A dynamical systems analysis of the kinematics of time-periodic vortex shedding past a circular cylinder

By J. M. Ottino[1]

Computer flow simulation aided by dynamical systems analysis is used to investigate the kinematics of time-periodic vortex shedding past a two-dimensional circular cylinder in the context of the following general questions: Is a dynamical systems viewpoint useful in the understanding of this and similar problems involving time-periodic shedding behind bluff bodies? Is it indeed possible, by adopting such a point of view, to complement previous analyses or to understand kinematical aspects of the vortex shedding process that somehow remained hidden in previous approaches? We argue that the answers to these questions are positive. What follows is the result of a collaborative work with K. Shariff from NASA-Ames.

We find that several kinematical aspects of the problem become clear when viewed in terms of dynamical systems concepts such as periodic points and their associated manifolds. The flow is shown to produce transversal heteroclinic and homoclinic intersections, and it is, therefore, mathematically chaotic; these findings easily translate into an understanding of the transport mechanisms operating in the flow. The analysis relies on the identification of two classes of periodic points: (i) parabolic periodic points associated with separating and attaching pathlines, which produce unstable and stable manifolds that are associated with zero time-averaged wall shear, and (ii) a period-one hyperbolic point located in the wake itself. Stretching and transport are largely dominated by manifolds belonging to period-one parabolic points attached to the cylinder and a period-one hyperbolic point in the wake. Computed manifolds agree well with experimentally obtained streaklines and give a good indication of the regions of maximum and minimum stretch. Passive tracers are deformed in such a way that they approximate the regions traced out by unstable manifolds, the fluid filaments that stretch the most originate from near the stable manifolds and are initially oriented normal to them..

1. Setting

The understanding of the flow behind a cylinder is important in fluid mechanics and transport processes from both theoretical and applied viewpoints. However, in spite of numerous studies, it is apparent that there are still a few basic points that deserve some clarification and that new approaches should probably be examined. One such possibility is the application of dynamical systems tools which have proved useful in the analysis of chaotic advection and mixing (for a review see Ottino 1990). Such a viewpoint does not seem to have been used in the context of wake flows.

1 Department of Chemical Engineering, University of Massachusetts at Amherst

Let us review a few of the most important experimental results. Consider a circular cylinder with diameter D placed in a stream of fluid moving with uniform speed U in the x-direction. For $R_e = \rho UD/\mu << 1$, the flow is symmetric with respect to both the x-axis and the y-axis; as the R_e increases, the flow loses y-symmetry, and two attached eddies form behind the cylinder which grow in size with increasing R_e until, at $R_e \approx 40$, the flow ceases to be steady and becomes time-periodic. Dye injection experiments show that when $R_e \approx 100$ eddies are shed periodically from the top and bottom part of the cylinder, all the vortices originating from the top rotate in one direction; all the vortices originating from the bottom rotate in the opposite direction while the whole pattern of vortices travels downstream but with a speed smaller than U. As R_e is increased above 200 or so, the flow develops three-dimensionality, time-periodicity is lost, and the flow ultimately produces a turbulent wake (see figures 40-48 in van Dyke, 1982).

Computations have been able to mimic several of these findings (a recent and fairly complete analysis is given by Eaton 1987). However, it is apparent that a few points remain imperfectly understood and that there is a need for further studies. For example, until rather recently, there seemed to be questions regarding the topology of the streamlines and the implications on transport in the time-periodic vortex shedding regime. In a recent work, Perry, Chong & Lim (1982) tried to get insight into the transport process by studying streamline patterns at various phases. They argued that once the vortex shedding process begins, instantaneous "alleyways" of fluid are formed which penetrate the cavity. Even though they acknowledged that the relationship between instantaneous streamlines and streaklines is far from trivial, they nevertheless put forward a picture of the folding of a sheet of dye in the Kármán vortex street in terms of a thought provoking "threading" diagram (figure 8 of their paper), which looks, to someone trained in dynamical systems, very much like a portrait of a manifold.

