Local isotropy in buoyancy-generated turbulence

By Jeffrey R. Chasnov

1. Motivation and objectives

Batchelor *et al.* (1992) recently considered the turbulent motion generated by buoyancy forces acting on random fluctuations in the density of an infinite fluid. This homogeneous buoyancy-generated flow field with zero mean density gradient was conceived as an idealized system which, like isotropic turbulence, may be useful as a vehicle for the general study of turbulence.

The Batchelor *et al.* study focused on large-scale density and velocity fluctuations and yielded power-law forms for the asymptotic decay of their mean-square values. An interesting discovered feature of the buoyancy-generated flow field was that no matter how small the initial buoyancy force, the fluid motion always becomes turbulent with an increasing Reynolds number at large times, in contrast to isotropic turbulence, where the Reynolds number decreases asymptotically.

An increasing Reynolds number of the flow at large times implies an active small-scale turbulence. Although buoyancy-generated turbulence is not isotropic but is axisymmetric at the largest scales where buoyancy forces are strongest, it may become isotropic at the smallest scales due to a local Kolmogorov-like cascade of energy and density-variance from large to small scales. The buoyancy-generated flow field thus presents to us a simple physical flow in which it is possible to study the turbulence cascade from anisotropic large scales to isotropic small scales. The question as to whether large-scale anisotropy may induce anisotropy in the small-scales even in high Reynolds number flows is a matter of current controversy (Yeung & Brasseur 1991; Waleffe 1992).

The Batchelor *et al.* study relied partly on theoretical analysis and partly on direct and large-eddy numerical simulations of the flow field. To this mix, we add here a two-point closure study based on the eddy-damped quasi-normal Markovian (EDQNM) closure model applied to axisymmetric turbulence. The EDQNM model has been shown to yield reasonably accurate quantitative results for a variety of problems in homogeneous turbulence (Lesieur 1987). The main advantage here in applying EDQNM to the buoyancy-driven flow field is the wide range of wavenumbers over which a solution of the EDQNM equations may be solved. Whereas a typical large-eddy simulation using 128^3 grid points has a wavenumber range of only 60, the EDQNM calculation can be easily run with a wavenumber range of several decades. Because of the growth in length scales in the buoyancy-driven flow field, this large wavenumber range allows for a solution of the flow field well into its asymptotic regime. Recent comparisons between large-eddy simulations and closure theory (Herring 1990) indicate that a time longer than that attainable by current large-eddy simulations is required to reach flow asymptotics and that conclusions based on large-eddy simulation results may be based only on an intermediate transient state.

In this research brief, we will briefly introduce the EDQNM equations for the buoyancy-generated flow field. We then present a Kolmogorov-like theoretical argument on the scaling of the small-scale spectra. This scaling is then confirmed by numerical solution of the EDQNM equations. We briefly conclude with possible future research directions.

2. Accomplishments

2.1. Spectral equations

The EDQNM approximation for homogeneous buoyancy-generated turbulence applied to a velocity field $\mathbf{u}$ and a buoyancy field ϕ results in time-evolution equations for four spectra of arguments k and η_k, where k is the magnitude of wavevector $\mathbf{k}$ and $\eta_k = k_z/k$, where k_z is the vertical component of the wavevector.

The four spectra F^1, F^2, F^3, and F^4 are most easily defined as the scalar functions which specify the velocity and buoyancy correlation tensors in Fourier space:

$$< u_i(\mathbf{k})u_j(\mathbf{k}') >= 4\pi k^2 \left[F^1(k,\eta_k)e_i^1(\mathbf{k})e_j^1(\mathbf{k}) + F^2(k,\eta_k)e_i^2(\mathbf{k})e_j^2(\mathbf{k})\right] \delta(\mathbf{k}+\mathbf{k}'), \quad (1)$$

$$< u_i(\mathbf{k})\phi(\mathbf{k}') >= 4\pi k^2 F^3(k,\eta_k)e_i^2(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}'), \quad (2)$$

$$< \phi(\mathbf{k})\phi(\mathbf{k}') >= 4\pi k^2 F^4(k,\eta_k)\delta(\mathbf{k}+\mathbf{k}'), \quad (3)$$

