Direct simulations of wall-bounded compressible turbulence

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1. Introduction

A study has been initiated on the effects of compressibility (Mach number) on turbulent boundary layers. An understanding of both qualitative (turbulence structures, physical processes) and quantitative (turbulence statistics) effects are desired. This understanding should lead to better turbulence models for applications involving supersonic wall-bounded flows. Direct numerical simulations of an idealized problem will be used to accomplish these objectives. Among several possibilities we chose plane Couette flow with constant-temperature walls as the first problem to be studied. The lack of a mean streamwise pressure gradient plus isothermal walls implies that both horizontal directions can be assumed to be homogeneous and that the flow can reach a statistically steady state. Together, these features greatly simplify the calculations and analyses of the results.

To date, an algorithm has been developed and implemented (but not yet fully tested) for the accurate solution of the Navier-Stokes equations with the assumptions noted above. The scales we use for nondimensionalizing the problem are the channel half-width (b), half the velocity difference between the walls (U_w), average density (ρ_a), wall temperature (T_w), and the fluid viscosity evaluated at the wall temperature (μ_w). In nonconservative form, the continuity, momentum and energy equations are

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial \rho}{\partial x_j} = 0,$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\gamma M^2} \frac{\partial T}{\partial x_i} + \frac{T}{\gamma M^2 \rho} \frac{\partial \rho}{\partial x_i} - \frac{1}{\rho Re} \frac{\partial \tau_{ij}}{\partial x_j} = 0,$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} + (\gamma - 1) T \frac{\partial u_j}{\partial x_j} - \frac{\gamma(\gamma-1)M^2}{Re} \frac{\tau_{ij}}{\rho} \frac{\partial u_i}{\partial x_j} + \frac{\gamma}{\rho\, Re\, Pr} \frac{\partial q_j}{\partial x_j} = 0,$$

where

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right), \qquad q_j = -\mu \frac{\partial T}{\partial x_j}.$$

The ideal gas relation $\rho T = \gamma M^2 p$ was used and the Reynolds and Mach numbers are defined by $Re = \rho_a U_w b / \mu_w$ and $M^2 = U_w^2 / \gamma R T_w$. The Prandtl number $Pr = c_p \mu^* / k^*$ and c_p are assumed to be constant throughout the flow.

2. Numerical method

Unlike homogeneous and free-shear flows, the simulation of wall-bounded turbulence requires the use of an implicit time-integration scheme for the diffusive and acoustic terms because of the small grid size near the wall. Furthermore, unlike finite difference methods, an efficient implementation of a spectral method requires that the terms treated implicitly have constant coefficients. Thus, the terms that one would like to advance implicitly are broken into constant-coefficient and variable-coefficient parts by adding certain linear terms to both sides of the equations. We also let $\tau_{ij} = \mu\sigma_{ij}$ and write the diffusion terms in nonconservative form. This yields

$$\frac{\partial \rho}{\partial t} + \rho_1 \frac{\partial u_j}{\partial x_j} = -(\rho - \rho_1)\frac{\partial u_j}{\partial x_j} - u_j \frac{\partial \rho}{\partial x_j},$$

$$\frac{\partial u_i}{\partial t} + \frac{1}{\gamma M^2}\frac{\partial T}{\partial x_i} + \frac{T_0}{\gamma M^2 \rho_0}\frac{\partial \rho}{\partial x_i} - \frac{\mu_0}{\rho_0 Re}\frac{\partial \sigma_{ij}}{\partial x_j} = -u_j\frac{\partial u_i}{\partial x_j} - \frac{1}{\gamma M^2}\left(\frac{T}{\rho} - \frac{T_0}{\rho_0}\right)\frac{\partial \rho}{\partial x_i} +$$
$$\frac{1}{\rho Re}\frac{d\mu}{dT}\sigma_{ij}\frac{\partial T}{\partial x_j} + \frac{1}{Re}\left(\frac{\mu}{\rho} - \frac{\mu_0}{\rho_0}\right)\frac{\partial \sigma_{ij}}{\partial x_j},$$

$$\frac{\partial T}{\partial t} + (\gamma - 1)T_0\frac{\partial u_j}{\partial x_j} - \frac{\gamma \mu_0}{\rho_0 Re\, Pr}\frac{\partial^2 T}{\partial x_j \partial x_j} = -\left(u_j - \frac{\gamma}{Re\, Pr}\frac{1}{\rho}\frac{d\mu}{dT}\frac{\partial T}{\partial x_j}\right)\frac{\partial T}{\partial x_j} -$$
$$(\gamma - 1)(T - T_0)\frac{\partial u_j}{\partial x_j} + \frac{\gamma}{Re\, Pr}\left(\frac{\mu}{\rho} - \frac{\mu_0}{\rho_0}\right)\frac{\partial^2 T}{\partial x_j \partial x_j} + \frac{\gamma(\gamma - 1)M^2}{Re}\frac{\mu\sigma_{ij}}{\rho}\frac{\partial u_i}{\partial x_j},$$

where the constants ρ_1, T_0, T_0/ρ_0, and μ_0/ρ_0 are used to optimize the stability of the time advancement scheme.