It was, in fact, this threading diagram that prompted us to pose the following question: Can we use dynamical systems tools to understand, reveal, or at least rationalize known or new aspects of the vortex shedding mechanism? We are of the opinion that the answer to this question is affirmative and that a dynamical systems approach provides a complementary viewpoint to previous analysis for the analysis of the shedding process. In order to apply these tools, we need first to numerically simulate the velocity field. Such simulations were carried at NASA-Ames by K. Shariff. The numerical method used was ARC2D (Pulliam 1985), and the Mach number was 0.1 in all cases. The computational analysis was restricted to $80 < Re_D < 180$, and the results show good agreement between the computational simulations and available experimental results such as vortex shedding frequencies as a function of the Reynolds number (Williamson 1989) and streakline experiments. The structure of the instantaneous streamlines agrees, in general, with previous observations: separation bubbles grow and decay, and the topology of the velocity field allows for the formation of open "alleyways" for transport of fluid. However, we find also that separation bubbles achieve a minimum size and do not disappear, that there are small secondary bubbles which periodically merge with the primary bubbles, and that the instantaneous streamlines have at most one saddle point and sometimes they have none.

Once the velocity field is obtained, the analysis proceeds along the following points: (i) establishing the existence and location periodic points; (ii) determining the character of the periodic points; (iii) computing their associated manifolds and possible intersections. The analysis is conducted over a range of Reynolds numbers to study local and global bifurcations (changes in the character of periodic points and changes in intersections of manifolds, respectively; this program program of analysis is, in fact, very similar to that proposed for the analysis of other chaotic flows; see Table 7.1 in Ottino 1989). In our particular case, the key point resides in establishing the fact that there is a period-one hyperbolic point in the wake flow and that manifolds belonging to different periodic points produce transversal intersections.

2. Dynamical systems background

Consider a two-dimensional volume preserving time-periodic velocity field, i.e.

$$dx/dt = u(x,y,t), \tag{1a}$$

$$dy/dt = v(x,y,t), \tag{1b}$$

with $\partial u/\partial x + \partial v/\partial y = 0$ and $u(x,y,t) = u(x,y,t+T), v(x,y,t) = v(x,y,t+T)$. The fixed points (or critical points) of the system (1a, b) correspond to the condition $u = v = 0$, and as is well known, due to incompressibility, only saddles and centers are possible. In our case, however, we do not deal with the velocity field itself but rather with its associated "flow" or "motion", that is, the solution of (1a, b) for all possible initial conditions (x_0, y_0). The flow is typically written as

$$x(t) = \Phi t(x_0), \tag{2}$$

and indicates that the initial condition $x_0 = (x_0, y_0)$ is found at position $x = (x, y)$ at time t. However, since the velocity field (1a, b) is time-periodic, it is sometimes more convenient to deal with a map describing the state of the system at times $T, 2T, 3T, \ldots, nT$. This is usually written as

$$x_n = f(x_{n-1}), \tag{3a}$$

or, equivalently,

$$x_n = f^n(x_0). \tag{3b}$$

Flows and maps have fixed points and periodic points. Thus, P is a *fixed point of the flow* $x = \Phi_t(x_0)$ if

$$P = \Phi_t(P) \tag{4}$$

for all times t . On the other hand, P is designed as a *periodic point of period* τ if

$$P = \Phi_\tau(P) \quad \text{but} \quad \Phi_\tau(P) \neq P \quad \text{for} \quad t < \tau, \tag{5}$$

i.e. the material particle that happened to be at the position P at that time $t = 0$ will be located in exactly the same spatial position precisely after a time nT.

The definitions for mappings (3a, b) are similar. A point P is a *fixed point of the mapping* $f(\cdot)$ if

$$P = f^n(P) \quad \textit{for all} \quad n, \tag{6}$$

on the other hand, we say that P is a *periodic point of order n* of the map $f(\cdot)$ if

$$P = f^n(P) \quad \text{but} \quad f^m(P) \neq P \quad \text{for any} \quad m < n, \tag{7}$$

i.e. the particle initially located at P returns to its initial location after *exactly n* iterations [It is important to note that, in general, a fixed point of a flow and its corresponding mapping are in general not the same, and in general, the fixed point of a mapping corresponds to a periodic point of the flow.]

