Single and double point modeling of homogeneous turbulence

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1. Introduction

Investigations carried out for revisiting homogeneous turbulent flows in the presence of mean shear, rotation, or external compression are summarized in this report. The simplest and most concise RDT (Rapid Distortion Theory) formulation, which includes a comprehensive linear stability analysis, is used for this purpose. Such a linear approach could be extended by a generalized EDQNM (Eddy Damped Quasi-Normal Markovian, Orszag, 1970) to two point closure in order to model non-linear interactions, especially when pure Coriolis effects are present. The results are discussed in connection with databases obtained by DNS (Direct Numerical Simulations), including previous CTR results and new calculations in progress. The main goal is to contribute to and significantly improve on the rational one-point closure models in progress at the CTR and at ECL.

In the case of a mean planar flow, including arbitrary rate of strain and rotation, previous studies suggest that the products of the spanwise (normal to the plane of the mean flow) integral length scales by associated Reynolds stress components $\mathcal{E}^l_{ij} = \overline{u_i u_j} L^l_{ij}$ are relevant quantities to be examined. Such quantities, referred to as *Quasi-2D energy components* are shown to have a very simple behavior in the inviscid RDT limit. Moreover, they play an important role for the study of streaks, or *jet structures*, predicted by Lee, Kim and Moin (1990) in the case of pure shear flows. Simple RDT solutions and non-linear effects exhibited by DNS could be used for developing transport equation models for the spanwise Quasi-2D energy components. The validation of this model will be done in the case of pure rotation (see Jacquin *et al.*, 1990, for example) and pure shear.

More generally, all of these works could support a one-point closure model recently proposed by Reynolds (1989, 1990), which includes transport equations for both the Reynolds stress tensor and the *structure tensor*. This model could complement a proposal by Cambon, Jacquin and Lubrano (1990) based upon splitting the Reynolds stress tensor anisotropy b_{ij} into a part unaffected by the "rapid" background rotation b^e_{ij} and a complementary part b^z_{ij}. The latter was shown to be damped by a strong system rotation.

Additional works concern the non-linear effects of the Coriolis force. New DNS (Mansour, Cambon and Speziale, 1990) are in progress in order to achieve comparisons with the generalized EDQNM model and the experiment (Cambon and Jacquin, 1989, Jacquin *et al.*, 1990). Good agreement is obtained for predicting

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the *non-linear non-isotropic* behavior of the integral length scales in an intermediary range of Rossby numbers (from 0.01 to 1). These results also complement our knowledge of the behavior of the Quasi-2D energy components, mentioned above. Speziale, who is also interested in these studies, proposed, moreover, to revisit the case of pure shear flow in a rotating frame. By looking only at RDT, we selected very simple cases for challenging the one-point closure models. Note that the effects of stable stratification and buoyancy (with or without mean shear) could be studied in the same theoretical framework (Itsweire *et al.*, 1990).

Regarding the theoretical understanding, we hope that the non-linear non-isotropic trends predicted by the EDQNM (in the case of pure rotation) could be retrieved by a more straightforward analysis (deriving a dynamical system) starting from the same expansion in terms of the eigenmodes of the linear regime (inertial waves). Such an approach was successfully used by Waleffe (1989) for studying non-linear stability of the elliptical flows. Moreover, a new experimental approach of *Elliptical flows* but also hyperbolic or linear (pure shear) in progress (Moulin, Leuchter and Geoffroy, 1989) could benefit from the theoretical support by Waleffe and me.

Finally, the effects of an external compression on a *solenoidal* fluctuating velocity field are also revisited by means of RDT. Such a study is relevant for predicting the drop of the cross-correlation coefficient when the turbulent flow passes through a shock wave. Non-isotropic upstream conditions (sheared turbulence) are needed to complement the previous calculations by Lee *et al.* (1991). In the same framework, the interaction between periodic external compression and swirl could lead to comparisons between RDT, DNS and EDQNM, following the stability analysis by Mansour and Lundgren (1990).

2. Formalism

2.1. Fluctuating velocity field

Classic RDT formulations in Fourier space can be made simpler and more general if one considers the initial value problem for the fluctuating velocity and pressure fields and if one reduces the number of components by taking explicitly into account the incompressibility constraint. By decomposing the Fourier mode of the fluctuating velocity field into two solenoidal modes, any RDT solution can be generated by a simple matrix $g_{\alpha\beta}$ with only four coefficients (Cambon, 1982, Cambon *et al.*, 1985).

This approach is also very close to the linear stability analyses using a *time dependent wave vector* $\underline{k}(t)$ (and, therefore, not based upon the classic assumption of space and time separation, the perturbation phase being $\underline{k} \cdot \underline{x} - \sigma t$), proposed in the last years (Craik and Criminale, 1986, Baily, 1986, Waleffe, 1989). It can be pointed out that the strong analogies between stability analyses and RDT are masked if one looks only at statistical quantities. For example, linear solutions of the equation governing the spectral tensor of double correlations were studied at Lyon, following Craya (1958). Craya proposed to use a local frame to reduce the number of components of the spectral tensor of double velocity correlations, but working with a covariance matrix (the spectral tensor) is more intricate and less general than working with the fluctuating field itself.

