Turbulent thermal convection in a differentially rotating channel

By W. Cabot

1. Motivation and objectives

Differentially rotating disks of gases and solids occur in several astrophysical systems, in particular in the inner parts of protostellar nebulae, of which our own solar system is thought to be a relic (see reviews in Black & Matthews, 1985). These "accretion disks" (see review by Pringle, 1981) are characterized by near centrifugal balance (i.e. nearly Keplerian orbits), a small vertical-to-radial aspect ratio, highly supersonic rotation speeds (the rotation rate being the dominant timescale), and very low effective Prandtl numbers due to very low densities that give rise to high radiative emissivities. Accretion disks are deduced to evolve on timescales many orders of magnitude faster than can be accounted for by angular momentum transport by molecular viscosity. Other mechanisms — like turbulent Reynolds stresses — are thus hypothesized to account for angular momentum transport. Turbulence is suspected because any sustained large-scale disturbance in the disk will have a very high Reynolds number. Unfortunately, there are as yet no reliable models to describe accretion disk turbulence, nor even many testable constraints from present astronomical observations. It is not even agreed on which mechanism is most responsible for generating and sustaining the postulated turbulence: mechanical stirring by infalling material, or thermodynamic instabilities, such as thermal convection, or magnetohydrodynamic instabilities (more than one may apply at different epochs in the protostellar evolution). The shear for Keplerian rotation is stable, and by itself cannot drive the turbulence, but in the presence of an instability it can generate Reynolds stresses that transport mass and angular momentum. As a result, as material in the disk falls down the gravitational well of the central body, gravitational potential energy becomes available to drive turbulence in a self-sustaining manner.

Lin & Papaloizou (1980) and Cabot et al. (1987) proposed *ad hoc* — and mutually incompatible — models of Reynolds stress production in protostellar disks due to thermal convection, and it is this particular problem that motivated the work described here. We have undertaken a program of making numerical "experiments" to test various models of Reynolds stress production and convective heat transfer in differentially rotating thermal convection, and to develop better models if need be. (This problem in principle has a wider interest than the astrophysical one, for it involves the complicated interactions of thermal convection, rapid rotation, and shear in a turbulent, compressible medium, which should provide a severe test for many turbulence models.) We also wish to determine if thermal convection can indeed generate self-sustaining turbulence in an accretion disk environment. Our objectives are (1) to study localized turbulence in circumstances approximating those found accretion disks using previously existing expertise in performing *direct numerical simulations* of turbulent, incompressible channel flows with low Reynolds number, (2) to determine the limitations of such calculations, and (3) to extend the type of numerical simulation (e.g., to include density stratification and compressibility effects and to accommodate higher Reynolds numbers with subgrid-scale modelling) so that the relevant physical effects are realistically captured.

2. Direct numerical simulations

The direct numerical simulation code of Kim et al. (1987) for an incompressible, semi-infinite channel flow was modified as described previously by Cabot (1989). The flow is homogeneous in the horizontal (x, z) directions (with periodic boundaries assumed) and inhomogeneous in the vertical (y) direction, bounded by impermeable walls. The simulation code now allows buoyancy in the vertical direction in the Boussinesq approximation; gravity can be uniform or variable. The flow must be either externally or internally heated, as the heat dissipation of kinetic energy is neglected in the internal energy equation in the incompressible limit. For the accretion disk problem, the flow is given a gravity proportional to the distance from midchannel and an imposed uniform internal heat source. The code includes imposed differential rotation about the vertical axis by integrating the governing equations in a comoving frame with a (locally) linear shear profile using Rogallo's (1981) transformation to remesh the distorted numerical grid. The boundary conditions imposed on the walls are that the vertical velocity component (v) vanishes, that the potential temperature (θ) is fixed, and that the horizontal velocity components (u, w), or their normal derivatives, vanish (no-slip or no-stress conditions, respectively). No-stress wall conditions are used almost exclusively for the accretion disk problem, as they produce weaker viscous boundary layers. (A few simulations performed with internal heating featuring an interior source and exterior sinks are observed to further reduce the viscous boundary layers.)

3. Current progress

3.1. Unsheared thermal convection

Direct numerical simulations of non-rotating thermal convection have been performed for external and internal heat sources, uniform gravity, and no-slip walls and compared favorably to prior laboratory and numerical simulation data. Sequences of simulations for different rates of uniform rotation were performed for internal heating, linearly varying gravity, and both no-slip and no-stress walls; the results of these simulations are described in Cabot et al. (1990).

