Numerical simulations of turbulent thermal convection with differential rotation

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1. Motivation and objectives

The solar nebula, from which the planets in our solar system formed, featured a disk of gas and dust grains in rapid, differential rotation, and at some stage was likely to have been unstable to thermal convection. This situation is suspected by many to lead to significant turbulent Reynolds stress production and angular momentum transport in such systems, and estimates of transport rates have been attempted from unsubstantiated phenomenological models (cf. Ruden & Lin 1986; Cabot *et al.* 1987). In order to determine the circumstances and physical conditions under which our own planetary system formed and to explain recent observations of young stellar systems, it is necessary to develop realistic models of heat and angular momentum transport for such flows. Developing an understanding of complicated flows featuring thermal convection, rotation, and shear is also of wide interest in stellar astrophysics and in planetary and terrestrial atmospheric studies.

In order to determine the nature of Reynolds stress production under localized solar nebula conditions, we have performed direct numerical simulations of channel flow (Cabot 1990a, Cabot *et al.* 1990b, Cabot & Pollack 1990c) that include Boussinesq thermal convection (incompressible except for buoyant density fluctuations) with linearly varying gravity and centrifugally stable differential rotation whose axis is aligned with gravity and whose gradient is perpendicular thereto (see Figure 1). These simulations are unrealistic in their near-incompressibility and uniformity of density, in their too high Prandtl number Pr and too low Reynolds number Re (required to resolve the flow fully), and in their impermeable "wall" vertical boundary conditions, although these have been relaxed to no-stress conditions. These simulations do, however, reveal that the turbulent Reynolds stress has a very complex behavior that is sensitive to the rotation and shear rates as well as the Reynolds number.

Our ultimate objective is to develop workable models based on the numerical simulations for constructing global solar nebula models; viz., we want to characterize, quantify, and develop relatively simple prescriptions for heat and angular momentum fluxes from given system parameters (e.g., ratios of rotation, shear, and convective lapse rates). Toward this end, our program has been (1) to attempt to understand the behavior of the direct numerical simulations of Boussinesq convection, which, despite the complexity of the results, is still an overly simplified approximation to the real system and should be more amenable to analysis. These results are also intended to be tested against turbulence models, especially those designed for atmospheric boundary layers, and may provide a basis for subgrid-scale models. (2) In order to make the numerical simulations more realistic with regard

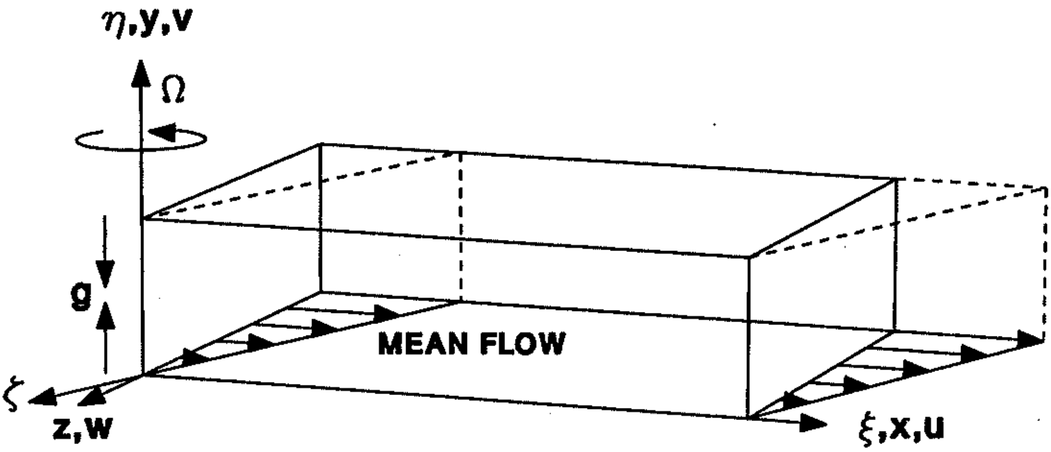

FIGURE 1. Local Cartesian coordinates of the numerical domain with the orientation of gravity (g) and rotation ($\mathbf{\Omega}$): co-rotating coordinates (x, y, z) and velocity components (u, v, w) correspond to azimuthal (streamwise), vertical, and radial (spanwise) coordinates in the solar nebula (channel flow). The co-moving coordinates (ξ, η, ζ) follow the mean flow, as illustrated by the dashed box.

to the solar nebula problem, our goal has been to develop a fully compressible code that will allow us to incorporate large density stratifications and realistic thermodynamic and radiative properties. (3) In order to explore the properties of these flows at the very high values of Re found in natural systems and the very low values of Pr found in most astrophysical contexts, we will need to employ large-eddy simulations for which we want to determine the most appropriate subgrid-scale model to incorporate.