Fixed and periodic points of area-preserving maps and flows can be classified as hyperbolic, elliptic, or parabolic, according to the eigenvalues λ_1 and λ_2 of the flow in the neighborhood of the periodic point.

$$\begin{aligned} &|\lambda_1| > 1 > |\lambda_2|, \quad \lambda_1\lambda_2 = 1, && \text{hyperbolic} \\ &|\lambda_k| = 1 \quad (k = 1,2) \quad \text{but} \lambda_k \neq 1, && \text{elliptic} \\ &\lambda_k = \pm 1 \quad (k = 1,2), && \text{parabolic.} \end{aligned}$$

The associated physical picture is as follows. A small circle surrounding an elliptic point returns to its original position and undergoes a net rotation; within periods, there is a periodic stretching and compression as well. On the other hand, a circle enclosing a hyperbolic point returns to its initial location stretched in one direction by an amount λ_1 and compressed in the other by $1/\lambda_1$; since the process is repetitive, this leads to outflow in one direction and inflow in the other. Finally, the parabolic case corresponds to simple shear.

In the case of hyperbolic points, the linearization captures the essential behavior of the full nonlinear system (Hartmann-Grobman theorem, see Guckenheimer & Holmes 1983, p. 13). Hyperbolic points have associated invariant sets called the stable $[W^s(P)]$ and unstable $[W^u(P)]$ manifolds defined as

$$W^s(P) = \{\text{all } x_0 \in R^2 \text{ s.t. } \Phi_t(x_0) \to P \text{ as } \; t \to \infty\} \tag{8a}$$

$$W^u(P) = \{\text{all } x_0 \in R^2 \text{ s.t. } \Phi_t(x_0) \to P \text{ as } \; t \to \infty\}, \tag{8b}$$

which are tangent to the eigendirections, ξ_1 and ξ_2, of the linearized problem in the neighborhood of P (the definitions for maps are entirely similar).

Manifolds admit a physical interpretation; for example, the unstable manifold corresponds to region of fluid traced by a blob of fluid injected at the periodic point or to the streakline traced by a dye injected with an injection apparatus that moves as a whole with the periodic point (see figure 5.8.3 in Ottino 1989). It should be noted as well that in general, manifolds move in time. If observations, however, are made at the period of the hyperbolic point, the manifold remains motionless.

By definition, manifolds are invariant; particles x_0 belonging to $W^s(P)$ and $W^u(P)$ do so permanently and cannot escape from them. A point belonging simultaneously to both the stable and unstable manifolds of two different fixed (or periodic) points P and Q is called a *transverse heteroclinic point.* If $P = Q$, the point is called homoclinic. One intersection implies infinitely many and implies sensitivity to initial conditions. This is one of the fingerprints of chaos, and it is entirely equivalent to the formation of what is called a horseshoe map (Smale 1967, Moser 1973).

The definition of stable and unstable manifolds can be extended to parabolic points. In this case, however, as opposed to the case of hyperbolic points, the linearized flow fails to capture the essential aspects of the nonlinear system. In general, much less is known about this case in spite of its importance in transport processes near solid walls (see for example, Danielson & Ottino 1990, Camassa & Wiggins 1990).