The rotational (and viscous) stabilization of the thermal convection is found to agree well with linear stability analysis. Turbulence is transported efficiently through the midchannel region where gravity vanishes, leading us to model the convective heat flux by modifying mixing length models for Bénard convection with a reduced effective Rayleigh number to account for the variable gravity and with rotational stabilization based on linear analysis. The models give the correct qualitative effects of rotation, but with poorer quantitative agreement with heat fluxes and turbulence intensities, missing by as much as factors of 2.

I have begun to apply some second-order closure models (e.g., k-ε and Reynolds stress models) to the case of thermal convection with no rotation, uniform gravity, uniform internal heating, and no-slip walls, which is convectively unstable on one side and stable on the other and features an entrainment region much like planetary boundary layers (PBLs). I have therefore focused on closure models successfully applied to PBLs, e.g., Zeman & Lumley (ZL, 1976), although there are some difficulties with all such models. In contrast to the channel simulation, PBLs have an impermeable, no-slip lower surface with fixed heat fluxes and a moving, no-stress upper surface; they also feature very high Reynolds numbers. This channel simulation therefore resembles a viscous PBL with a lid on top. Aside from requiring low Reynolds number corrections, this poses a problem in applying boundary conditions using the ZL model. As matching conditions in the near-wall vicinity to the inviscid interior, ZL use similarity solutions based on the distance from the fixed wall and the positive buoyancy production term; but this approach breaks down near a fixed wall with negative buoyancy production, as occurs in the channel. I am currently exploring other ways to specify matching conditions. A related problem with some second-order closures (generally attributed to deficiencies in modelling the pressure terms) is their inability to predict realistic near-wall horizontal velocity intensities that peak due to deflection by an impermeable wall ("the splatting effect").

3.2. Differentially rotating thermal convection

Sequences of simulations with differential rotation, internal heating, linearly varying gravity, and no-stress walls have been performed most recently with (1) fixed epicyclic frequency $\kappa \equiv [2\Omega(2\Omega + S)]^{1/2}$ (measuring the mean angular momentum gradient) while varying the ratio of shear rate S to rotation rate Ω, and (2) a fixed Keplerian ratio of S to Ω ($-3/2$) while varying κ. The orientation of the rotation, shear, and gravity are depicted in Figure 1. The latter sequences for Keplerian rotation are being performed at three different Reynolds/Prandtl numbers with fixed Péclét number to determine the effects of different viscosities for the same heating. Statistical samples are extracted when the numerical grid is orthogonal, which occurs once per b_{33}/S, where $b_{33} \geq 1$ is the ratio of the streamwise to spanwise box size. This limits us to perform simulations with moderate to rapid shear rates in terms of convective scale times.

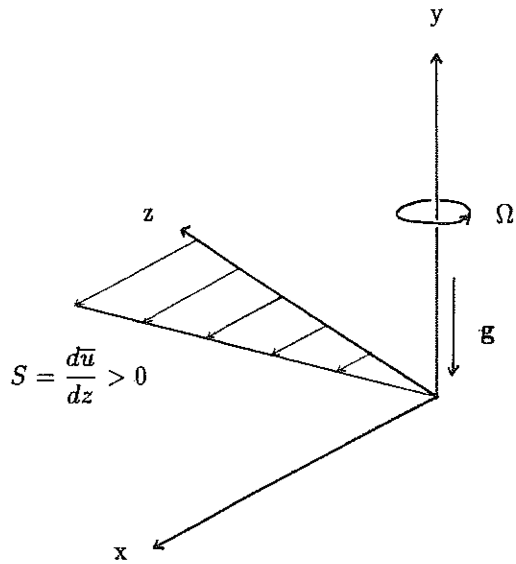

FIGURE 1. Orientation of rotation Ω, (positive) linear shear S, and gravity g with respect to numerical simulation coordinates.

