Progress in understanding the renormalization group skewness and $K-\varepsilon$ models

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1. Motivation and objectives

The immediate goal of this work is to understand and validate the Yakhot-Orszag model for the velocity-derivative skewness and model equation for the rate of energy dissipation $\mathcal{E}$ (Yakhot and Orszag (YO), 1986). This report is a summary of a more detailed manuscript in preparation (Smith and Reynolds (SR), 1990a). Our purpose is to clarify some limitations of the theory by careful examination of key assumptions and approximations, and thereby to encourage its improvement. Our focus is as follows:

1. We reformulate their recursive solution for the velocity field of the removed scales as a perturbation analysis and show that it is unlikely to be quantitatively accurate at the perturbation levels involved.
2. We examine the effect of using the low wavenumber limit of the modified viscosity in the solution for the high wavenumber modes. This work suggest that their approach significantly overestimates the low wavenumber limit.
3. We introduce the concept of "displacement value" characterizing the contribution of the dissipative high wavenumber portion of the spectrum to various integrals in the theory.
4. We correct an identifiable algebraic error in the velocity derivative skewness, finding $\bar{S} = -0.59$ instead of their value of -0.488.
5. We correct several errors in the development of the transport equation for the rate of energy dissipation $\mathcal{E}$. The most important of these corrections gives zero as the coefficient of YO's term proportional to the rate of production of energy P_K. There may exist a non-zero term responsible for the production of $\mathcal{E}$ at higher-order in the approximations involved. However, we show that its structure is different from the structure of the $\mathcal{E}$-production term reported by YO and in current turbulence models. Hence, the RG method has not yet placed the $\mathcal{E}$-transport model equation on solid ground.

In spite of these difficulties, it seems clear that YO have introduced an important approach worthy of further exploration. Although the method does not provide a derivation of the $\mathcal{E}$-transport model equation presently in use, the correct RG $\mathcal{E}$-transport equation may be useful and should be developed.

2. Accomplishments

2.1 The perturbation expansion for the high wavenumber modes

YO use RG techniques to develop a theory for the large scales in which the effects of the small scales are represented by modified transport coefficients. The effect of the very large scales on eddies in the inertial range is represented by a random force chosen to produce the correct form of the inertial range spectrum when opposed by the modified viscosity. The force is assumed Gaussian, white noise in time, homogeneous in time and space, and isotropic in space. YO assume that the statistics of the inertial range of homogeneous turbulence forced in this manner will be representative of the inertial range of turbulence sustained by inhomogeneities. They refer to this assumption as "the correspondence principle."

The dynamical equations for the large-scale field are derived by averaging over an infinitesimal band of the small scales to remove them from explicit consideration. This procedure yields infinitesimal modifications in the equations for the large scales. The removal process is iterated, and these corrections accumulate to give finite changes. YO retain only the modifications of the viscosity, arguing that other modifications are unimportant to the large-scale dynamics.

The iterative averaging is carried out in $(d+1)$-dimensional Fourier space in which the Fourier amplitudes are functions of the $(d+1)$-vector $\hat{\mathbf{k}} \equiv [\mathbf{k}, \omega]$, where $\mathbf{k}$ is the d-dimensional wavevector and ω is the frequency. At each stage, the averaging requires knowledge of the statistics of the scales being removed. YO generate these statistics by repeated substitution of the equation for $\hat{\mathbf{v}}^>$ into the equation for $\hat{\mathbf{v}}^<$, which is originally given in terms of $\hat{\mathbf{v}}^<$, $\hat{\mathbf{f}}^<$ and $\hat{\mathbf{v}}^>$. Since $\hat{\mathbf{v}}^>$ is in terms of $\hat{\mathbf{v}}^<$, $\hat{\mathbf{f}}^>$ and $\hat{\mathbf{v}}^>$, the statistics of $\hat{\mathbf{v}}^>$ are given by the assumed statistics of $\hat{\mathbf{f}}^>$.