Townsend (1976) looked at the fluctuating field, but he did not use a local frame. Similarly, the use of the vorticity field, following Batchelor and Proudman (1954), presents interest only in the case of irrotational mean flows. For more detailed appreciation of RDT application, the reader is referred to the earlier reviews by Hunt (1978), Aupoix (1987) and Cambon (1989). I just would like to recall that a complete approach to homogeneous turbulent flows in the presence of a mean flow with uniform and arbitrary rates of strain and rotation (including a first evidence of the so-called "elliptical flows instability" for unbounded eddies) was carried out in my thesis where RDT and EDQNM were revisited. This general approach (including a numerical code) supports all the studies presented in this report. The formalism is given in what follows.

Let $\hat{u}_i(\underline{k},t)$ be the 3D Fourier transform of the fluctuating velocity field. It is convenient to introduce an orthonormal frame $(\underline{e}^1, \underline{e}^2, \underline{k}/k)$ attached to the wave vector $\underline{k}$, so that

$$\hat{u}_i(\underline{k},t) = \hat{\varphi}^1(\underline{k},t)e_i^1(\underline{k}) + \hat{\varphi}^2(\underline{k},t)e_i^2(\underline{k}) + \hat{\varphi}^3\frac{k_i}{k} \tag{1}$$

In this frame, the Fourier transform of the fluctuating vorticity field is also expressed as

$$\hat{\omega}_i(\underline{k},t) = -Ik\hat{\varphi}^2(\underline{k},t)e_i^1(\underline{k}) + Ik\hat{\varphi}^1(\underline{k},t)e_i^2(\underline{k}) \tag{2}$$

according to its derivation in spectral space

$$\underline{\hat{\omega}} = I\underline{k} \times \hat{u}_i(\underline{k},t) \tag{3}$$

where $k = \|\underline{k}\|$ and $I^2 = -1$.

The component $\hat{\varphi}^3$ along $\underline{k}$ corresponds to the *dilatation part* of the velocity field. It will be neglected in the following, except in subsections 3.3 and 3.4. The two solenoidal modes are orthogonal to each other and located in the plane normal to the wave-vector $\underline{k}$. As the definition of a system of spherical coordinates, the precise characterization of $\underline{e}^1$ and $\underline{e}^2$ requires to choose a preferential fixed axis $\underline{n}$, referred to as the *polar axis*. In accordance with Herring (1974), one has

$$\underline{e}^1 = \frac{\underline{k} \times \underline{n}}{\|\underline{k} \times \underline{n}\|}; \quad \underline{e}^2 = \frac{\underline{k}}{k} \times \underline{e}^1 \tag{4}$$

The first mode corresponds to the solenoidal part of the transverse planar flow (i.e. normal to $\underline{n}$), and the second one collects both the axial velocity (i.e. parallel to $\underline{n}$) and the residual (dilatation) part of the planar field. From equations (1), (2), and (4), it is clear that

$$\hat{\omega}_{\|} = -Ik_{\perp}\hat{\varphi}^1$$

$$\hat{u}_{\|} = -\frac{k_{\perp}}{k}\hat{\varphi}^2$$

(the notations $v_{\|} = v_i.n_i$ and $v_{\perp} = \sqrt{v^2 - v_{\|}^2}$ being systematically used). In accordance with the above equations and the terminology used by Reynolds, the first solenoidal mode $\hat{\varphi}^1\underline{e}^1$ could be christened *axial vortical mode* and the second one $\hat{\varphi}^2\underline{e}^2$, *axial jet mode.* Note that such a decomposition is current in geophysical turbulence (Riley (1981)) but is always considered from a very specific point of view (not as a general and convenient mathematical basis for any solenoidal velocity field!).

Regarding the homogeneous incompressible turbulence in the presence of uniform mean velocity gradients $\lambda_{ij} = \overline{U}_{i,j}$, the velocity field is governed by the following system of equations:

$$\begin{pmatrix} \dot{\hat{\varphi}}^1 \\ \dot{\hat{\varphi}}^2 \end{pmatrix} + \nu k^2 \begin{pmatrix} \hat{\varphi}^1 \\ \hat{\varphi}^2 \end{pmatrix} + \begin{pmatrix} e_i^1\lambda_{ij}e_j^1 & e_i^1(\lambda_{ij} - \lambda_{ji})e_j^2 \\ 2e_i^2\lambda_{ij}e_j^1 & e_i^2\lambda_{ij}e_j^2 \end{pmatrix} \begin{pmatrix} \hat{\varphi}^1 \\ \hat{\varphi}^2 \end{pmatrix} = \begin{pmatrix} R^1 \\ R^2 \end{pmatrix} \quad (5)$$

or

$$\dot{\hat{\varphi}}^\alpha + \nu k^2 \hat{\varphi}^\alpha + m_{\alpha\beta}\hat{\varphi}^\beta = R^\alpha$$