where the three vectors $\mathbf{e}^1(\mathbf{k})$, $\mathbf{e}^2(\mathbf{k})$, and $\mathbf{k}/k$, form an orthonormal basis which span the wavespace, and $\mathbf{e}^1(\mathbf{k})$ and $\mathbf{e}^2(\mathbf{k})$ are given by

$$\mathbf{e}^1(\mathbf{k}) = \frac{\mathbf{k}\times\mathbf{j}}{|\mathbf{k}\times\mathbf{j}|}, \qquad \mathbf{e}^2(\mathbf{k}) = \frac{\mathbf{k}\times\mathbf{e}^1(\mathbf{k})}{k|\mathbf{k}\times\mathbf{j}|}, \quad (4)$$

where $\mathbf{j}$ is the unit vector in the direction of the gravitational field.

The exact unclosed equations governing the time-evolution of the homogeneous buoyancy-generated turbulence spectra are

$$\frac{\partial F^1(k,\eta_k)}{\partial t} + 2\nu k^2 F^1(k,\eta_k) = T^1(k,\eta_k), \quad (5)$$

$$\frac{\partial F^2(k,\eta_k)}{\partial t} + 2\nu k^2 F^2(k,\eta_k) = T^2(k,\eta_k) - 2g\sqrt{1-\eta_k^2}F^3(k,\eta_k), \quad (6)$$

$$\frac{\partial F^3(k,\eta_k)}{\partial t} + (\nu + D)k^2 F^3(k,\eta_k) = T^3(k,\eta_k) - g\sqrt{1-\eta_k^2}F^4(k,\eta_k), \quad (7)$$

$$\frac{\partial F^4(k,\eta_k)}{\partial t} + 2Dk^2 F^4(k,\eta_k) = T^4(k,\eta_k), \quad (8)$$

where ν and D are the molecular transport coefficients associated with the velocity and buoyancy fields, g is the gravitational constant, and the T's are the spectra of the non-linear transfers. Note that with our definition of the axisymmetric energy spectra F^1 and F^2, the buoyancy force appears explicitly only in the time-evolution equation for F^2.

The EDQNM approximation provides a means to express the transfer spectra solely as functionals of the F's. Although a derivation of the EDQNM forms for the transfer spectra is too long to present here, we note that the transfer spectra require evaluation of triple integrals over wavenumber magnitudes p and q (such that $\mathbf{p}+\mathbf{q}=\mathbf{k}$) and the azimuthal angle of $\mathbf{q}$. Furthermore, the integrands contain products of the various F's as functions of (k,η_k), (p,η_p), and (q,η_q). In addition, they contain two phenomenological rates associated with the velocity and buoyancy fields, which have been fixed so as to recover the reasonable values $\alpha_E = 1.8$ and $\alpha_G = 0.7$ for the inertial and inertial-convective subrange constants, respectively, in isotropic turbulence.

2.2. Small-scale spectra

In this section, we apply Kolmogorov-like arguments — assuming such arguments to be valid — to the small-scale spectra of buoyancy-generated turbulence . We begin by defining spectra E(k), H(k), and G(k), of wavenumber magnitude only, by

$$E(k) = \frac{1}{2}\int_0^1 d\eta_k \left[F^1(k,\eta_k) + F^2(k,\eta_k)\right], \qquad (9)$$

$$H(k) = -\int_0^1 d\eta_k \sqrt{(1-\eta_k^2)}F^3(k,\eta_k), \qquad (10)$$

$$G(k) = \int_0^1 d\eta_k F^4(k,\eta_k). \qquad (11)$$

For an isotropic turbulence, $E(k)$ is the usual energy spectrum, $G(k)$ is the spectrum of a passive scalar field, while $H(k)$, the spectrum of $< u_3\phi >$, vanishes identically.