One can show that the repeated substitution of $\hat{\mathbf{v}}^>$ is equivalent to a series expansion in which zeroth-order effects are linear and nonlinear effects are higher-order corrections,

$$\hat{\mathbf{v}}^> = \hat{\mathbf{v}}^{(0)} + \lambda \hat{\mathbf{v}}^{(1)} + O(\lambda^2) \tag{1}$$

$$\hat{v}_i^{(0)}[\hat{\mathbf{k}}] = \hat{f}_i^>[\hat{\mathbf{k}}] \tag{2}$$

$$\hat{v}_i^{(1)}[\hat{\mathbf{k}}] = \frac{i}{2} G_\nu[\hat{\mathbf{k}}] P_{imn}[\mathbf{k}] \int_*^{k_c} \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \Big(\hat{v}_m^<[\hat{\mathbf{q}}]\hat{v}_n^<[\hat{\mathbf{k}}-\hat{\mathbf{q}}] + \hat{v}_m^<[\hat{\mathbf{q}}]\hat{f}_n^>[\hat{\mathbf{k}}-\hat{\mathbf{q}}] + \hat{v}_n^<[\hat{\mathbf{k}}-\hat{\mathbf{q}}]\hat{f}_m^>[\hat{\mathbf{q}}] + \hat{f}_m^>[\hat{\mathbf{q}}]\hat{f}_n^>[\hat{\mathbf{k}}-\hat{\mathbf{q}}] \Big) \tag{3}$$

where

$$G_\nu[\hat{\mathbf{k}}; k_c] \equiv (-i\omega + \nu_T[k, \omega; k_c] k^2)^{-1}. \tag{4}$$

Here k_c is the current cutoff wavenumber above which all scales have been previously removed, $\int_*$ indicates a $(d+1)$-dimensional integral, $P_{ijk}[\mathbf{k}]$ is a compound projection operator, G_ν is called the "propagator", and $\lambda = 1$ is the ordering parameter. Square brackets [ ] will be used throughout to indicate functional dependence. The modified viscosity $\nu_T[k,\omega;k_c]$ is the term through which the high wavenumber modes affect the retained scales. YO assume that when $k_c = \Lambda_o$ is a Kolmogorov scale, all significant scales are retained and then $\nu_T[k,\omega;\Lambda_o] = \nu_o$ is the (constant) molecular viscosity. YO do not discuss the dependence of ν_T on k and ω, but it is implicit in their analysis.

Use of the above perturbation expansion can be expected to give quantitatively correct results only if the Reynolds number of the removed eddies, *based on the modified viscosity*, is small. We denote the Reynolds number of the removed eddies by $R_c \equiv v_c l_c/\nu_T$ where the velocity scale $v_c = D_o^{1/2} l_c/\nu_T^{1/2}$, time $t_c = l_c^2/\nu_T$, and length $l_c = 1/k_c$ are characteristic of the high wavenumber band to be eliminated and $D_o \propto \mathcal{E}$ is the amplitude of the two-point force correlation.

The requirement that R_c be small is not met very well because the removed scales are in the inertial range. One may estimate the actual value of R_c at each iteration of the fine-scale elimination (SR). One finds that $R_c = O(10)$ at each iteration of the removal process, which seems large for a perturbation parameter and suggests the possibility of quantitative inaccuracy.

2.2 The modified viscosity

The differential equation describing how the modified viscosity changes with cut-off wavenumber is developed by averaging over infinitesimal bands as described above in section 2.1. The modified viscosity at any low wavenumber Λ is calculated by integrating this equation from the first removal, where the viscosity was the fluid viscosity ν_o and the cutoff wavenumber was the Kolmogorov cut-off Λ_o.