(Greek indices taking the value 1 or 2 only) in which R^α are non-linear terms. The superimposed dot represents a material derivative, so that

$$\dot{k}_i + \lambda_{ji}k_j = 0 \quad (6)$$

The solution of the latter equation is

$$k_i = F_{ji}^{-1}(t,0)K_j = 0$$

in which the role of the mean distortion gradient tensor

$$F_{ij}(t,0) = \frac{\partial x_i}{\partial X_j} \quad (7)$$

$$F_{ij}(0,0) = \delta_{ij}$$

is clearly displayed. Recall that the mean trajectories in physical and in spectral space are respectively

$$x_i = F_{ij}X_j \quad ; k_i = F_{ji}^{-1}K_j$$

so that $k_ix_i = K_iX_i$ (wave conservation law). Capital letters stand for material coordinates of the mean flow (also called Rogallo's space). In the case of RDT, the relation $l_i(t) = F_{ij}(t,0)l_j(0)$ allows calculation of the deviatoric tensor $\delta_{ij}/3 - l_il_j/l^2$ proposed by Reynolds(1990) as a model for the deviatoric part of the structure tensor y_{ij}.

Solving equations (5) with $R^\alpha = 0$ is the simplest and the most general way for obtaining RDT. Non-linear terms can be reintroduced *in a statistical way* by a convenient two point closure. For the record, the formal solution of equation (5) is given as follows:

$$\hat{\varphi}^\alpha(\underline{k},t) = g_{\alpha\beta}(\underline{k},t,0)\hat{\varphi}^\beta[\underline{\tilde{F}}(t,0).\underline{k}, \quad 0] + \int_0^t g_{\alpha\beta}(\underline{k},t,t')R^\beta[\underline{\tilde{F}}(t,t').\underline{k}, \quad t']dt' \quad (8)$$

where $\tilde{F}_{ij} = F_{ji}$. Linear as well as non-linear approaches start from equation (8). The matrix $g_{\alpha\beta}$ generates the basic linear solution (first term in the right hand side of (8)) and is also involved if an *explicit form of the non-linear terms* is given (second term in the right hand side).

2.1 Statistical approach

The spectral tensor of double correlation $\hat{U}_{ij}(\underline{K}, t)$ is obtained by correlating $\hat{u}_i^*$ with $\hat{u}_j$. In the local frame $(\underline{e}^1, \underline{e}^2, \underline{k}/k)$, it has only four non-zero components in the case of incompressible turbulence. These four components $\Phi^{\alpha\beta}$ correspond to $\hat{\varphi}^{\alpha *}.\hat{\varphi}^{\beta}$. Regarding the reduced spectral tensor at fixed $\underline{k}$, in the local frame $(\underline{e}^1, \underline{e}^2)$, the anisotropy is accurately characterized by using a set of variables (e, Z).

$$\begin{pmatrix} \Phi^{11} & \Phi^{12} \\ \Phi^{12*} & \Phi^{22} \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} + \begin{pmatrix} -\Re e Z & \Im m Z \\ \Im m Z & +\Re e Z \end{pmatrix} \tag{9}$$

(Z is a complex term, having a real and an imaginary part). This decomposition of type trace-deviator exhibits the invariants of the spectral tensor (namely e and $\|Z\|$, with the *unique* realizability constraint $e > \|Z\|$). Moreover, a very simple expression of $\hat{U}_{ij}$ is found in the *fixed frame of reference*

$$\hat{U}_{ij}(\underline{k}, t) = e(\underline{k}, t).P_{ij}(\underline{k}) + \Re e[Z(\underline{k}, t) N_i(\underline{k}) N_j(\underline{k})] \tag{10}$$

Recall that $e = \frac{1}{2}\hat{U}_{ii}$ and $P_{ij} = \delta_{ij} - k_i k_j / k^2$ is the classic projector. One also has

$$Z = \frac{1}{2}\hat{U}_{ij} N_i^* N_j^*; \quad \underline{N} = \underline{e}^2 - I\underline{e}^1 \tag{11}$$

If the turbulence is isotropic, Z is null everywhere and e is only dependent on the modulus of $\underline{k}$. By reintroducing the averaged energy spectrum $E(k, t)$ (integral of e over spherical shells of radius k), it is now possible to distinguish two kinds of anisotropy

$$\hat{U}_{ij} = \frac{E}{4\pi k^2} P_{ij} + (e - \frac{E}{4\pi k^2}) P_{ij} + \Re e(Z N_i N_j) \tag{12}$$

Regarding the three terms in the right hand side, the first one is the pure isotropic part, the second represents the anisotropy due to the angular dependence of the spectral distribution of energy, and the third reflects the polarization of this energy at fixed $\underline{k}$. Any one point correlation could be calculated in terms of these three contributions. So the Reynolds stress tensor is obtained by an integral of the spectral tensor over $\underline{k}$-space.

$$\overline{u_i u_j}(t) = \int\int\int \hat{U}_{ij}(\underline{k}, t) d^3\underline{k}$$