2. Accomplishments

2.1 Direct numerical simulation results for Boussinesq convection

Sequences of direct numerical simulations with Boussinesq thermal convection subjected to centrifugally stable differential rotation (described in Cabot 1990a and Cabot & Pollack 1990c) were completed and their turbulence statistics were compiled. Most sequences used Keplerian rotation, which is typical of centrifugally balanced solar nebula disks. The simulation was carried out with a channel code (Kim, Moin & Moser 1987) that was modified to follow the sheared flow (Rogallo 1981). The key results were: (1) The vertical convective heat flux is largely independent of the horizontal shear *per se*, though it does depend indirectly on it through the epicyclic frequency κ (where $\kappa^2 \equiv 2\Omega[2\Omega + S]$ for a rotation rate Ω

and horizontal shear rate S), which measures the effects of Coriolis forces; (2) the net shear production rate for Keplerian rotation is positive (as expected) at low κ but becomes negative at higher critical values of κ (which increase for increasing Re); and (3) very long streamwise structures develop at high κ nearly independent of Re for fixed Péclét number Pe.

The relative insensitivity of the vertical heat flux to details of horizontal shearing suggests that this can be approximated in physical systems with models that take account of suppression of convection by centrifugally stable rotation. For example, a simple mixing length model for the heat flux with uniform rotation was suggested by Cabot *et al.* (1990b); one would merely need to generalize the rotational vorticity 2Ω to κ in this model.

The behavior of the shear production rate at rapid rotation and shear rates has yet to be understood or modeled adequately. A crude model by Kichatinov (1986), based on linearized equations, appears to reproduce the negative shear production rate at rotation rates comparable to an unspecified turbulence time τ. However, the assumptions that go into the model ($|S| \ll \Omega \ll \tau$) are all violated in our flow. Moreover, Kichatinov's model is constructed for infinite Re, whereas our results apply to low Re, and the trend for our shear production rates to become negative shows signs of disappearing asymptotically at high Re.

Inhomogeneous features of the Reynolds stress $-\overline{uw}$ and shear production rate $-S\overline{uw}$ also need to be explained. Negative shear production at high κ is mostly generated in the outer wall regions (see Figure 2). There are regions around $|y| = 0.6$ that always have positive shear production; these roughly coincide with regions of maximal buoyancy production. The interior regions develop less significant negative shear production rates in low-Re, high-κ simulations, and tend even more to zero at higher Re. The degree of correlation between u and w is never very large, varying between ± 0.3. A sequence of simulations was also performed with heat sinks in the exterior regions so that the outer half of the channel is convectively stable. This better approximates the stable radiative exterior that exists in solar nebula disks and reduces the direct effects of the impermeable walls on the interior convection. Reynolds stress production in the interior of these simulations (see Figure 3) is similar to that across the full width of the fully convective channel (cf. Figure 2). The "splatting" effect at the impermeable walls (ascribed to pressure effects or kinematic blocking) that was seen in the fully convective channel simulations is also present to a lesser degree near the convectively stable-unstable boundary ($|y| \approx 0.5$). This is also near where the most negative correlations between u and w (and the most negative shear production) occur, suggesting a possible link.

The asymptotic behavior with respect to Re cannot be determined conclusively at rapid Keplerian rotation rates due to the appearance of "quasi-two-dimensionality" (Q2D) in which very long streamwise wavelengths develop in one or more of the fluctuating quantities such that the turbulent field cannot be resolved numerically. Linear analysis (cf. Knobloch 1984, 1985) predicts that shear will reduce fluctuations with streamwise variations through the direct action of shearing and indirectly by contracting spanwise wavelengths (in the co-moving frame) such that they are

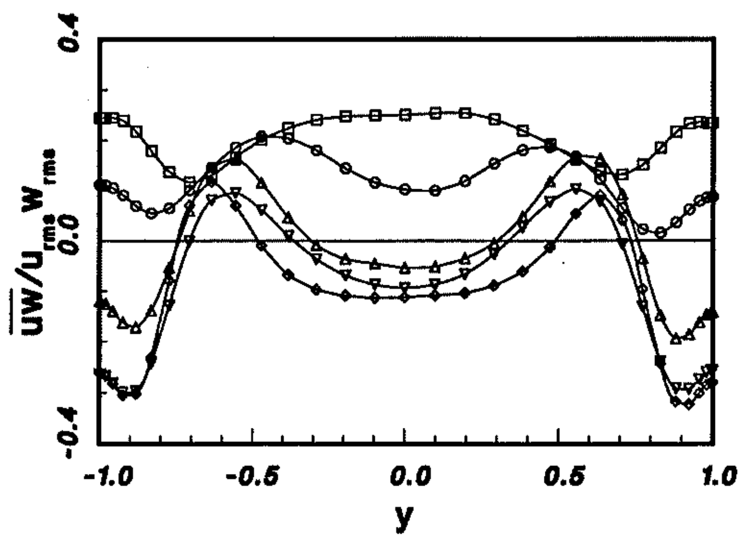

FIGURE 2. Vertical profiles of Reynolds stress correlation coefficients for the fully convective Keplerian rotation sequence with $Re = 559$ for different κ: □ 0.141, ○ 0.253, △ 0.447, ◇ 0.612, and ▽ 0.707.