The modified viscosity is in general a function of the retained wavenumber, the frequency and the cutoff wavenumber: $\nu_T = \nu_T[k,\omega;k_c]$. The modified viscosity is calculated to be independent of k and ω only in the "distant-interaction" limit $k << k_c$, $|i\omega| << \nu_T k_c^2$. It is important to realize that the distant interaction approximation includes only the effect of triad interactions between two modes in the removed shell and a retained mode at very small wavenumber. Hence, the modified equation will be most accurate at low wavenumber and least accurate near the new cutoff wavenumber, *where it will be used in the perturbation solution for the next band of removed wavenumbers.* Thus the YO procedure assumes

$$\nu_T[k_c,\omega;k_c] = \nu_T[0,0;k_c] \tag{5}$$

For the remainder of this section, we focus only on the k-dependence of ν_T and suppress the ω-dependence by writing $\nu_T[k;k_c]$.

Many studies suggest that a realistic eddy-viscosity model exhibits a plateau far from the cutoff ($k << k_c$) and increases to several times its plateau value very near the cutoff ($k \approx k_c$). This "cusp-up" behavior of the modified viscosity is predicted by EDQNM (ref) and the Test-Field model of Kraichnan (ref), and is consistent with direct numerical simulations (ref).

In order to study the effect of assumption (5), we instead use

$$\nu_T[k_c; k_c] = \zeta \nu_T[0; k_c] \tag{6}$$

where ζ is a constant. The differential equation for $\nu_T[0; k_c]$ is

$$\frac{d\nu_T[0; k_c]}{dk_c} = -\frac{\beta}{\nu_T^2[k_c; k_c]k_c^3} = -\frac{\beta}{\zeta^2 \nu_T^2[0; k_c]k_c^3} \tag{7}$$

YO's approximate differential equation is

$$\frac{d\nu_T[0; k_c]}{dk_c} = -\frac{\beta}{\nu_T^2[0; k_c]k_c^3} \tag{8}$$

In the limit of small k_c, the solution to (8) overestimates the solution to the correct differential equation (7) by a factor $\zeta^{2/3}$.

We can estimate ζ by comparing YO's results and EDQNM (ref). We denote $\nu_T[0; k_c] = A_0 \mathcal{E}^{1/3} k_c^{-4/3}$. EDQNM predicts $A_0 = 0.28$ for YO's value of the Kolmogorov constant $C_K = 1.6$, whereas YO predict $A_0 = 0.49$. These numbers suggest $\zeta = 2.3$.

The same assumption (5) is made in the differential equations to derive the skewness and the modified source term in the model $\mathcal{E}$-equation. Thus some quantitative error is expected in the YO skewness value and $\mathcal{E}$-model coefficients.

2.3 The concept of the "displacement value"

YO assume that the Fourier modes vanish when $k > \Lambda_o$, beyond which there is no significant physics. In their analysis, Λ_o is an ultraviolet cutoff of the inertial range. This is a reasonable assumption when dealing with the kinetic energy, since the Kolmogorov modes contain only negligible energy. However, turbulence quantities related to velocity derivatives, such as the dissipation, receive significant contribution from modes beyond the inertial range. Therefore, it seems essential in performing RG on such moments to allow for undetermined physics beyond the inertial range. We introduce the term " displacement value" to represent the cumulative effect of modes beyond the ultraviolet cutoff. The displacement value of the kinetic energy is taken to be zero.

Since interactions among wavenumbers above Λ_o are not modeled by the YO theory, only moments that are negligibly small above the ultraviolet cutoff Λ_o and, hence, have displacement value zero can be calculated using their method.

For moments of velocity derivatives, the displacement values are nonzero and dominant. Although the YO method does not provide values of individual velocity-derivative moments, YO use it to provide estimates for Reynolds-number independent combinations of derivative moments. They argue that such combinations can be evaluated from the statistics of RG-filtered turbulence acted on by the modified viscosity, which at large Reynolds numbers is independent of Reynolds number.