3.2.1. *Fixed* κ, Re; varying S/Ω

Simulations with scale Reynolds number $Re = 559$, scale Péclét number $Pe = 112$, and scale epicyclic frequency $\kappa = 0.14$ (i.e., with fixed convective and centrifugal stabilization properties) were performed for $s \equiv S/2\Omega = 0$, -0.25, -0.50, -0.75, and -0.90 ($s = -1$ being the critical value for marginal centrifugal stability). The roughly uniform vertical profile of the Reynolds stress $-\overline{uw}$ for this value of κ is found to increase nearly homologously with shear, and its correlation $\overline{uw}/u_{rms}w_{rms}$ increases from 0 to about 0.28 for $s = 0 \rightarrow -1$; $\overline{uw}$ varies roughly as $|S|^{1/2}$ in this range ($S = \kappa s/[1+s]^{1/2}$). The cause of this apparent scaling is not yet known. Despite different degrees of elongation of convective cells in the streamwise direction and other anisotropy characteristics of u and w, statistical convection properties, such as the vertical convective heat flux, the buoyancy production rate, the convection correlation $\overline{\theta v}/\theta_{rms}v_{rms}$, and the vertical velocity variance, are only slightly affected by the presence of the shear; the turbulence time scale $\tau = q^2/\varepsilon \approx 10$ (where q^2 is twice the turbulent kinetic energy) also varies little. However, the shear-to-turbulence ratio $|S|\tau$ is only about 0.5 to 4 in these simulations, and most of the variation is seen at the high end of this range. It may therefore be instructive to do another fixed-κ sequence with higher shear rates. A characteristic of more rapid rotation and shear rates is that $\overline{uw}$ becomes negative in regions near midchannel and at the walls, and a fixed-κ sequence in this regime would especially test the homology of the Reynolds stress profile for different values of s.

3.2.2. *Fixed (Keplerian) S/Ω; varying κ, Re*

Three sequences of Keplerian rotation simulations with $Re = 559$, 1000, and 1789 were performed for $\kappa = 0.14$, 0.25, 0.45, 0.61, and 0.80. The scale Péclét number $Pe = 112$ for all simulations. It was necessary to increase the mesh size from $64 \times 33 \times 64$ for $Re = 559$ (where 33 is the vertical) to $96 \times 49 \times 96$ for $Re = 1000$ and $128 \times 65 \times 128$ for $Re = 1789$. (Some runs are still in progress, and some results are presented for more coarsely meshed grids than indicated above and/or with minimal statistics.)

There are two distinct trends that are evident from these simulations. The first is tendency for the Reynolds stress $-\overline{uw}$ to change sign near the wall and midplane regions for rapid rotation, where there is significant rotational stabilization of the convective flow, such that the net shear production of turbulent kinetic energy becomes negative. This is seen in Figure 2, which depicts the ratio of net shear to net buoyancy production in the channel and the average Reynolds stress correlation versus epicyclic frequency. The epicyclic frequency κ_o where the Reynolds stress changes sign is seen to be a sensitive function of Re. The rough progression of $\kappa_o \approx 0.4$, 0.6, 0.7 for $Re = 559$, 1000, 1789 may suggest asymptotic dependences for κ_o like $\alpha \exp(-\beta/Re)$ with $\alpha \approx 1$ and $\beta \approx 500$ or $a/[1 + (b/Re)^2]$ with $a \approx 0.75$ and $b \approx 500$, which imply that there is always negative net shear production for $\kappa > 1$ at this value of Pe. However, higher-Re data is needed to confirm the accuracy of this estimate, and runs with different Pe are needed to determine its dependence on convective efficiency.

The second major trend seen in these sequences of simulations is the tendency of the channel flow to become two-dimensional (but still three-component) for $\kappa > \kappa_1$, where $\kappa_1 \leq 0.8$ regardless of the value of Re. In such simulations the streamwise autocorrelation functions for v and θ (and u and w at some depths in the channel) remain constant and large (typically ≥ 0.6) at large separations; namely, most of the power in these variables is found at (or very near) $k_x = 0$. Increasing the box size in the streamwise direction has no ameliorating effect on the autocorrelation functions. As a result, the spectral code is unable to resolve the streamwise direction, making any statistics therefrom untrustworthy. The energy spectra in the spanwise direction tend to have a lot of power concentrated at a particular (finite) value of k_z. The onset of this regime is perhaps governed by a critical Richardson number of some sort composed of the shear rate and a convective time scale (perhaps depending on Pe). The shear causes the flow to become two-dimensional in the linear analysis, and it is only the effect of buoyancy production of turbulence that counters that trend. The shear-to-turbulence ratio $|S|\tau$ appears to be between about 7 and 9 (depending on Re) at the onset of two-dimensionality.