From equation (12), a decomposition into three terms is easily derived,

$$\overline{u_i u_j} = \overline{q^2}(\frac{\delta_{ij}}{3} + b_{ij}^e + b_{ij}^z) \tag{13}$$

The deviatoric part b_{ij} of the Reynolds stress tensor is, therefore, split into two parts. b^e_{ij} reflects the *dimensionality* of the spectral tensor and b^z_{ij} reflects its *componentiality*. In accordance with equation (12) and with the spectral derivation of the structure tensor Y_{ij} used by Reynolds (1989), the following exact relation is found:

$$\overline{q^2} b^e_{ij} = -\frac{1}{2}(Y_{ij} - Y_{ll}\frac{\delta_{ij}}{3}) = -\frac{\overline{q^2}}{2} y_{ij} \tag{14}$$

Note that the structure tensor is also connected with the structure of the vorticity field. From equations (2) and (11) it is possible to derive a splitting equivalent to (14) for the vorticity correlations tensor $\overline{\omega_i \omega_j}$ *by only changing* e *into* $k^2 e$ *and* Z *into* $-k^2 Z$.

Finally, the Quasi-2D energy components are given by integration over a plane

$$\mathcal{E}^l_{ij} = \overline{u_i u_j} L^l_{ij} = \pi \int\int \hat{U}_{ij}\mid_{k_l=0} d^2\underline{k} \tag{15}$$

The two indices i and j, which refer to the components of the velocity fluctuation, are not summed, whereas l shows in what direction the integral length scale L^l_{ij} is calculated. As for the velocity and vorticity correlations tensor, a splitting in terms of e and Z can be found, *but the distribution in the wave plane* $k_l = 0$ *is only emphasized.*

From the equation which governs $\hat{U}_{ij}$ (usually referred to as Craya's equation), it is possible to derive an equation for the set (e, Z). The same result is more easily obtained from equation (5) and leads to

$$\dot{e} + 2\nu k^2 e + L_e = T_e$$

$$\dot{Z} + 2\nu k^2 Z + L_z = T_z \tag{16}$$

The detailed form of the linear terms L_e and L_z with respect to e, Z and Z^* (derived from $m_{\alpha\beta}$ in (5)) is not given for the sake of brevity. T_e and T_z represent spectral transfer terms (including also "slow" pressure effects) mediated by non-linear interactions.

3. Revisiting homogeneous shear flows

3.1 Behavior of the integral length scales

It is possible to show with the formalism presented in Section 2 that the products of the spanwise integral length scales by associated Reynolds stress components (see equation (15)) are conserved in the inviscid RDT limit for any planar mean flow, provided that the initial data are isotropic. These 2-D energy components

$$\mathcal{E}^3_{ij} = \overline{u_i u_j} L^3_{ij} = \text{ constant}$$

(where 3 corresponds to the spanwise direction) are derived from the spectral tensor of double correlations by integrating in the plane, $k_3 = 0$, in which no effect of stretching by the mean flow is found. Recall that the incompressible planar mean flow is characterized by the following velocity gradient matrix (Craya, 1958):

$$\overline{U}_{i,j} = \lambda_{ij} = \begin{pmatrix} 0 & D+\Omega & 0 \\ D-\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{17}$$

In the case of a pure shear $D = \Omega = S/2$, a simple solution is also found for the streamwise quantities, which display the spectral distribution at $k_1 = 0$.

$$\mathcal{E}^1_{11}(t) = \overline{u_1 u_1}(t) L^1_{11}(t) = (1 + \frac{(St)^2}{2}) \frac{\overline{q}^2}{3}(0) L^1_{11}(0)$$

$$\mathcal{E}^1_{ii}(t) = \overline{u_i u_i}(t) L^1_{ii}(t) = \text{ constant}; \qquad i = 2,\ 3$$

Thus the increase in the component $\overline{u_1^2}$ with a quasi-linear law (for large St) corresponds to a decrease in L^3_{11} and an increase in L^1_{11} in the same proportion. The appearance of jet structures–or streaks–in the streamwise direction with a decreasing distance in the spanwise direction can be connected to these simple RDT solutions. Indeed, one can assume that L^1_{11} gives the length of the streaks (in the streamwise direction) and that L^3_{11} gives the distance across the streaks (in the spanwise direction). The DNS databases by Moon Lee (Lee, Kim and Moin, 1990) are in very good agreement with the RDT behavior of the streamwise quantities $\mathcal{E}^1_{ij}$ in the case of high shear rate.