Two examples of Reynolds-number-independent combinations of derivative moments are the velocity-derivative skewness $\bar{S}$ and the "source" terms $\mathcal{Y}$ in the $\mathcal{E}$-transport equation. The RG models for $\bar{S}$ and $\mathcal{Y}$ are discussed in sections 2.4 and 2.5, respectively.

2.4 The RG skewness model

The velocity-derivative skewness is defined

$$\bar{S} \equiv \frac{\langle(\nabla_1 v_1)^3\rangle}{\langle(\nabla_1 v_1)^2\rangle^{3/2}} \equiv \frac{A}{B^{3/2}}. \tag{9}$$

Estimates of A and B are provided by a spectrum model of the form

$$E[k] = 0 \qquad \text{for} \qquad k < \Lambda_f$$

$$E[k] = C_K \mathcal{E}^{2/3} k^{-5/3} \exp[-\alpha(k\eta)^2] \qquad \text{for} \qquad \Lambda_f \leq k \leq \infty \tag{10}$$

where $\eta = (\nu_o^3/\mathcal{E})^{1/4}$, $\Lambda_o = .2/\eta$, Λ_f is the integral scale, and we take $C_K = 1.6$ for consistency with the YO theory. This form of the exponential tail has been predicted by various theoretical considerations and elsewhere (Smith and Reynolds, 1990b) we show that it is the preferred exponential form of the Kolmogorov spectrum.

Using the model spectrum (10), one can show (SR) that $A = O(R_T^{3/2})$ and $B = O(R_T)$ where $R_T \equiv \mathcal{K}^2/(\nu_o\mathcal{E})$ is the turbulent Reynolds number. Also, the displacement value accounts for 70% of B. The skewness for isotropic turbulence is found to be -0.694 in agreement with observations at moderate Reynolds numbers.

The RG differential equations for $A^>$ and $B^>$ are found from the Fourier transforms $\hat{A}$ and $\hat{B}$ by the averaging over infinitesimal bands of high-wavenumber modes as described in section 2.1. The distant-interaction approximation leads to (see section 2.2)

$$\frac{dA^>[k_c']}{k_c'} = \frac{a\mathcal{E}^2}{\nu_T^3[0,0;k_c'](k_c')^3} \tag{11}$$

$$\frac{dB^>[k_c']}{k_c'} = -\frac{b\mathcal{E}}{\nu_T[0,0;k_c']k_c'} \tag{12}$$

where a and b are constants. Notice that error is introduced in (11) and (12) by using $\nu_T[0,0;k_c']$ instead of $\nu_T[k_c',\omega;k_c']$.

Integration of (11) and (12) over $k_c \leq k \leq \Lambda_o$ gives $A^>[k_c]$ and $B^>[k_c]$. Taking $k_c = 0$ corresponding to elimination of the entire inertial range, one finds

$$A^>[0] = -0.0044\frac{\mathcal{E}^3}{\mathcal{K}^3}R_T^{3/2} + A_o \tag{13}$$

$$B^>[0] = 0.022\frac{\mathcal{E}^2}{\mathcal{K}^2}R_T + B_o \tag{14}$$

where A_o and B_o are the displacement values. One sees that the RG method yields the proper scaling of the individual moments A and B. However, their values cannot be determined because the displacement values A_o and B_o are unknown.

An estimate for the skewness is made from RG-filtered turbulence having scales $k < \Lambda$, acted on by the modified viscosity $\nu_T[0,0;\Lambda]$. One approximates $A^<[\Lambda]$

and $B^<[\Lambda]$ from the RG expressions for $A^>[k_c]$ and $B^>[k_c]$. To do this, the differential equations (11) and (12) are integrated over $0 \le k'_c \le \Lambda$, where $\Lambda/\Lambda_o \to 0$ corresponding to $R_T \to \infty$,

$$A^<[\Lambda] \approx a\mathcal{E}^2 \int_0^\Lambda \frac{dk'_c}{(k'_c)^3 \nu_T^3[0,0;k'_c]} \tag{15}$$

$$B^<[\Lambda] \approx b\mathcal{E} \int_0^\Lambda \frac{dk'_c}{k'_c \nu_T[0,0;k'_c]}. \tag{16}$$