Note from Figure 2b that the maximal Reynolds stress correlations appear to be not much greater than 0.2 (and probably less than 0.25) for all cases, although the maxima are not that well defined by the simulation data. Vertical

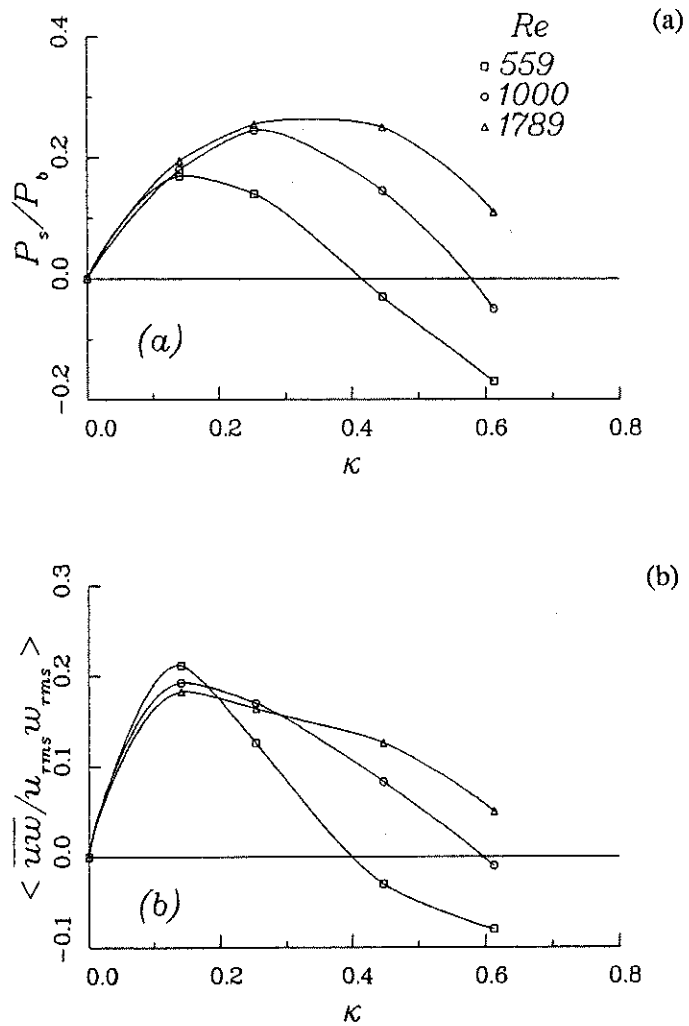

FIGURE 2. (a) The ratio of volume-averaged shear production rate P_s to buoyancy production rate P_b and (b) the volume-averaged Reynolds stress correlations at different epicyclic frequencies κ for Keplerian rotation (shear rate $S = -1.5\kappa$).

profiles of the Reynolds stress correlation are shown in Figure 3a for $Re = 559$ at different κ, in which the progression of positive to negative values at the walls and midchannel is seen. Note the region of positive values around $y = \pm 0.6$, which persists for all shear rates; this is probably not directly related to the buoyancy production peaks at about $y \doteq \pm 0.7$, but is rather a reaction to impermeable walls, like "splatting", regulated by pressure effects. The regions of positive $\overline{uw}$ move to $y = \pm 0.7$ for $Re = 1789$, and the wall regions of negative $\overline{uw}$ have thicknesses that scale roughly as $Re^{-1/2}$ and are comparable to the