Regarding the spanwise quantities at high shear rate $\mathcal{E}^3_{ij}$, a significant increase is found for $\overline{u_1 u_1} L^3_{11}$, whereas $\overline{u_3 u_3} L^3_{33}$ remains almost constant. Recall that both these terms are constant in RDT with isotropic initial data, where $\mathcal{E}^3_{11} = \frac{1}{2}\mathcal{E}^3_{33}$. The different behavior of $\mathcal{E}^3_{11}$ observed in DNS leads to a crossover of the two former quantities ($\mathcal{E}^3_{11} > \mathcal{E}^3_{33}$) about $St = 7$. Such a crossover is also observed in a low shear rate case, although a strong decrease in the $\mathcal{E}^3_{ij}$ prevails at the beginning of the evolution. The significant departure from isotropy (valid for very low shear rapidity) and RDT (valid for very high shear rapidity) presents a strong analogy with the non-linear non-isotropic behavior predicted for an intermediary range of Rossby numbers in the case of pure rotation. So a transport equation for the spanwise quasi-2D energy components is in progress. The closure model for non-isotropic non-linear terms in the transport equations for $\mathcal{E}^3_{ij}$ could be validated in pure shear and pure rotation. Note that the total transverse (with respect to the direction of the rotation axis, and thus spanwise) energy

$$\mathcal{E}^3_{\perp} = \pi \int\int \Phi^{11} \mid_{k_3=0} dk_1 dk_2 = \overline{u_1 u_1} L^3_{11} + \overline{u_2 u_2} L^3_{22}$$

displays exactly the contribution of the transverse mode of energy in the wave-plane normal to the mean rotation axis, which plays an essential role in the theoretical and experimental approach by Jacquin *et al.* (1990). Recall that the spectral energy is concentrated in $\Phi^{11} \mid_{k_\parallel=0}$ in the case of pure 2D turbulence.

3.2. Shear flow in a rotating frame

This case presents a particular interest in turbomachinery. It questions the current one-point closure models, which are almost exclusively sensitive to the Richardson number. Recall that the stability analysis in terms of the Richardson number ignores the complex redistribution effects of the pressure. Previous RDT calculations (for example Bertoglio, 1981) and DNS show that the Richardson number is not the uniquely relevant parameter, even for predicting the evolution of the turbulent kinetic energy. So C. Speziale proposed to revisit RDT, EDQNM, DNS, especially in order to compare different ratios $r = \Omega/S$ (i. e. rotation rate divided by shear rate) which give the *same* Richardson number $2r(1-2r)$.

A first investigation of the two coupled equations which govern the two incompressible modes for two different ratios r_1 and r_2 with $r_1+r_2=\frac{1}{2}$ clearly shows that the role of the "vortical" mode seems to be interchanged with the role of the "jet" mode (when changing r to $(\frac{1}{2}-r)$. Simpler equations are found by choosing the polar axis in the vertical direction, or $n_i = \delta_{i2}$, although the mean vorticity is in the spanwise direction. The vortical mode under consideration is, therefore, a *vertical* mode. Equation (5) leads to

$$\dot{\hat{\varphi}}^1 + (S-2\Omega)\frac{k_3}{k}\hat{\varphi}^2 = 0 \tag{17.1}$$

$$\dot{(k\hat{\varphi}^2)} + 2\Omega k_3\hat{\varphi}^1 = 0 \tag{17.2}$$

in the linear inviscid limit. The role of r is clearer by working with the dimensionless time St. A single second order equation is easily found

$$\ddot{(k\hat{\varphi}^2)} - 2\Omega(S-2\Omega)(\frac{k_3}{k})^2(k\hat{\varphi}^2) = 0$$

Although the Richardson number seems to be the unique parameter in the latter equation, the individual role of the two solenoidal modes must be taken into account in the complete linear solution. These effects are particularly simple in the "marginal stability case" $Ri = 0$; for $r = 0$, the case of pure shear in an inertial frame is retrieved. The other "symmetric" case with $r = 1/2$ corresponds to a pure strain, the principal axes of which are being continuously rotated, when looking at the inertial frame, in which the total mean vorticity is null. As a consequence, the RDT solution for the fluctuating vorticity is an incredibly simple Cauchy solution

$$\omega_i(\underline{x},t) = F_{ij}(t,0)\omega_j(\underline{X},0)$$

in the case $r = 1/2$. Detailed solution of equations (17) also show that the increase in turbulent kinetic energy is stronger in the case of $r = 1/2$ than in the case of pure shear flow. Moreover, the structure of the vorticity correlations tensor is very different in the two cases. The increase in the turbulent vorticity (in $(St)^2$) is also stronger in the case $r = 1/2$. These results could explain why a DNS carried out by Rogallo seems to indicate opposite behavior with respect to RDT (a stronger increase in kinetic energy in the case $r = 0$ than in the case $r = 1/2$). Indeed, if the rapidity of the shear is not large enough, the increase in the turbulent vorticity, which strongly affects the dissipative scales, can significantly counterbalance the production of kinetic energy.

3.3. Stratified turbulence in the presence of shear

If we consider the framework of Boussinesq approximation and homogeneity, a complete analogy could be found with the former case. By taking into account the two previous incompressible modes of the fluctuating velocity field and by adding a pseudo-compressible mode connected to the fluctuating temperature θ (along the wave-vector) in eqn. (5),

$$\hat{\varphi^3} = \frac{\beta g}{N}\hat{\theta},$$

the RDT and EDQNM formalism are almost unchanged with respect to the flows in presence of system rotation (see Cambon, 1989). This approach suggests revisiting the databases by Holt (1990) in the same way.