Then the RG skewness is formed as

$$\bar{S} = \frac{A^<[\Lambda]}{(B^<[\Lambda])^{3/2}} = -0.59 \tag{17}$$

where $A^<[\Lambda]$ and $B^<[\Lambda]$ are both $O(1)$ quantities. The RG prediction $\bar{S} = -0.59$ is in good agreement with experiments at moderate Reynolds numbers. YO made an algebraic error in their calculation of the constant a which defines $A^>$, which led them to find $\bar{S} = -0.488$ instead of the correct value -0.59 (SR).

2.5 The RG model $\mathcal{E}$-transport equation

The RG model $\mathcal{E}$-transport equation is derived by applying the iterative-averaging technique to the evolution equation for the instantaneous value of the dissipation rate ϕ in homogeneous flow,

$$\phi \equiv \nu_o(\nabla_j v_i)^2. \tag{18}$$

Then $\mathcal{E} = <\phi>$ under the assumption that the dissipative scales are locally homogeneous.

Differentiation in time of (18), followed by substitution from the Navier Stokes equations, gives

$$\frac{\partial\phi}{\partial t} = -v_j\nabla_j\phi + \chi_o\nabla_j\nabla_j\phi - \overbrace{2\nu_o^2(\nabla_m\nabla_j v_i)^2}^{\mathcal{Y}_{\rm I}[{\bf x},t]}$$
$$-\overbrace{2\frac{\nu_o}{\rho}(\nabla_j v_i)(\nabla_j\nabla_i p)}^{\mathcal{Y}_{\rm II}[{\bf x},t]} - \overbrace{2\nu_o(\nabla_j v_i)(\nabla_j v_m)(\nabla_m v_i)}^{\mathcal{Y}_{\rm III}[{\bf x},t]} \tag{19}$$

where $\chi_o = \nu_o$.

The RG calculation of the modified "source" terms $\mathcal{Y}_{\rm I} = <\mathcal{Y}_{\rm I}[{\bf x},t]>$, $\mathcal{Y}_{\rm II} = <\mathcal{Y}_{\rm II}[{\bf x},t]>$ and $\mathcal{Y}_{\rm III} = <\mathcal{Y}_{\rm III}[{\bf x},t]>$ follows the RG procedure to find velocity moments, outlined for the velocity-derivative skewness in section 2.4 above. Since the source term is observed to be independent of Reynolds number at high Reynolds number, the RG modified source term is calculated from the RG-filtered turbulence field acted on by the modified viscosity ν_T.

At this writing, our results for the RG modified source term differ from YO's results. We find a different decay rate for isotropic flow and a different form for the term responsible for the production of $\mathcal{E}$ in anisotropic flow. In the limit of high Reynolds number, we find

$$\frac{\partial \mathcal{E}}{\partial t} = -U_j \nabla_j \mathcal{E} + \nabla_j \chi_T \nabla_j \mathcal{E} - 5.65 \frac{\mathcal{E}^2}{\mathcal{K}} + \Pi^* \tag{20}$$