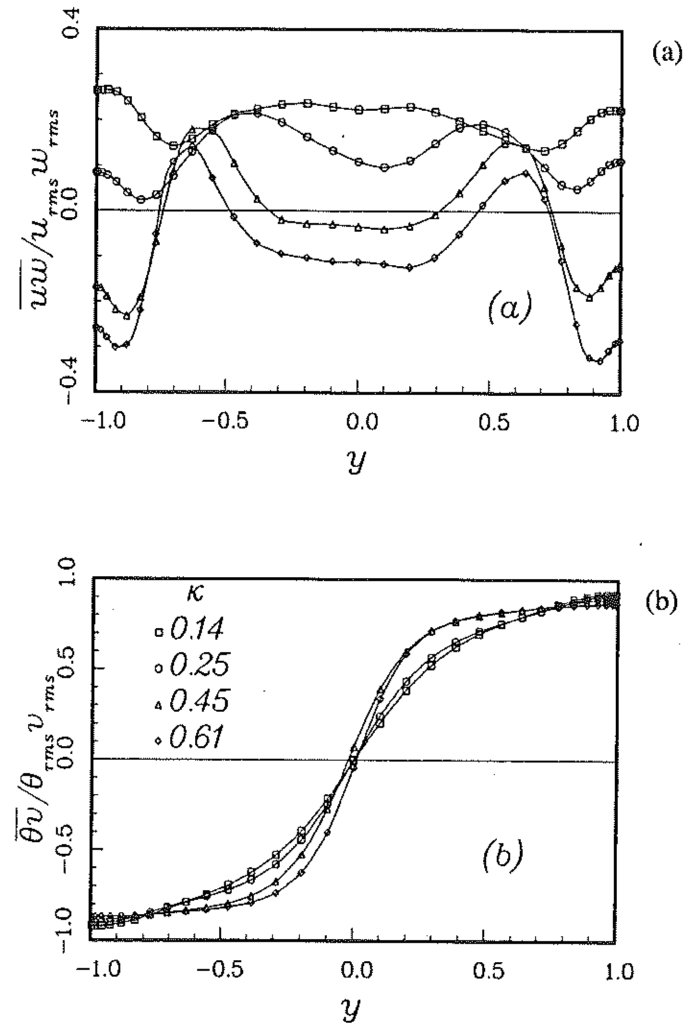

FIGURE 3. Vertical profiles of (a) Reynolds stress and (b) convective flux correlations for Keplerian rotation and $Re = 559$.

viscous length $(\nu q^2/\varepsilon)^{1/2} = (\tau/Re)^{1/2}$. We also note again that properties of vertical heat convection are not greatly altered by the Keplerian shear. For example, Figure 3b shows the vertical profiles of the convection correlations for the same cases as Figure 3a.

Statistics for terms in the governing equations of $\overline{uu}$, $\overline{vv}$, $\overline{ww}$, and $\overline{uw}$ are being accumulated, which include pressure-strain rates. The normal pressure-strain components tend to become concentrated near the walls for larger rotation/shear rates. The pressure-strain component for $\overline{uw}$ remains more evenly distributed and largely balances the production term in the interior; about a viscous length from the walls the production features positive peaks that are not balanced by pressure-strain (but rather by diffusion), and this appears to be directly related to the positive-$\overline{uw}$ regions seen in Figure 3a. I have yet to explain adequately what controls the appearance of the negative-$\overline{uw}$ regions, especially in the interior where we expect (hope) that wall effects are minimal. It is clear, though, that near-wall models will be crucial for an accurate overall representation of such flows. The dissipation rates for the normal velocity components follow the trends of the normal velocities; this is also true for $\overline{uw}$ at lowest Re, with $\overline{uw}$ and its dissipation becoming negative at the walls, but at the higher Re the dissipation rate stays positive definite, even though $\overline{uw}$ goes negative, and is distributed like dissipation rates for $\overline{uu}$ and $\overline{ww}$. This has implications for modelling, for it shows how the low-Re component of dissipation behaves differently than the high-Re asymptote.

4. Discussion

From the preceding work, we are able to make a few tentative conclusions about the nature of thermal convection in a differentially rotating, centrifugally balanced disk, although there are several deficiencies in the incompressible simulations when it comes to describing the actual physics in protostellar accretion disks. The incompressible calculations have, however, provided a number of insights into the course that future endeavor should take.

4.1. Present conclusions

Most of the properties of vertical heat transport by convection are not drastically affected by the rotational shear (at least in the regime of three-dimensional turbulence), which means that results from simulations with uniform rotation can be used more widely for differential rotation cases with the same epicyclic frequency κ. Since $\kappa \geq 1$ in protostellar disks, we conclude that convective efficiencies are much less (by over an order of magnitude) than for no rotation, as was assumed in the mixing length models by Lin & Papaloizou (1980).