Without shear, the straightforward study of linear and non-linear effects of the dispersive gravity waves very close to the inertial waves could lead to improve both the physical understanding and the modeling of stably stratified flows. Note that only vague and unconvincing considerations about the "collapse" were derived from the more recent DNS, or LES in this case. (see Metais and Lesieur, 1990).

If a mean vertical shear is also present, the unique difference with the case of shear in a rotating frame is that the Richardson number takes into account the ratio $r = N/S$ in a monotonic form, $Ri = r^2$. N is the Brumt-Waisala frequency, which plays the same role as 2Ω.

3.4. Towards compressible turbulent shear flows

The shear flow is one of the cases with constant mean velocity gradients, which is consistent with the homogeneity of a compressible fluctuating field. Recall that the skew part of the tensor $\dot{\lambda}_{ij} + \lambda_{il}\lambda_{lj}$ must be zero in incompressible turbulence. Verifying such a condition is equivalent to following the Helmholtz equation for the mean vorticity. Moreover, the symmetric part must also vanish in the compressible case in order to eliminate the position-dependent term $\rho'(\dot{\lambda}_{ij} + \lambda_{il}\lambda_{lj})x_j$ in the equation governing the fluctuating velocity field u_i. ρ' is the density fluctuation. The databases of G. Blaisdell suggest considering a "true" dilatation term in the fluctuating velocity field and suggest looking at the associated new terms in the spectral tensor. In the local orthonormal frame (e_i^1, e_i^2, $e_i^3 = k_i/k$), the general decomposition (1) into three orthogonal modes is applied (including the axial vortical mode, the axial jet mode, and the dilatation mode).

Regarding the spectral tensor (or the covariance matrix) *in the same frame*, we are concerned with six components ($\Phi^{ij} \quad i \le j \le 3$). The components $\Phi^{\alpha\beta}$ associated with the solenoidal part are collected into a solenoidal 3-D energy spectrum $e^s(\underline{k},\ t)$ and a deviator $Z^s(\underline{k},\ t)$, as previously (equation (10)). Three new components correspond to a dilatation 3-D energy spectrum, $e^d(\underline{k},\ t) = \frac{1}{2}\Phi^{33}$ and two cross-correlations $\Phi^{\alpha 3}$ between the dilatation mode and the solenoidal modes. As a first assumption, the contribution of these cross-correlations are neglected in the one point correlations. Damping effects of the acoustic waves could be invoked in accordance with the behavior of the integrals

$$f(ct) = \int_0^\infty h(k)\exp(\pm Ikct)dk$$

for large values of ct (c is the speed of sound). Accordingly, a generalized splitting of the Reynolds stress tensor is easily obtained:

$$\overline{u_i u_j} = q^{2s}\frac{\delta_{ij}}{3} + q^{2s}(b_{ij}^{es} + b_{ij}^{zs}) + Y_{ij}^d$$

where b_{ij}^{es} is exactly minus half the deviatoric part of the structure tensor introduced by Reynolds (y_{ij}), and Y_{ij}^d is exactly the structure tensor associated with the dilatation part of the energy spectrum.

$$Y_{ij}^d = \int\int\int 2e^d(\underline{k},t)\frac{k_i k_j}{k^2} d^3\underline{k}; \qquad e^d = \frac{1}{2}\hat{U}_{ln}\frac{k_l k_n}{k^2}$$

If one assumes that the solenoidal field evolves almost independently on the dilatation one, one can keep unchanged closed equations for the quantities with "s" and only add a new equation for Y_{ij}^d. DNS databases available at the CTR could be used to validate such a model.

4. Revisiting non-linear non-isotropic effects of pure rotation

The linear effects of the Coriolis force are easily characterized by looking at equations (5) and (16). The polar axis is now chosen to coincide with the system rotation axis ($n_i = \delta_{i3} = \Omega_i/\Omega$). Regarding the fluctuating field, linear combinations of the dependent variables in eqn. (5), $\hat{\varphi^1} \pm I\hat{\varphi^2}$, lead to a diagonal form of the matrix $m_{\alpha\beta}$, which exhibits the two eigenvalues $\pm 2I\Omega k_{\parallel}/k$. It is, therefore, very simple to work with the eigenmodes of the inertial waves. Recall that the eigenpulsation

$$\sigma(\underline{k},t) = 2\Omega\frac{k_{\parallel}}{k}$$

gives the dispersion relation. Regarding double correlations, e and Z correspond to quadratic products of the eigenmodes. In equation (16), $L_e = 0$ and $L_z = 2I\sigma$. The linear solution of equation (16) is, therefore,

$$e(\underline{k},t) = e(\underline{k},0)\exp(-2\nu k^2 t); \quad Z(\underline{k},t) = Z(\underline{k},0)\exp(4I\Omega t\frac{k_{\parallel}}{k})\exp(-2\nu k^2 t) \quad (18)$$

As a consequence, the quantities involving e, such as $\overline{q^2}$ and b_{ij}^e, are unaffected by a rapid rotation. This fact reflects that the Coriolis force produces no energy. The quantities involving Z in a 3D integral, such as b_{ij}^z, are damped in accordance with the behavior of the integral

$$f(\Omega t) = \int_{-1}^{1} h(\mu)\exp(4I\Omega t\mu)d\mu$$

for large values of Ωt. This fact reflects the *angular dispersivity* (influence of $\mu = k_{\parallel}/k$) of the inertial waves, and is called "phase-randomization" by Reynolds.