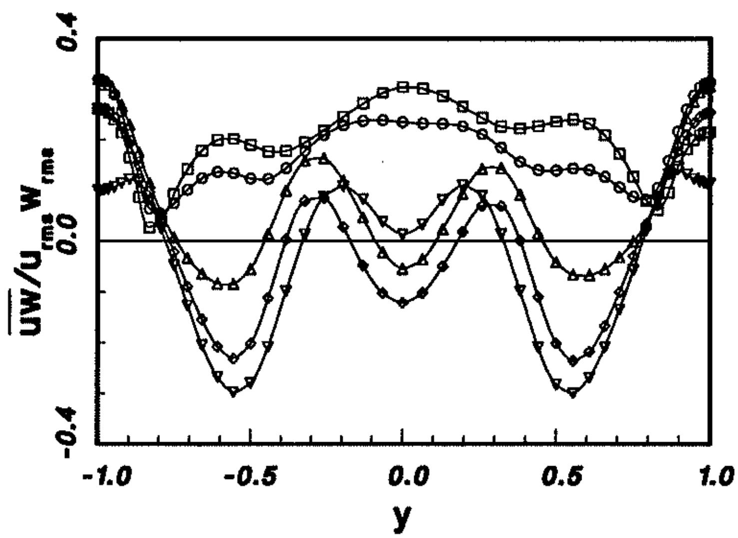

FIGURE 3. Same as Figure 2, but with convectively stable exterior regions ($|y| > 0.5$).

preferentially dissipated. In the long-time limit, all streamwise variations vanish and the shear production is negative perforce for centrifugally stable flows (Cabot & Pollack 1990c). It appears — qualitatively — that nonlinear redistribution of turbulence energy between wavevectors can proceed efficiently at low κ where there is little stabilization due to Coriolis forces, but that at higher κ, the nonlinear redistribution is hindered sufficiently to allow the linear tendencies to prevail. This picture has some resemblance to the effects that rapid uniform rotation is found to have on nonlinear transfer in turbulence (e.g., through EDQNM analysis by Cambon & Jacquin 1989). However, little progress has been made yet in positively identifying and quantifying the factors that control the onset of Q2D. Both the vertical inhomogeneity and the nonseparability induced by the shear make the problem largely unamenable to spectral analyses, such as EDQNM. Though linear analysis can be applied, it fails to give useful or reliable information about the final turbulence statistics.

For given Pe, Q2D sets at high critical epicyclic frequency κ_c with little (and perhaps inverse) dependence on Re. Q2D sets in between $\kappa = 0.612$ and 0.707 for $Re = 559$ (based on the basal convective lapse rate and channel halfwidth); preliminary results from a very high Reynolds number simulation with $Re = 3200$ indicate that Q2D has marginally set in at $\kappa = 0.612$. This suggests that $\kappa_c \sim Re^{-\alpha}$ with $0 \leq \alpha \leq 1/12$. Also note that oscillatory convection becomes the most rapidly growing disturbance in linear perturbation analysis at values of κ lower by about half, but also depending weakly and inversely on Re ($\kappa_c \sim Re^{-1/6}$). This also suggests that Q2D is associated with strong inertial wave production.

2.2 Direct numerical simulations with full compressibility

A fully compressible channel code (Thompson 1990) has been developed and direct numerical simulations for thermal convection with linearly varying gravity and moderate density stratification have been performed. The results are found to agree very well with the Boussinesq simulations. The r.m.s. Mach number of the turbulence was found for moderate convection ($Pe \sim 100$) to be about 0.3, which is too low for significant compressibility effects. The turbulence kinetic energy was found to be distributed relatively evenly across the channel, as for the corresponding Boussinesq cases, despite a decline in density by a factor of about 3 from midchannel to wall, indicating a tendency for turbulence intensities to vary as the inverse square root of the density. This suggests that highly stratified flows may exhibit very high turbulence intensities in the exterior regions, perhaps leading to shock(let?) formation.