Turbulent shear stresses that can, in principle, transport angular momentum are produced by thermal convection in the presence of differential rotation, but they are found to be very sensitive to the rotation and shear rates, as well as the viscosity. There is a possibility that the flow may develop negative shear production of turbulent kinetic energy, which is a pathological (i.e. unsustainable) situation in accretion disks, as well as becoming nearly two-dimensional. Some of this behavior is undoubtedly due to weakening of the thermal convection by rotational stabilization, but there are also clearly strong viscous/wall effects that need to be disentangled. Nevertheless, the present simulations suggest that the conversion of thermal convection to Reynolds stress is less efficient than commonly assumed in protostellar disk models with a maximal Reynolds stress correlation for thermal convection of about 0.25 and a maximal value of $\overline{uw}/q^2$ of about 0.06, which is an order of magnitude less than in the standard solar nebula model of Lin & Papaloizou.

4.2. Deficiencies in the numerical simulations

4.2.1. Incompressible flows

Codes that simulate incompressible flows, including those in the Boussinesq approximation, cannot take into account the large density variations and acoustic waves that occur in nature. In accretion disks the regions of interest cover several density scaleheights, and some disturbances are likely to develop strong acoustic components, if not weak shocks. Another property of density stratification is the occurrence of convectively stable exterior regions where radiation-dominated emissivities grow as ρ^{-2} and cause the temperature gradient to decline to subadiabatic levels; this stable buffer zone in principle can damp exterior disturbances and make them less susceptible to less-than-physical (i.e. numerically convenient) boundary conditions. This would mitigate the direct effects of impermeable walls on the turbulence properties, which has already been seen to be a problem in the incompressible simulations.

4.2.2. Low Reynolds numbers

Accretion disks feature very large Reynolds numbers (but moderate Péclét numbers), and this means that direct numerical simulations are susceptible to unrealistic viscous effects. Low Reynolds numbers also stabilize convection in the direct numerical simulations at lower rotation rates than are expected in accretion disks. It is therefore desirable to attain higher Reynolds numbers (preferably in conjunction with more realistic boundaries) in order to minimize viscous effects, as well as to provide more stringent tests on asymptotic relations, such as for the Reynolds stress production in §3.2. In order to accomplish this goal, modelling of the subgrid scales is required.

4.2.3. Self-consistent energy balance in protostellar disks

An important question for convective accretion disks is whether or not they can quasistatically sustain thermal convection by tapping the gravitational energy released from torqued disk material, and what internal heating distribution arises. In the incompressible governing equations, terms in the internal energy equation involving the adiabatic temperature gradient, the pressure work, and the heat dissipation of kinetic energy are formally neglected. Internal heating is imposed on the flow. This makes it impossible to determine consistently the energy balance in the disk.

For simulations of thermal convection in a compressible channel flow (e.g., see Thompson, 1989), where all of the previously neglected internal energy terms are included, there is still a problem in making a realistic, self-sustaining balance. The heating distribution in the channel with no imposed heat source will equal the heat dissipation of turbulent kinetic energy less the pressure work, with net heating equal to the net shear production, since the pressure-work precisely balances the buoyancy production of turbulent kinetic energy. The energy source (sink) for a positive (negative) net shear production is supplied in the channel by the work done to maintain the imposed rotational shear rate. It is not clear what vertical profile of heating would result, nor even if a self-sustaining state exists. The problem is that one cannot relate this heating to the release of gravitational energy, which depends on the radial (spanwise) gradient of the stress, when the spanwise direction is assumed to be homogeneous in the channel flow. In a real accretion disk, the local heating is supplied by a combination of comparable amounts of gravitational energy and annular stresses that maintain the Keplerian rotation. To do this problem consistently, we clearly need to abandon homogeneity in the spanwise direction.

4.3. Future directions

4.3.1. Modelling

I am pursuing second-order (Reynolds-stress) modelling of the incompressible simulation results, starting at the simple case of uniform gravity and uniform heating in the channel (with the attempted development of better wall matching conditions and near-wall models). Next I plan to move to cases with vertically varying gravity, initially testing the simple replacement of derivatives of the convective flux with derivatives of the buoyancy production rate containing the variable buoyancy term. If these prove successful, then modelling of cases with uniform rotation and differential rotation can proceed. Such tests should be useful for testing the limits of current models and, if reasonably successful, could be used to approximate some properties of accretion disks. Since simplified forms of second-order closure models are commonly used as subgrid-scale models, these tests could conceivably help to verify or improve them.