Nevertheless, the quantities which involve Z in the wave-plane $k_{\|} = 0$, such as $\mathcal{E}^3_{ij}$, are not affected by the linear Coriolis effects. These linear effects are now well known (see also Veeravalli, 1989).

Non-linear effects are reflected by T_e and T_z in the system of equations (16). The influence of the rotation on these terms is crucial for predicting the behavior of an initially isotropic turbulence. Indeed, the linear terms are null, at least in a preliminary phase. The generalized EDQNM model used by Cambon and Jacquin (1989) (for closing T_e and T_z) predicted very spectacular non-isotropic behavior of the integral length scales in agreement with results from experiment and DNS (Roy and Dang, 1985). Non-isotropic features in physical and spectral space began to be verified by 128^3 DNS (Teissedre and Dang, 1987). New DNS are in progress at the CTR in order to complement this information. The most important anisotropy criterion is chosen to be,

$$\mathcal{A} = \mathcal{E}^3_{33} - (\mathcal{E}^3_{11} + \mathcal{E}^3_{22}) = \overline{u^2_{\|}} L_{\|} - 2\overline{u^2_{\perp}} L_{\perp} \tag{19}$$

A dimensionless form could also be proposed by using $\overline{q^2} L_0$. Regarding the Reynolds stress tensor, the axisymmetric trend is quantified by

$$\mathcal{B} = \overline{u^2_3} - \frac{1}{2}(\overline{u^2_1} + \overline{u^2_2}) = (\overline{u^2_{\|}} - \overline{u^2_{\perp}}) = \frac{3}{2}\overline{q^2} b_{33} \tag{20}$$

The spectral derivation also leads us to separate the two contributions of the latter criterion, according to $\mathcal{B}/\overline{q^2} = \frac{3}{2}(b^e_{33} + b^z_{33})$.

Now, it is easy to explain why the anisotropic features occur only for an *intermediary* range of Rossby numbers. -If the Rossby number is too large, the specific effect of rotation is weak and the initially isotropic behavior prevails. -If the Rossby number is too small, the non-linear effects are assumed to be weak (in relative value, with respect to the linear ones), but the true feature is that the rotation tends to inhibit the level (in *absolute* value) of triple correlation. The phase scrambling of *cubic* products of fluctuating velocity components gives the simplest explanation of this phenomenon. So a regime of pure viscous RDT is obtained. Hence, an isotropic behavior is again retrieved. Previous DNS by Speziale, Mansour & Rogallo (1987) confirms these results at very low Rossby number.

The best indicator of the peculiar anisotropy, generated by non-linear interactions, is $\mathcal{A}$. This criterion, which is rigorously null in the isotropic case, is unaffected in any case by linear (RDT) Coriolis effects because it involves only the polarization part of the spectral energy Z in the transverse wave-plane ($\underline{k}$ normal to $\underline{\Omega}$). Considering Rossby number built on the axial integral length scale L^3_{33}, the new DNS by Mansour *et al.* 1990 predicts that the criterion is weak for initial Rossby numbers larger than the unity or smaller than 0.01. A maximum is found to be about 0.1. The upper limit of the intermediary range (about 1) is in good agreement with the experimental and EDQNM results (Cambon and Jacquin (1989), Jacquin *et al.* (1990)). The capture of the lower limit by DNS is also a surprisingly good result, regarding the risk of numerical inaccuracies (especially on the lengthscales) in the case of very strong rotation rates.

The DNS and the EDQNM (new computations carried out at the CTR) are also in agreement in predicting the rise of a *weak but significant axisymmetry of the Reynolds stress tensor, only due to the contribution of the structure tensor.* By using Reynolds' notations, it is found that

$$b_{33} = -\frac{1}{2}y_{33} > 0$$

This tendency corresponds to a relative concentration of the spectral energy in the transverse wave-plane. Unfortunately, it is weak, especially when looking at the classic invariants of b_{ij} (quadratic and cubic). The order of magnitude of the difference between the axial and a transverse component of the Reynolds stress tensor ($\mathcal{B}$ indicator) is no larger than 5 per cent of the trace (or $\frac{3}{2}b_{33} = 0.05$). Nevertheless, the different anisotropic trends remain consistent with the first phase of a transition towards two-dimensionality in accordance with our previous theoretical analyses. Regarding physical interpretation, the rise of a positive value for b^e_{33}, which reflects a preferential concentration of e in the wave-plane $k_\parallel = 0$, corresponds to a decrease in the axial variability $\partial/\partial x_\parallel \searrow$. Such a trend is often deduced from the Proudman-Taylor theorem, but one recalls that this theorem was proposed in the zero Rossby number limit. Although the exact RDT contradicts this theorem (and the dogma that a strong rotation makes two-dimensional any turbulent flow), weak but significant non-linear terms (and possibly boundary conditions) are able to reduce the axial variability, in agreement with the rise of columnar structures.