The separation point point corresponds to a change of sign in the vorticity, $\partial u/\partial y = 0$, but it is evident that a linear expansion is unable to reveal this fact. A second order expansion about a point $x = x_0, y_0 = 0$ at a no-slip wall gives

$$u = A_{12}y + 2A_{112}(x - x_0)y + A_{122}y^2 \tag{9a}$$

$$v = -A_{112}y^2, \tag{9b}$$

where the coefficients

$$A_{12} = (\partial u/\partial y)_{x_0,y_0}, \tag{10a}$$

$$A_{112} = (1/2)(\partial^2 u/\partial x\partial y)_{x_0,y_0}, \tag{10b}$$

$$A_{122} = (1/2)(\partial^2 u/\partial y^2)_{x_0,y_0}, \tag{10c}$$

are, in general, functions of time which can obtained by forcing the expansion to satisfy the Navier-Stokes equations and suitable boundary conditions. The vorticity at the wall is given by

$$\omega|_{y=0} = [\partial v/\partial x - \partial u/\partial y]_{y=0} = -[2A_{112}(x - x_0) + A_{12}], \tag{11}$$

and the condition for $\omega = 0$ is met at

$$x_s - x_0 = -A_{12}/2A_{112}. \tag{12}$$

In general, the separation point x_s moves as a function of time. If $A_{112} < 0$, x_s corresponds to a point of separation and if $A_{112} > 0$, x_s corresponds to a point of attachment. The angle of separation is given by $\tan\Theta = 3\mu(\partial\omega/\partial x)/(\partial p/\partial x)$ where μ is the viscosity and p is the pressure (Oswatitsch 1958, Lighthill 1963). In terms of the coefficients of the expansion (9), $\tan\Theta = -3A_{112}/A_{122}$. In this case, the manifold can be obtained by placing a blob at $x = x_s, y = 0$. These results for steady flow may be generalized to time-periodic flow, and this will be discussed elsewhere (Shariff, Pulliam & Ottino, 1990).

3. Periodic points and manifolds

The entire wall of the cylinder consists of parabolic points, and the points of special interest are those corresponding to a change of sign in time-averaged shear stress, i.e. separation and attachment points of the time-averaged flow. It can be shown that manifolds emanate from such points. There is as well a periodic point in the wake itself; this is a period-one hyperbolic point. It is important to stress that this point is completely unrelated to the critical points in the velocity field itself. An important difference is that periodic points do not disappear; on the other hand, critical points in the velocity field itself can appear or disappear as the topology of the streamlines changes in time.