Original arguments by Kolmogorov and by Corrsin and Obukhov (Monin & Yaglom 1975) state that universal small-scale spectra for E(k) and G(k) may be constructed from the energy cascade rate ϵ, the kinematic viscosity ν, and the scalar-variance cascade rate ϵ_θ. We assume here that the Prandtl number (Schmidt number) $\sigma = \nu/D$ is unity so that we need not consider small and large Prandtl number effects. By dimensional arguments, universal small-scale spectra $\hat{E}$ and $\hat{G}$ are defined by

$$E(k) = \left(\epsilon\nu^5\right)^{1/4}\hat{E}(\hat{k}), \qquad G(k) = \epsilon_\theta\left(\frac{\nu^5}{\epsilon^3}\right)^{1/4}\hat{G}(\hat{k}), \qquad (12)$$

where

$$k = \left(\frac{\epsilon}{\nu^3}\right)^{1/4}\hat{k} \qquad (13)$$

defines a dissipation wavenumber $k_d = (\epsilon/\nu^3)^{1/4}$, which separates the inertial subrange from the viscous subrange.

It is unclear what form the non-dimensional buoyancy-flux spectrum $\hat{H}(\hat{k})$ should take, since an additional dimensional parameter, namely g, may enter into its specification. Nevertheless, we proceed by non-dimensionalizing $H(k)$ without use of g, call the resulting non-dimensional spectrum $\hat{h}(\hat{k})$,

$$H(k) = \epsilon_\theta{}^{1/2}\left(\frac{\nu^5}{\epsilon}\right)^{1/4} \hat{h}(\hat{k}), \tag{14}$$

and hope for some further guidance from the equations of motion for the non-dimensional spectra.

These equations are determined from Eqs. (5)-(8) to be

$$\frac{\partial \hat{E}(\hat{k})}{\partial \hat{t}} + 2\hat{k}^2 \hat{E}(\hat{k}) = \hat{T}_E(\hat{k}) + B\hat{h}(\hat{k}), \tag{15}$$

$$\frac{\partial \hat{h}(\hat{k})}{\partial \hat{t}} + \frac{1+\sigma}{\sigma}\hat{k}^2 \hat{h}(\hat{k}) = \hat{T}_h(\hat{k}) + B\xi\hat{G}(\hat{k}), \tag{16}$$

$$\frac{\partial \hat{G}(\hat{k})}{\partial \hat{t}} + \frac{2}{\sigma}\hat{k}^2 \hat{G}(\hat{k}) = \hat{T}_G(\hat{k}), \tag{17}$$

where $t = \hat{t}\sqrt{\nu/\epsilon}$ defines the non-dimensional time (we neglect here the time-dependence of ϵ), the $\hat{T}$'s are the non-dimensional transfer spectra, $\xi = 2/3$ if the scalar field is isotropic at wavenumber $\hat{k}$, and B is a non-dimensional number, defined by

$$B = \frac{g\sqrt{\epsilon_\theta \nu}}{\epsilon}. \tag{18}$$

The number B may also be expressed as a ratio of a buoyancy wavenumber to the dissipation wavenumber

$$B = (k_b/k_d)^{2/3}, \tag{19}$$

where

$$k_b = \left(\frac{g^6 \epsilon_\theta^3}{\epsilon^5}\right)^{1/4}. \tag{20}$$

For $\hat{k}$ of order unity and $B \ll 1$, buoyancy effects on the flow become small and we expect the last term of Eq. (15) to be negligible, yielding an energy spectrum free from the effects of buoyancy. However, the last term of Eq.(16) cannot be neglected because without it the isotropic equations are recovered and the non-dimensional buoyancy-flux would vanish identically. This suggests the scaling $\hat{h}(\hat{k}) = B\hat{H}(\hat{k})$ so that a universal function for the buoyancy-flux should be defined by

$$H(k) = g\epsilon_\theta \left(\frac{\nu^7}{\epsilon^5}\right)^{1/4} \hat{H}(\hat{k}), \tag{21}$$

which yields the following equations for the universal small-scale spectra:

$$\frac{\partial \hat{E}(\hat{k})}{\partial \hat{t}} + 2\hat{k}^2 \hat{E}(\hat{k}) = \hat{T}_E(\hat{k}) + B^2 \hat{H}(\hat{k}), \tag{22}$$

$$\frac{\partial \hat{H}(\hat{k})}{\partial \hat{t}} + \frac{1+\sigma}{\sigma}\hat{k}^2 \hat{H}(\hat{k}) = \hat{T}_H(\hat{k}) + \xi \hat{G}(\hat{k}), \tag{23}$$

$$\frac{\partial \hat{G}(\hat{k})}{\partial \hat{t}} + \frac{2}{\sigma}\hat{k}^2 \hat{G}(\hat{k}) = \hat{T}_G(\hat{k}). \tag{24}$$