Noting that the left-hand side of these equations are to be advanced implicitly, the algorithm for solving them is as follows. First, perturbation variables T' and u_1' are defined by $T = 1 + T'$, $u_1 = U(y) + u_1'$, ($U(y)$ may be any function satisfying the boundary conditions on the velocity; we take $U(y) = y$ for the Couette flow case) so that homogeneous Dirichlet boundary conditions obtain for T', u_1', u_2 and u_3. All five variables are expanded in Fourier series in the horizontal directions, *e.g.*,

$$\rho(x, y, z) = \sum_{m=-N_x/2}^{N_x/2-1} \sum_{n=-N_z/2}^{N_z/2-1} \tilde{\rho}_{mn}(y) \exp\big(i(\alpha_m x + \beta_n z)\big),$$

where the horizontal wavenumbers are $\alpha_m = 2\pi m/L_x$ and $\beta_n = 2\pi n/L_z$. L_x and L_z are the periodic box lengths in the x and z directions, and the coefficients are assumed to be conjugate symmetric. The vertical functions are expanded using Legendre polynomials,

$$\tilde{\rho}_{mn}(y) = \sum_{l=1}^{N_y} \hat{\tilde{\rho}}_{mnl} P_{l-1}(y),$$

$$\bar{\psi}_{mn}(y) = \sum_{l=1}^{N_y} \hat{\bar{\psi}}_{mnl}\big(P_{l-1}(y) - P_{l+1}(y)\big),$$

where ψ denotes the four variables satisfying Dirichlet boundary conditions. The only constraint on ρ analogous to the boundary conditions on the other variables is that it satisfies global conservation of mass:

$$\int_0^{L_z}\int_{-1}^{1}\int_0^{L_x} \rho(x,y,z)\,dx\,dy\,dz = \text{constant}.$$

Using the above expansion and the Legendre orthogonality property,

$$\int_{-1}^{1} P_i(y)P_j(y)\,dy = \frac{2}{2j+1}\delta_{ij},$$

this constraint reduces to $\hat{\bar{\rho}}_{001} = 1$. Numerically, it is imposed by simply not advancing the density equation corresponding to this mode. Ordinary differential equations in time are obtained by implementing a Galerkin method. After substituting the above expansions into the governing equations, one multiplies by the corresponding basis functions and integrates over the domain.

The method of Spalart (private communication) is used for time-integration of the ODE's. This method combines the explicit third order Runge-Kutta method of Wray (1987) with a new implicit scheme. The latter has the same structure as Crank-Nicolson applied at each substep, except it has different coefficients. Like Crank-Nicolson it is second order, but the stability properties for modes with large eigenvalues are better: instead of an amplification factor approaching -1, it has one of about 0.5 (depending on the value of a free parameter). For the implicit part, all five variables are coupled linearly. However, all the horizontal Fourier modes decouple from each other, as well as the real and imaginary parts of each mode, and the odd and even Legendre modes. This yields about $2N_xN_z$ systems of bandwidth 15 and order $\frac{5}{2}N_y$ that must be solved at each substep. Fortunately, because of symmetries involving the real and imaginary parts of the coefficients, and plus and minus z wavenumbers, only $\frac{1}{2}N_xN_z$ systems actually need to be inverted, each with four right-hand sides. One of the significant advantages of Legendre polynomials over Chebyshev is that the best formulation of the latter would yield systems with bandwidths of 20. Another is that the eigenvalues of these systems are much smaller using Legendre polynomials. This is important because numerical stability will be a limiting factor at sufficiently high Mach numbers. The nonlinear and variable-coefficient linear terms on the right-hand sides are evaluated by transforming to and from physical space and are then advanced explicitly. Note that time advancement in wavespace allows an arbitrary amount of dealiasing by using more collocation points than modes. As opposed to the incompressible case, full dealiasing is not practical since the density appears in the denominator of several terms.

The appropriate model problem to test the time advancement scheme is

$$u_t = \lambda u + \lambda \sigma u,$$

where λ is a complex constant and σ is a real constant (imaginary λ models the acoustic problem and real λ models the diffusion problem). We want to advance the first term on the right-hand side with an implicit scheme and the second with an explicit scheme. For the "standard" coefficients in the Runge-Kutta method, a straightforward stability analysis shows that as $\Delta t \lambda \to \infty$ in the complex plane, the scheme is stable as long as $-0.517 < \sigma < 0.165$. The constants ρ_1, etc., are chosen to make this so, at least in the near-wall region.