The decay rate for isotropic flow is 5.65, in poor agreement with observations and the $\mathcal{K} - \mathcal{E}$ models in current use. The term Π^* is responsible for production of $\mathcal{E}$ in anisotropic flow. The Fourier transform $\hat{\Pi}^*[\hat{\mathbf{k}}; k_c', k_c]$ has the form (SR)

$$\hat{\Pi}^* = \lambda^2 \frac{\mathcal{E}}{\nu_T^2} \frac{(k_c')^{d-y-4} - k_c^{d-y-4}}{4-y-d} \int_{*2}^{k_c'} \frac{d\hat{\mathbf{r}} d\hat{\mathbf{q}}}{(2\pi)^{2d+2}}$$

$$\times \Big(O(r^2/k_c^2) + O(rq/k_c^2) + O(q^2/k_c^2) \Big)_{jlbt} (k_j - r_j) \hat{v}_j^<[\hat{\mathbf{k}} - \hat{\mathbf{r}}] \hat{v}_b^<[\hat{\mathbf{q}}] \hat{v}_t^<[\hat{\mathbf{r}} - \hat{\mathbf{q}}]. \tag{21}$$

where $(\ )_{jlbt}$ is a function of the angles of $\mathbf{k}_c$, $\mathbf{r}$ and $\mathbf{q}$, and $\int_{*2}$ indicates two $(d+1)$-dimensional integrals. YO reported

$$\frac{\partial \mathcal{E}}{\partial t} = -U_j \nabla_j \mathcal{E} + \nabla_j \chi_T \nabla_j \mathcal{E} - 1.7 \frac{\mathcal{E}^2}{\mathcal{K}} + \Pi \tag{22}$$

where the decay rate 1.7 is in good agreement with observations and the $\mathcal{K} - \mathcal{E}$ models in current use. The Fourier transform $\hat{\Pi}[\hat{\mathbf{k}}; k_c', k_c]$ of the production term is given as (YO)

$$\hat{\Pi} = -i \frac{d-2}{d(d+2)} \frac{\tilde{B}_d \mathcal{E}}{\nu_T^2} \frac{(k_c')^{d-y-2} - k_c^{d-y-2}}{2-y-d}$$

$$\times \int_{*2}^{k_c'} \frac{d\hat{\mathbf{r}} d\hat{\mathbf{q}}}{(2\pi)^{2d+2}} (k_j - r_j) \hat{v}_i^<[\hat{\mathbf{k}} - \hat{\mathbf{r}}] \hat{v}_j^<[\hat{\mathbf{q}}] \hat{v}_i^<[\hat{\mathbf{r}} - \hat{\mathbf{q}}]. \tag{23}$$

where $\tilde{B}_d$ is a constant. We found that integrals of the form (23) exactly cancel, and that the power of the wavenumber in the integrand must be at least two (SR). We are in correspondence with YO over these differences.

3. Future Plans

We will continue our correspondence with Yakhot and Orszag until we agree on the correct form of the RG $\mathcal{E}$-transport model equation. This may involve calculations to higher-order in the distant-interaction limit in order to derive a term responsible for $\mathcal{E}$-production.

We may also consider an exponential roll-off to the inertial range spectrum, instead of an abrupt ultraviolet cutoff, by adjusting the statistics of the force. Then displacement values could be calculated using RG and may have interesting consequences for the RG skewness and $\mathcal{K} - \mathcal{E}$ models.

A longer term project is the reformulation of the RG method to produce the cusp-up behavior of the modified viscosity (see section 2.2). The cusp-up behavior appears essential for a realistic eddy viscosity model (Kraichnan, 1976, Chollet and Lesieur, 1981, Domaradzki, Metcalfe, Rogallo and Riley, 1987). It has been shown (Zhou and Vahala, 1989) that the present RG modified viscosity (in the absence of the triple nonlinearity) cusps down instead of up: ν_T decreases instead of increases as k increases to the cutoff wavenumber k_c. One would like to correct this unphysical behavior.

Finally, we hope to derive the RG modified viscosity in the presence of a strong shear. This calculation will involve a forcing which is anisotropic at lowest order. We intend to find the appropriate form of the force correlation from the direct numerical simulation data for homogenous shear flow. One must also retain the sweeping and straining terms due to the mean shear in the equations of motion. These terms may be too cumbersome to allow for analytic RG, but present no difficulty for numerical implementation of an RG algorithm.

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