For $B \ll 1$, the energy spectrum $\hat{E}(\hat{k})$ and buoyancy spectrum $\hat{G}(\hat{k})$ reduce to their isotropic form, whereas a new universal spectrum of the buoyancy-flux $\hat{H}(\hat{k})$ is defined.

The scaling of the three spectra in the inertial subrange may be obtained by requiring $\hat{E}(\hat{k})$, $\hat{G}(\hat{k})$, and $\hat{H}(\hat{k})$ to satisfy power-law behaviors such that viscosity ν cancels explicitly. In this way, we find the usual Kolmogorov and Corrsin-Obukhov spectra

$$E(k) = \alpha_E \epsilon^{2/3} k^{-5/3}, \qquad G(k) = \alpha_G \epsilon_\theta \epsilon^{-1/3} k^{-5/3} \tag{25}$$

and an additional inertial buoyancy-flux spectrum

$$H(k) = \alpha_H g \epsilon_\theta \epsilon^{-2/3} k^{-7/3}, \tag{26}$$

seen to be directly proportional to g. The buoyancy-flux spectrum $H(k)$ is observed to decrease faster than $\sqrt{E(k)G(k)}$ with increasing k as is reasonable for a return-to-isotropy of the small scales. We note here that a $k^{-7/3}$ spectrum for $H(k)$ has been previously predicted for a stably stratified flow (Lumley 1964) and also for the spectrum of the cross-correlation $< uv >$ in homogenous-shear turbulence (Leslie 1972).

2.3. The turbulence cascade

An interesting consequence of the non-zero inertial buoyancy-flux spectrum, Eq. (26), is its effect on the inertial cascade of energy from large to small scales. For wavenumbers in the inertial range, the buoyancy force continually adds energy to the cascade and an equation for the variation of the cascade rate with wavenumber may be determined to be

$$\frac{\partial \epsilon(k)}{\partial k} = gH(k). \tag{27}$$

Following Lumley's (1964) work on stably-stratified flows, we assume that the cascade rate ϵ which enters into the scaling of the universal spectra may be taken to be $\epsilon(k)$, in contrast to isotropic turbulence, where it is independent of k. (However, in our buoyancy-generated flow, the density-variance cascade rate ϵ_θ which enters into the scaling of the universal spectra is, in fact, independent of k in the inertial subrange.)

Substituting Eq. (26) for $H(k)$ into Eq. (27), we find

$$\epsilon(k) = \epsilon_\infty \left[1 - \frac{5\alpha_H}{4}\left(\frac{k_b}{k}\right)^{4/3}\right]^{3/5}, \tag{28}$$

where $\epsilon_\infty = \epsilon(\infty)$ is the energy dissipation rate and k_b is the buoyancy wavenumber defined in Eq. (20) with ϵ replaced by ϵ_∞. It is clear from Eq. (28) that the cascade rate increases with increasing k and converges asymptotically to the energy dissipation rate. It is also clear that the concept of an inertial cascade must break down at $k \sim k_b$, so that inertial range behavior may only be expected for $k_b \ll k \ll k_d$. From Eq. (19), this implies the existence of an inertial subrange only for $B \ll 1$. Corrected inertial subrange scaling due to buoyancy may now be determined by use of $\epsilon(k)$, Eq. (28), for ϵ in Eqs. (25) and (26).