The compressible code at present follows acoustic waves explicitly, which puts a limit on the timestep to achieve numerical stability. For situations in which convective motions are highly subsonic — which may well be the case for rapidly rotating convection —, this limit is overly restrictive in terms of computational costs. We are presently exploring alternative systems of equations that filter sound waves (widely known as "anelastic" or "soundproofed" equations); or implicit timestepping of acoustic modes ("muffling"?) whereby long timesteps can be used that are numerically stable but inaccurate. The latter alternative is more operationally flexible in terms of treating full or filtered compressibility, but it remains to be seen if it is the most cost-effective.

2.3 Subgrid-scale modeling for large-eddy simulations

The Smagorinsky model for the subgrid-scale (SGS) Reynolds stress is purely dissipative: it extracts energy from the resolved turbulent field. However, the residual SGS Reynolds stress dissipation was computed by Piomelli *et al.* (1990) from direct numerical simulations of channel flow, *inter alia*, with a variety of filters, and it was found that the mean dissipation is generally a small residual of positive and negative terms. At points in transitional flow, energy may actually backscatter into the resolved scales from the unresolved scales. This suggests that more realistic SGS models should include large stochastic terms that allow backscatter.

A model that in principle allows for realistic fore- and backscatter was proposed by Germano *et al.* (1990) based on extrapolating the behavior of SGS stress from the highest resolved wavenumbers in a numerical simulation fitted to a Smagorinsky-esque model. The spatially local version of the model was found to be numerically ill conditioned; thus, well conditioned averages over homogeneous planes were used instead. The advantages of this approach are (1) that no constant need be specified *a priori* for the Smagorinsky model; (2) the new, extrapolated constant can have any sign, so that the SGS model need not be absolutely dissipative in the mean; and (3) the proper wall behavior for the SGS dissipation is predicted without the need for *ad hoc* wall-damping functions. A large-eddy simulation (LES) of channel flow was performed incorporating this model and was followed through transition to turbulence; the new model was found to give comparable or improved results compared with LES's using a standard Smagorinsky model with wall-damping functions. Work was begun incorporating a more general version of this model in a finite-difference channel code.

3. Future plans

3.1 Modeling of Boussinesq convection

We hope to apply one-point closure models to the results from Boussinesq thermal convection with rapid rotation and horizontal shear in order to test their ability to reproduce the observed heat flux properties and the tendency to give net negative shear production at rapid rotation and low Reynolds number. We also want to reproduce the vertical profiles of the observed Reynolds stress profiles. If successful, such a model, modified to include a mean density profile, could be applied to results with density stratification provided other compressibility effects were negligible. Two-point (spectral) closure models are likely to be intractable when applied to these sheared, inhomogeneous flows, but perhaps simpler test situations (e.g., locally homogeneous) can be made tractable to such analyses and made to yield some insight to the long-wavelength behavior associated with rapid rotation and shear.

The Boussinesq simulation code will also likely be used as a testing ground for LES's with large (or infinite) Reynolds numbers and very low (or zero) Prandtl numbers with moderate Péclét numbers. However, it may prove easier to modified extant LES codes to include thermal convection and differential rotation. We will also consider testing SGS Reynolds stress equations, based on one-point closure models, that can readily take account of buoyancy production effects. We wish to test the possibility, though, that SGS buoyancy models may not be important in low-*Pr* flows because thermal conductivity preferentially destroys thermal fluctuations at small (subgrid?) scales; if this were the case, the SGS modeling could be greatly simplified for our purposes.

3.2 Compressible simulations

We will proceed with a limited number of fully compressible simulations of turbulent channel convection, largely to assess the importance of acoustic effects in the flow and to test the validity of the assumptions that go into deriving anelastic or soundproofed systems of equations that attempt to approximate low-Mach-number flows. When we believe that we have determined the most efficacious means of sound-filtering, we will modify the existing code and perform some direct numerical simulations in order to validate it numerically as well as physically by comparison with fully compressible results.

The introduction of more realistic thermodynamics, including large density stratification, will lead naturally to a convectively stable exterior region; this may largely mitigate the effects of inaccurate wall boundary conditions. We will also successively introduce uniform rotation and differential rotation. The latter will probably require some significant modifications in the code in order to follow the mean sheared flow. As with the Boussinesq code, we will hope to implement an SGS model in order to perform LES's for high-*Re*, low-*Pr* situations; if the Mach numbers in the flow are sufficiently low, then SGS models developed for incompressible cases may be sufficient. We will be keenly interested in learning if long-wavelength behavior develops in the compressible simulations at rapid rotation as it did in the Boussinesq ones. If it does, it may not be possible to construct models in a parameter range useful to solar nebula modeling with these channel simulations; rather, numerical simulations of curved channels with large azimuthal coverage will be necessary.

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