallat (1989). Notice the localization in both wavenumber and physical space of the different events.

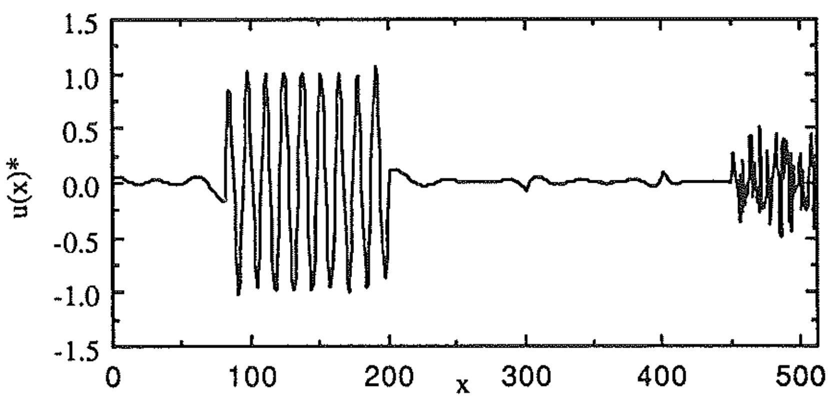

FIGURE 7. Reconstruction of the signal using scales between $m = 1$ and $m = 4$ (high-pass filtering).

set it to zero after some value of $| x |$, then the discrete orthonormality is not exactly obeyed. In other words, finite domain truncation leads to a loss of discrete orthonormality. Daubechies (1988) shows that one can construct orthonormal wavelets of compact support which are different from the Lemarie-Battle wavelets. However, such wavelets do not possess symmetry (Daubechies

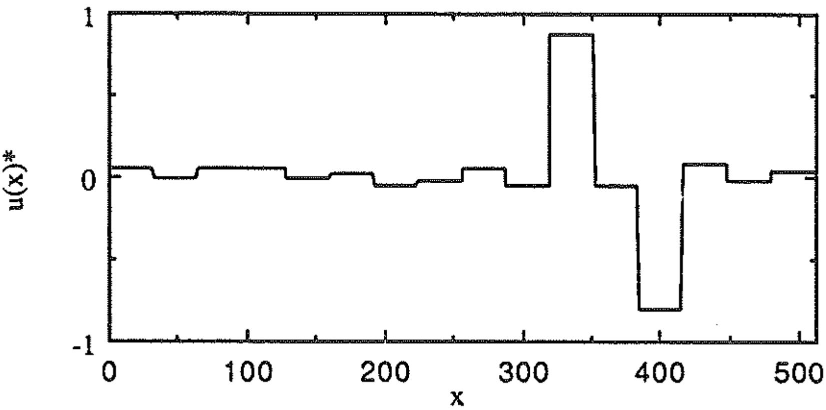

FIGURE 8. Reconstruction of the signal using scales between $m = 5$ and $m = 9$ (low-pass filtering). Since the reconstruction only uses modes down to scales of size 32, the result is a coarse-grained version of the signal.

(1988) even proves that the Haar basis is the *only* system with symmetry). It turns out that non-symmetric bases are a problem in many respects for the applications envisaged in this work. Essentially, the coefficients corresponding to some portion of the signal appear shifted from that position. Therefore, in the present work we will use the Lemarie-Battle wavelets. It is necessary to point out that the deviations from exact orthonormality due to truncation are negligible in practice. Also, the fast transform procedure of Mallat (1989) is implemented. A generalization of the algorithm to three dimensions will be done.

5. Future plans

The main objective of this work is to compute $\Pi(r, \vec{x}, t)$ of Eq. (22) from full numerical solutions of turbulent flows that are available in data bases at certain times t_0. Then the degree of spatial intermittency of $\Pi(r, \vec{x}, t_0)$ will be quantified for different values of r. We will compare the statistics of $\Pi(r, \vec{x}, t)$ with ε_r, the rate of dissipation averaged over a domain of size r, which is the quantity usually used for studies of intermittency. This is a dissipative quantity, whose integral over domains of sizes pertaining to the 'inertial range' is usually thought to represent statistical features of the inertial range. By comparing the dynamically relevant quantity $\Pi(r, \vec{x}, t)$ with ε_r, we hope to clarify this issue. Also, the statistics of 'breakdown' coefficients defined as

$$M = \frac{\Pi(r_1, \vec{x}, t)}{\Pi(r_2, \vec{x}, t)} \tag{30}$$

will be quantified. It will be tested whether a cascade model constructed in such a way as to display the measured statistics of M is consistent with our present knowledge of intermittency of the dissipation. A similar study will be made in the context of scalar dissipation and flux of scalar variance to smaller scales. This will lead to a better physical and statistical understanding of the energy cascade and of intermittency.

Other more long-term objectives are the study of Eq. (18) and in particular of the quantity $I_{ijk}(r,\vec{x};r',r'',\vec{x}',\vec{x}'')$. The problem of subgrid modelling in the present context is to find approximations to the right-hand side of Eq. (18) whenever there are interactions between the resolved scales (say $r \geq r_0$) and the smaller ones. A guide to such considerations could be given by the work of Nakano (1988), who applied DIA to the wave-packet representation. Another line of inquiry could be to attempt a real-space renormalization group analysis of Eq. (18).

In general, the hope is that models deduced from the behavior of wavelet coefficients may capture the physics of turbulence in a more natural way than those based on Fourier modes. However, at this point the manipulations appear to be much more complicated in the wavelet representation, and so its real usefulness remains to be proven.

REFERENCES

ARGOUL F., ARNEODO A., GRASSEAU G., GAGNE Y., HOPFINGER E., & FRISCH U. 1989 **338**, *51*

BATCHELOR G. K. & TOWNSEND A. A. 1949 Proc. R. Soc. Lond. **A199**, 238

COIFMAN R. 1989 Preprint Yale University

DAUBECHIES I. 1988 *Comm. Pure Appl. Math.* **XLI**, 909

FARGE M. & RABREAU G. 1988 *C. R. Acad. Sci. Paris* **II 307**, 1479

FRISCH & PARISI 1985 in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics. North-Holland, New York. p. 84*

GROSSMANN A. & MORLET J. 1984 *SIAM J. Math. Anal.* **15**, 723

KOLMOGOROV A. N. 1941 *C. R. Acad. Sci. U.S.S.R.* **30**, 301

KOLMOGOROV A. N. 1962 *J. Fluid Mech.* **13**, 82

KRAICHNAN H. 1974 *J. Fluid Mech.* **62**, 305

KRONLAND-MARTINET R., MORLET J. & GROSSMANN A. 1987 Int. J. Patt. Rec. and Art. Int. **1**, 97

MALLAT S. J. 1989 *IEEE Patt. Anal. and Mach. Int.* **11**, 674

MANDELBROT B. B. 1974 *J. Fluid Mech.* **62** 331

MENEVEAU C. & SREENIVASAN K. R. 1987 *Phys. Nucl. B (Proc. Suppl.)* **2**, 49

MENEVEAU C. & SREENIVASAN K. R. 1987 *Phys. Rev. Lett.* **59** 1424

MENEVEAU C. & SREENIVASAN K. R. 1989 preprint
NAKANO T. 1988 *Phys. Fluids* **31**, 1420
NOVIKOV E. A. 1971 *Appl. Math. Mech.* **35**, 231
PAUL T. 1985 *J. Math. Phys.* **25**, 3252
SIGGIA E. D. 1977 *Phys. Rev. A* **15**, 1730