2.4. EDQNM small-scale spectra

The ideas just developed have been tested by numerical solution of Eqs. (5)-(8) using EDQNM forms for the transfer spectra. Initial conditions are such that $F^4(k, \eta_k)$ is taken to be independent of η_k with spectra proportional to k^2 at small wavenumbers. The spectra F^1, F^2, F^3 are assumed to be initially zero. Physically, this corresponds to a fluid initially at rest with a given homogeneous buoyancy field containing large-scale fluctuations. Other initial conditions have also been run ($F^4 \sim k^4$ at small wavenumbers; F^1 and F^2 with given non-zero spectra, etc.), and for time-evolutions well into the asymptotic regime, it is observed that the scaling of the small scales becomes independent of the initial conditions.

The approach of the small-scale turbulence to isotropy can be observed in Fig. 1 where we have plotted $F^1(k, \eta_k = 0)$, $F^1(k, \eta_k = 1)$ and $F^2(k, \eta_k = 0)$, $F^2(k, \eta_k = 1)$ at a time well into the asymptotic regime. Small-scale isotropy occurs if, at large wavenumbers, the spectra become independent of η_k and $F^1 = F^2$, as is indeed observed. The large-scale anisotropy of the flow is also clearly evident where these four spectra diverge at low wavenumbers. In fact, it is easy to show that the behaviors of F^1 and F^2 as $k \to 0$ are very different in that $F^1 \sim k^4$ whereas $F^2 \sim k^2$, the former due to non-linear transfer, while the latter due to the direct effects of the buoyancy force.

We have tested the scaling laws given by Eqs. (12) and (21). Unscaled spectra $E(k)$, $G(k)$, and $H(k)$ at two different times in the asymptotic regime are shown in Fig. 2, whereas the scaled spectra $\hat{E}(\hat{k})$, $\hat{G}(\hat{k})$, and $\hat{H}(\hat{k})$ are shown in Fig. 3. A near-perfect collapse of the small-scale spectra at the two times is evident, in agreement with our earlier analysis. Also in Fig. 3, we show power-law behaviors corresponding to $k^{-5/3}$ and $k^{-7/3}$. The slight deviations of the EDQNM spectra from these predicted power laws need to be understood, although they may only be due to numerical errors.

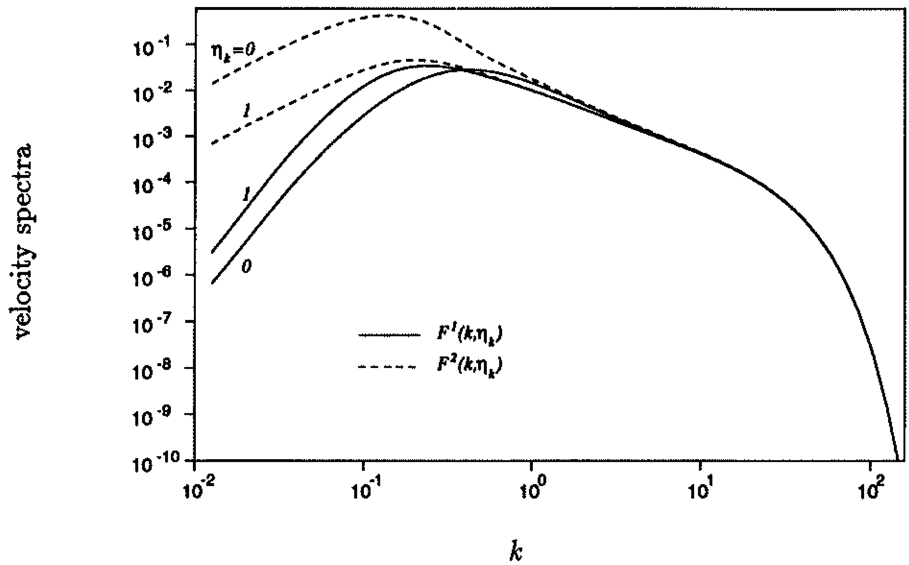

FIGURE 1. Approach of small-scale turbulence to isotropy.