Moreover, the strong negative value of $\mathcal{A}$ indicates a polarization of the spectral energy in the more energetic wave-plane $k_\parallel = 0$. Accordingly, the axial vortical mode is found to dominate in this plane at least in the low wave-numbers range.

Both of these trends contribute to align the turbulent vortices with the system rotation axis, but they are not in agreement with the decrease in the axial velocity ($u_\parallel \searrow$). (See Cambon, 1990, for a more detailed analysis of the 2D tendencies in MHD, rotated, and stably stratified turbulence). During the evolution, the constant decrease of the Rossby number and the increasing action of the viscous effects tend to block these transient mechanisms. Accordingly, I think that only a very high Reynolds number and the presence of a forcing term (in the wave vectors range $k_\parallel = 0, \quad k < k_0$) would lead to complete transition towards 2D in the intermediary range of Rossby numbers.

5. Solenoidal turbulence undergoing external compression

5.1. Contribution to a study of boundary layer-shock wave interaction

A first approach of the effects of a strong one-dimensional compression on a non-isotropic turbulence was carried out by using a solenoidal homogeneous RDT. The upstream conditions are built by a preliminary application of a pure shear, so that an important cross-correlation coefficient (between u_1 and u_2) is created. The shock is then considered as a strong compression in the streamwise direction. The RDT solution shows a strong decrease of the absolute value of the cross-correlation coefficient and even a change of sign. If the "rapid" pressure-rate of strain correlations are ignored or badly modeled (as in a current Reynolds stress model), this coefficient is conserved. Note that a new model, that includes the structure tensor of W. C. Reynolds, is in good agreement with RDT in this case. Regarding the role of the cross correlations in the balance of kinetic energy in a boundary layer, it is obvious that this mechanism must be correctly predicted. The idea of this RDT calculation was prompted by Mathieu (private communication), regarding experimental results obtained at ONERA.

5.2. Periodic compression with swirl

The validity of a solenoidal model in the presence of a strongly compressed mean flow requires moderate values of the "turbulent" Mach numbers. A previous analysis of Mansour justifies this framework of assumptions which is implicitly used in most of the turbulence models for reciprocating engines. Following a recent summer workshop at Lyon (Pepit workshop, July 1989), a particular emphasis is made in my laboratory about the interactions between external compression and rotation, so the stability analysis by Mansour and Lundgren (1990) is of great interest. The parameters are the ratio of the swirl rate to the period of the coaxial compression. An instability behavior is shown in narrow domains of angular dependence (in wave-space). First RDT calculations with a code based upon equations (5) (see the following section) and a DNS code (Mansour), in which the non-linear terms are omitted, seem to validate each other.

6. Towards a general one-point closure model

A very good agreement in the guidelines is found when one compares recent progress in the modeling of turbulent flows in the presence of rotation carried out in France (Jacquin *et al.*, Cambon *et al.*, 1990) with the works of the CTR, especially by W. C. Reynolds and S. Kassinos. The idea of introducing two contributions of the Reynolds stress tensor anisotropy, according to eqn. (13), leads to exactly the same amount of information that the use of the structure tensor together with the Reynolds stress tensor provides.

The validation in homogeneous turbulence of the most general model, recently proposed by Reynolds, could be made with the support of the numerical code "Thanatos", which could provide the spectra of the terms involved in the new single point closure model. This code takes into account any mean velocity gradient matrix, with possible time dependency and external compression, and solves the solenoidal RDT problem (time advancing eqn. (5) using a Runge-Kutta high order scheme). An EDQNM model using a parameterization of T_e and T_z (eqn. (16)) in terms of angular harmonics is included but not yet completely validated.

Specific EDQNM models used for studying non-linear interactions of internal waves in sections 3.3 and 4 (pure rotation or stable stratification) are more complicated, although they are restricted to quasi axisymmetric turbulence. They do not use parameterization of angular dependence, and they are too expansive (in computational time) and too cumbersome to handle to be implemented in the general case.

Note that the specific expansion in terms of angular harmonics is used in eqn.(16) only for decomposing the general transfer terms T_e and T_z, in accordance with a 3D EDQNM formulation. It involves spherical coefficients (depending only on the modulus of the wave vector) of e and Z, which easily generates the spectra of both the structure tensor and the Reynolds stress tensor.

We hope that the use of two anisotropy tensors, namely $b^e_{ij} = -\frac{1}{2}y_{ij}$ and $b^z_{ij} = b_{ij} + \frac{1}{2}y_{ij}$, will improve the classic single point closure models, in which b_{ij} is used as the only anisotropy indicator. Regarding only RDT, the classic procedure leads to a connection between the rapid part of the pressure-rate of strain correlations and the production terms in the balance of the Reynolds stress tensor. It is questioned in the presence of system rotation, or more generally by the effect of any body forces generating waves (such as the buoyancy forces), and even in the case of pure straining processes applied to initial data influenced by rotation (as in section 5.1).

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