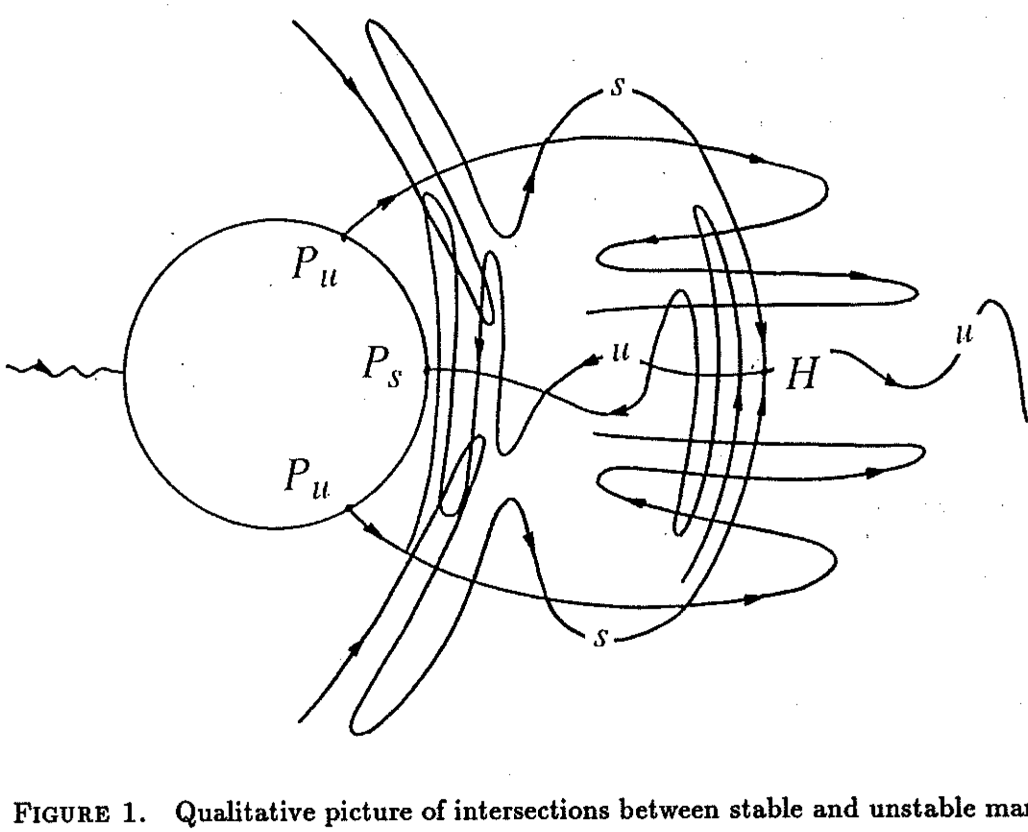

FIGURE 1. Qualitative picture of intersections between stable and unstable manifolds in the time-shedding regime.

Stable and unstable manifolds can produce heteroclinic and homoclinic intersections. In this particular flow, four types of transversal intersections are possible:

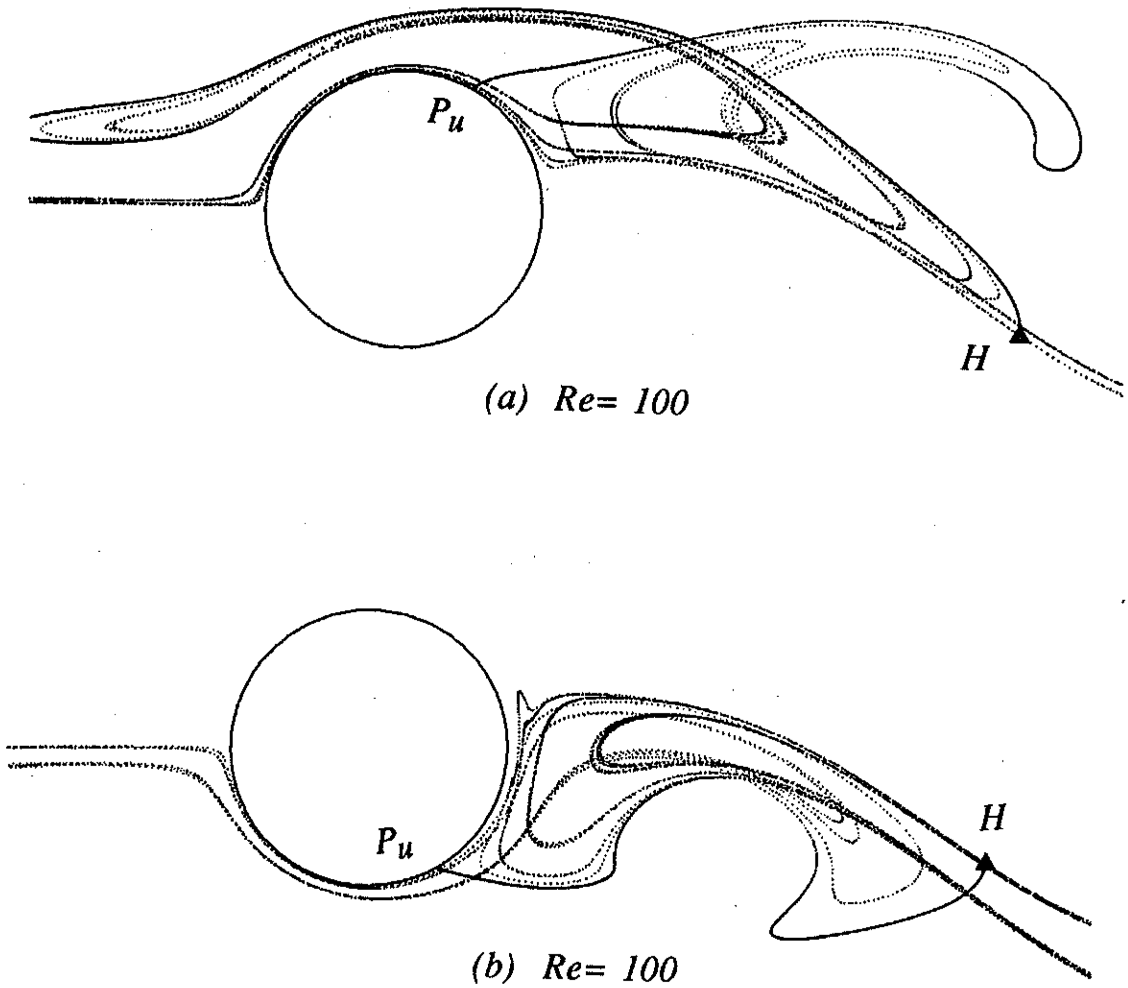

FIGURE 2. Intersections between unstable manifolds associated with (parabolic) separation points attached to the cylinder (denoted P) and stable manifolds associated with a hyperbolic points (denoted H) at Re=100; (a) manifold structure corresponding to the upper wake, (b) manifold structure corresponding to the lower wake.

heteroclinic intersections are produced by intersections of stable manifolds of the period-one hyperbolic with unstable manifolds of the periodic points attached to the cylinder as well as by unstable manifolds of the hyperbolic point intersecting the stable manifolds attached to the surface of the cylinder, *homoclinic* intersections are produced by crossings of stable and unstable manifolds belonging to the hyperbolic point as well as those of parabolic points attached to the cylinder. The complete manifold picture of the system is, however, more complex since there are additional period-one hyperbolic points close to the surface of the cylinder as well as higher order periodic points; they, however, seem to contribute much less to the gross aspects of the transport in the flow.

The manifold structures of the periodic points described in the previous sections results in the qualitative picture shown in figure 1 (the picture has been deformed, without changing topology, to improve clarity). Such a picture is characteristic of chaotic systems and contains information regarding transport of material in the flow (Rom-Kedar, Leonard & Wiggins 1990). Figure 2 shows computed pictures corresponding to $R_e = 100$, whereas figure 3 shows the changes in structure at $R_e = 180$. For the sake of clarity, figures 2(a) and 3(a) show the manifold structure corresponding to the upper wake; figures 2(b) and 3(b) show the manifold structure corresponding to the lower wake.

The manifold structure of figures 2 and 3 provides a template for stretching and transport and provides a qualitative picture for the stretching and folding of a streakline in the wake. Previous work suggests that dye patterns are well represented by the unstable manifolds belonging to low order hyperbolic periodic points (Swanson & Ottino 1990, Shariff 1989). This expectation is confirmed in the present work, and unstable manifolds associated with low order periodic points provide an excellent template for the study of evolution of stretching in the flow; most of the stretching originates from regions in the neighborhood of the stable manifold, and the orientations corresponding to the maximum stretch are oriented normal to the manifold.

Another quantity of kinematical interest is the stretching history, $\nabla v : mm$, following a fluid element, which is related to the Lyapunov exponents (Khakhar & Ottino 1986). This quantity enters in the convection diffusion equation which governs the molecular diffusion of the dye comprising the streakline (Ottino 1989, p. 276). Regions with large values of $\nabla v : mm$ correspond to large stretching along m, which tends to minimize the spread of molecular diffusion normal to the streaklines but at the same time tends to make the streakline less visible since the thickness of the dye streak is significantly reduced. On the other hand, low or negative values of $\nabla v : mm$ blur the picture. Such a situation arises near the tip of folds since as the dye folds there is a change of sign of $D : mm$. A comparison of computational results and experiments shows that it is precisely in these regions where molecular diffusion effects become important. Another quantity of interest for which experiments are available is the computation of distribution of residence times of fluid elements behind the cylinder. A detailed presentation of all the above results is in preparation (Shariff, Pulliam & Ottino, 1990).