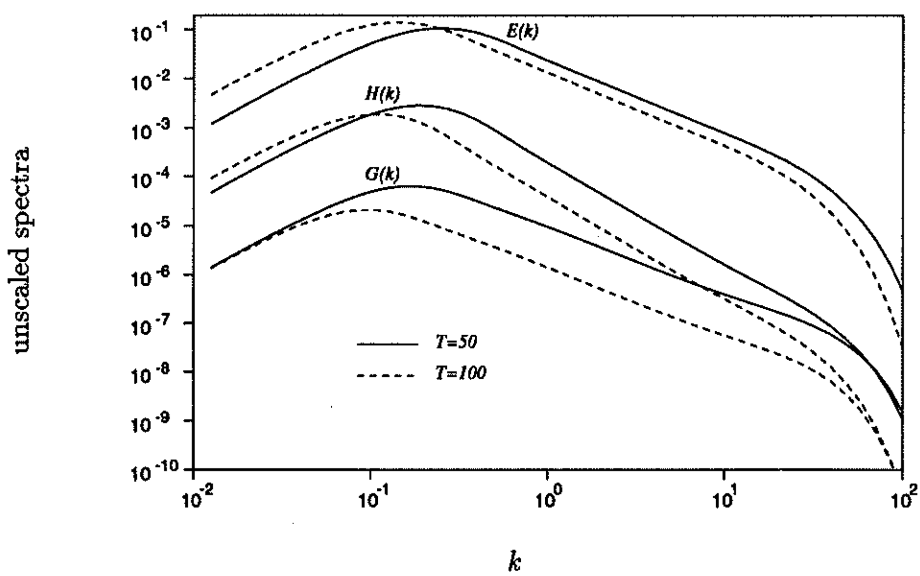

FIGURE 2. Unscaled spectra at two different times.

3. Future plans

It is of interest to apply some of the ideas developed for homogeneous buoyancy-generated turbulence to homogeneous stratified flows. The stably stratified case is perhaps the most relevant to mixing in the oceans and the atmosphere. It is easy

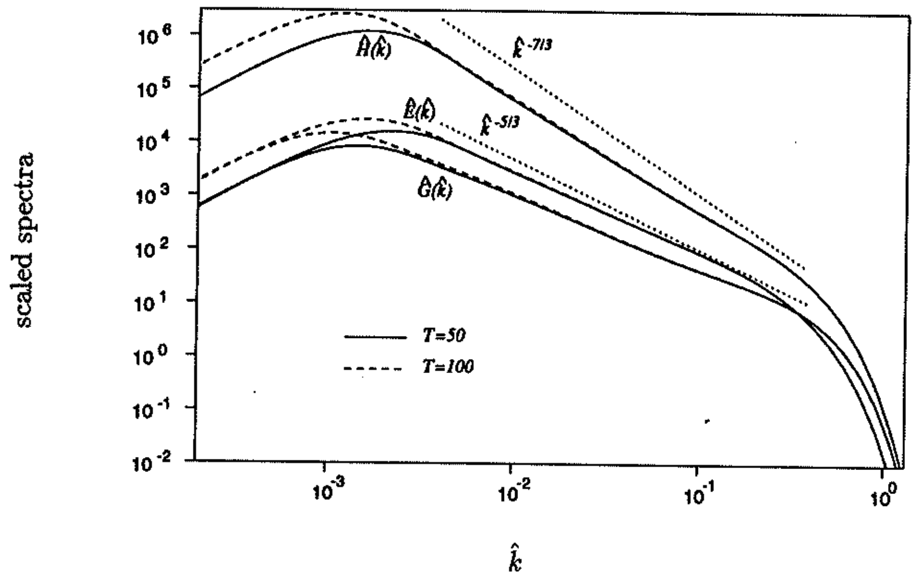

FIGURE 3. Scaled spectra at two different times.

to modify the EDQNM equations for this problem, and some interesting analytical and numerical results (which include large-eddy numerical simulations) have already been obtained for the case of negligible buoyancy (Chasnov 1992). When both buoyancy and mean-stratification are important, the complications arising from the generation of gravity waves must be considered.

Some of the ideas developed for homogeneous buoyancy-generated turbulence may also be applicable to homogeneous shear-flow. (For example, both of these flows have four defining spectra, an increasing Reynolds number at large-times, and possibly a small-scale $k^{-7/3}$ behavior for the co-spectra). It will be of interest to see how far possible analogies between these two types of flows may be exploited.

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