3. Linear stability

As an aid in testing the nonlinear code, as well as for its inherent interest, a linear stability code was developed. Since the viscosity of air is not constant, there is no general analytical solution for the laminar mean flow. The nonlinear equations for the mean flow are solved with the same method as above, except time advancement is replaced with an iterative scheme. The full 3-D equations are then linearized around this base state and the perturbations are assumed to be of the form $\psi(x,y,z,t) = \hat{\psi}(y) \exp i(\alpha x + \beta z + \lambda t)$. Applying the same method in y yields an eigenvalue problem of order $5N_y$ for the complex growth rate λ. This was solved using a routine from EISPACK.

Glatzel (1989) solved a simpler version of this problem. He assumed 2-D perturbations and the limit $Pr \to 0$. The latter assumption implies $T =$ const. and thus eliminates the energy equation. Although the incompressible case is stable for all Re, he found two different instabilities for supersonic Mach numbers. One is a resonance between two acoustic modes and is an inviscid instability. The other is a resonance between a viscous mode (related to Orr-Sommerfeld modes) and an acoustic mode. The lowest critical Reynolds number was 83.54 at $M = 4.11$ and $\alpha = .971$, for an acoustic-acoustic mode. Glatzel's results were reproduced with the present stability code.

Shown in Fig. 1 are the critical Reynolds number and corresponding wavenumber as a function of Mach number for air (we assume $Pr = 0.7$ and $\mu = T^{.7}$). The critical Reynolds numbers are much higher than in the case considered by Glatzel (the lowest Re_c is 1395.72 at $M = 4.18$ and $\alpha = 2.037$). Further investigation shows that increasing Pr and increasing the exponent in the temperature dependence of viscosity are both strongly stabilizing. Neither trend is understood. Intuitively, one would expect that increasing the diffusion of heat (lower Pr) would stabilize the flow. Also, the stabilizing effect of a variable viscosity is considerably greater than the amount of variation in the viscosity itself. It appears that in both cases the stabilization occurs because of changes to the mean profiles. The few 3-D perturbations tested were found to be more stable than 2-D perturbations. A more exhaustive search for unstable 3-D modes will be made.

REFERENCES

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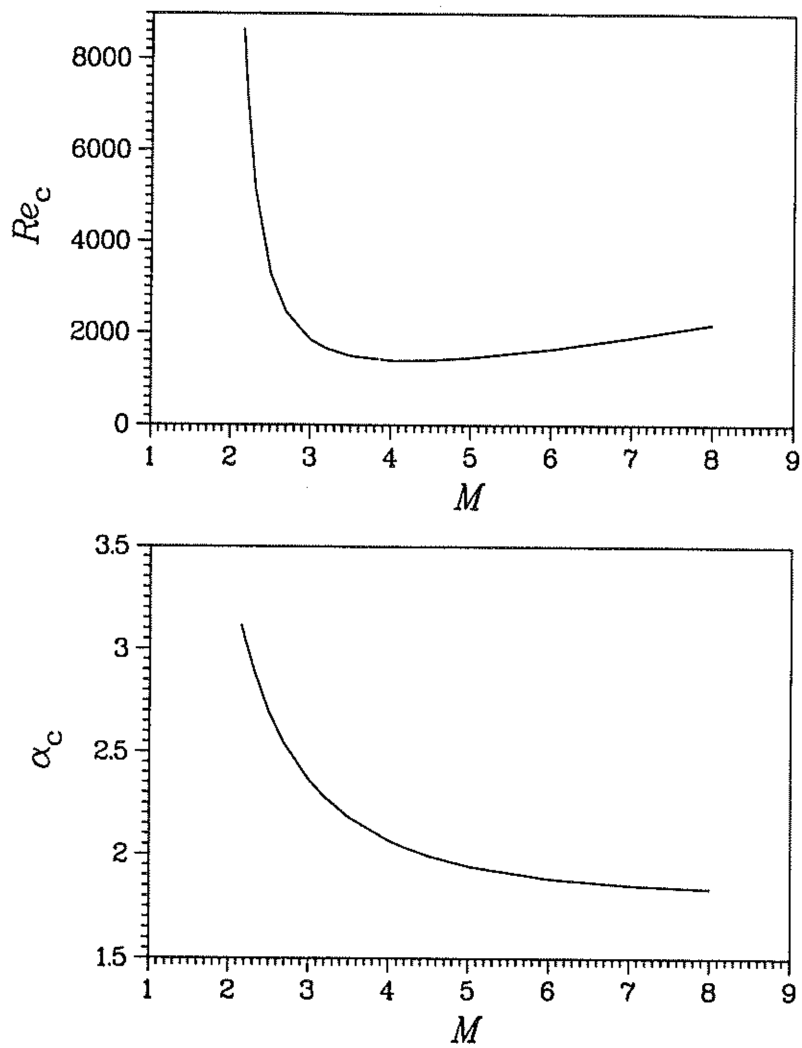

FIGURE 1. Critical Reynolds number as a function of Mach number for air (top), and corresponding critical wavenumber (bottom).