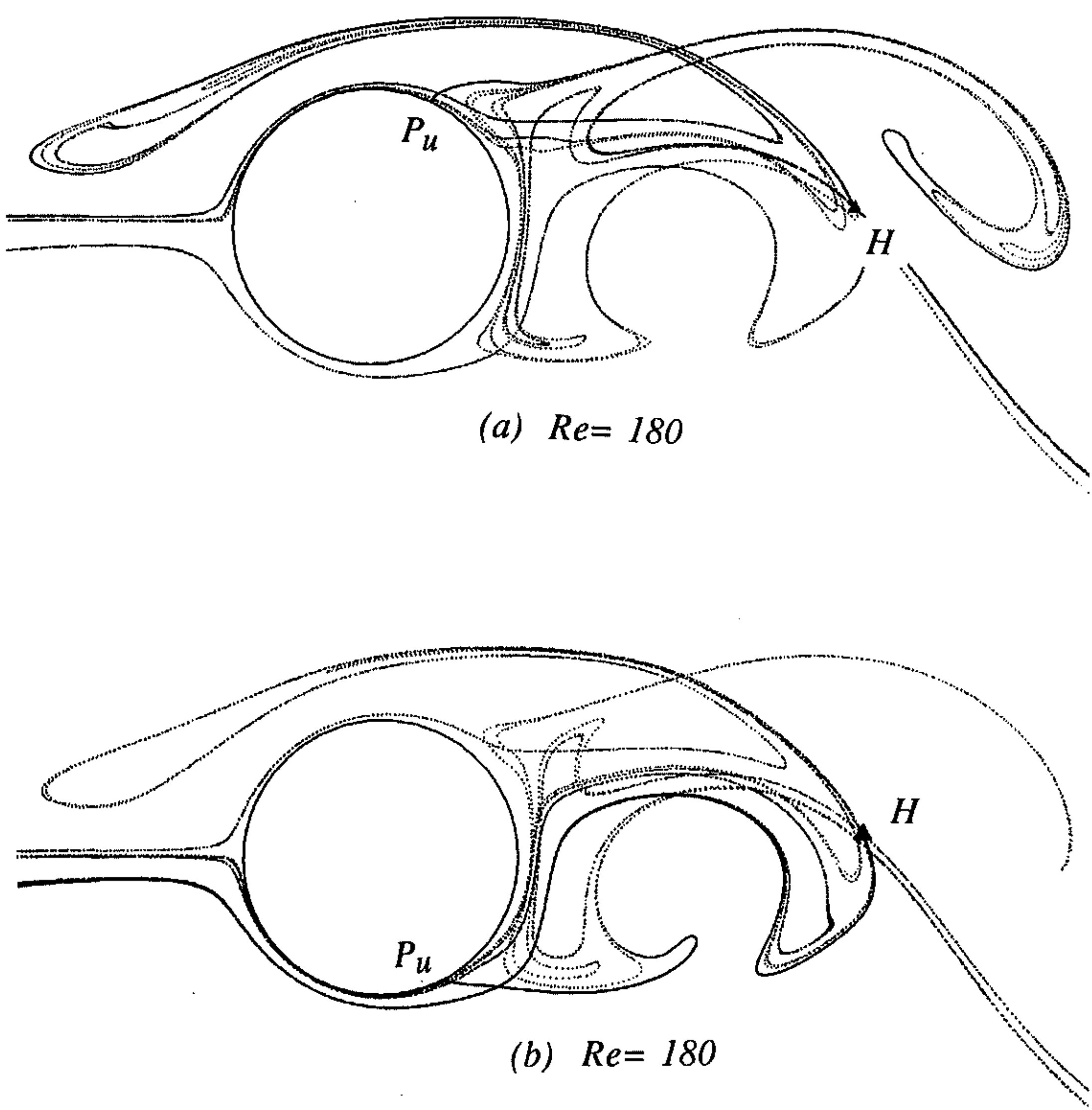

FIGURE 3. Intersections between unstable manifolds associated with (parabolic) separation points attached to the cylinder (denoted P) and stable manifolds associated with a hyperbolic points (denoted H) at Re=180; (a) manifold structure corresponding to the upper wake, (b) manifold structure corresponding to the lower wake.

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