In principle, density stratification in the Boussinesq approximation can be included in a straightforward way in the modelling, but general compressible effects will require careful consideration of dilitation and shock effects, like the model proposed by Zeman in this volume.

It must also be determined if two-dimensional flows occurring at rapid shear rates can be made tractable to (perhaps simpler) modelling or different numerical simulation techniques.

4.3.2. Direct numerical simulations

Although useful direct numerical simulations of the incompressible flow for application to protostellar disks have been nearly exhausted, there are a few trial runs that could prove interesting. First, a few simulations with uniform rotation have been done with a centrally peaked heat source and large heat sinks near the walls (approximating the effects of a large adiabatic gradient), which provides a convectively stable exterior and greatly weakened viscous wall layers. It would be interesting to repeat some of these simulations with differential rotation and compare the production of turbulent shear stress to simulations with uniform heating in order to gauge the qualitative effects of the walls. Second, to test for self-sustaining states, simulations could be attempted for developed, internally heated convection in which the imposed internal heat source is replaced with some function whose net heating is comparable to it, but which is directly proportional to the shear production rate. The same test with a compressible channel code can be made more easily, since the frictional heating due to turbulence is already consistently included, and once the imposed internal heat source is turned off the convection can conceivably feed off of the work done to impose the constant shear.

Once the compressible channel code being developed by Thompson (1989) is in production, we will be able to repeat similar sequences in parameter space that have been done with the incompressible code, and we will be able to assess realistic density stratification and compressibility effects in protostellar disks. As noted above, though, the exact energy balance in a real protostellar disk requires relaxation of spanwise homogeneity, and the concomitant use of periodic boundary conditions, in order to generate torques. A scheme needs to be devised to specify spanwise (radial) gradients in a consistent way. It may also not be much more trouble to include previously neglected curvature terms.

Finally, a wider range simulations with density stratification and/or compressibility, not confined to such a narrow application as discussed here, would prove useful fodder for, e.g., second-order closure models. We would have more confidence in applying or extending those models that have proven themselves under a wide range of circumstances.

4.3.3. Large-eddy simulations

In order to obtain more realistically high Reynolds numbers and to make determine better high-Re asymptotic behavior, we must inevitably perform large-eddy simulations in which the smallest scales are modelled rather than resolved. For buoyancy-driven (incompressible) flows, commonly used subgrid-scale (SGS) models are ones based on second-order closure models (e.g., Schmidt & Schumann, 1989) and on buoyancy-modified Smagorinsky models (e.g., Mason, 1989). It would be of interest to incorporate these into numerical simulations for Boussinesq convection in order to gain expertise with SGS modelling and to extend our results to higher Reynolds numbers. Because we are interested in very low Prandtl number flows, thermal fluctuations become unimportant at much lower wavenumbers than velocity fluctuations, which could either cause resolution problems due to the scale dichotomy or simplify matters by allowing resolution of all relevant thermal scales and modelling of smaller velocity scales.

An important consideration is testing the ability of SGS models to provide accurate results at moderate to high Reynolds numbers. Unfortunately, there are no astrophysical flows that can be resolved well enough to provide accurate data for such testing. (This is why we're using numerical simulations in the first place!) For buoyancy-driven flows, planetary boundary layers have received a lot of attention, and data exists for internally heated fluids in a uniform gravity. Laboratory experiments with rotating Bénard convection have also been performed. Other terrestrial or laboratory flow fields with rotation and shear effects need to be found or devised to test SGS models.

The development and testing of *compressible* SGS models is crucial to simulating accretion disks and other astrophysical systems in which acoustic waves are believed to transport energy to rarefied regions and deposit heat through shocks. Shocks in low-viscosity media by nature are narrow and difficult to resolve numerically. Again finding pertinent, terrestrially realizable test-cases will be the rub.

4.3.4. Magnetohydrodynamics

Finally, magnetic fields are known to be important in stars, energetic accretion disks, and in the collapse of molecular clouds to protostars. The importance of magnetic fields in protostellar disks is still problematic. Simulating the interaction of magnetic fields with convected and/or sheared turbulent flow in different systems is thus of fundamental interest, but something that has been largely neglected heretofore.

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