Non-linear interactions in homogeneous turbulence with and without background rotation

By Fabian Waleffe

1. Motivation and objectives

This project started from a discussion with Dr. Srinivas Veeravalli about the effect of rotation on turbulence during the CTR 1990 summer program. There seemed to be conflicting reports on the effect of rotation. Some expected a Taylor-Proudman reorganization of the flow at strong rotation rate, but this was refuted by linear analysis and direct numerical simulations (DNS) owing to the existence of *inertial waves* (Speziale *et al.* 1987). Linear analysis and DNS suggested an isotropization of the flow, but some experiments (e.g. Hopfinger *et al.* 1982, Veeravalli 1990) and analyses (Cambon & Jacquin 1989) showed a tendency towards anisotropy. These effects appear at small Rossby numbers, and it seemed that a weakly non-linear analysis could shed some light on the problem.

At the same summer program, Domaradzki *et al.* (1988, 1990 a,b) continued their study of triadic transfers. Their conclusions were that turbulent transfers are dominated by non-local interactions with local energy transfer. This is only partly consistent with the common wisdom that local interactions with local energy transfer dominate the inertial cascade. Brasseur *et al.* (1991 a,b,c) then called on this predominance of non-local interactions to refute the Kolmogorov assumption of local isotropy at the small scales. The inertial wave decomposition showed features observed in the simulations. A deeper analysis was undertaken in search of a better understanding of triad interactions and of the significance of the numerical results.

2. Accomplishments

The helical (or inertial wave) decomposition of the velocity field clearly identifies two types of triadic transfers depending on whether the small scale helical modes have helicities of the same or the opposite sign. Only one type of interaction shows local transfer when the triads are non-local. In those cases, the local cascade to higher wavenumber must always be accompanied by a feedback on the large scale. An *instability principle*, suggested by the stability characteristics of triad interactions, has been introduced and predicts the direction of the energy transfers. These predictions agree with DNS and the Test Field Model.

Although the transfer from the medium to the longest leg becomes dominant in non-local triads, the change in the energies of the long legs is not large. This is because of a cancellation occurring when summing over several triads. In particular, one must always consider the two triads involving the large scale and its conjugate. The net result of the large local transfers is an advection in wave space. The cascade (or flux) of energy through a given wavenumber is not dominated by the large local transfers either. This is a consequence of the necessary feedback on the large scale. For each large local transfer through a wavenumber, there are many feedbacks on the large scale. In an infinite inertial range, it can be shown that the net effect is actually a *reverse cascade* to the large scales. One must conclude that 'non-local interactions with local transfers' are not dominant in turbulence. The rejection of the Kolmogorov hypothesis of local isotropy at the small scales is thus ill-founded. The other type of triad interactions, which do not have the local transfer character when the triads are non-local, are the interactions responsible for the inertial cascade to larger wavenumber. Their structure shows strong similarity with the *elliptical instability* [Waleffe 1991].

The main effect of background rotation is to restrict triad interactions to *resonant* ones. The *instability principle* still applies and, coupled with the triad resonance condition, it predicts a transfer of energy towards wavevectors perpendicular to the rotation axis. That tendency is observed in experiments, DNS, and an EDQNM model.

2.1. Homogeneous turbulence

2.1.1 Helical decomposition

The flow of an incompressible fluid in a periodic box of side L is conveniently represented by its Fourier series

$$\vec{u}(\vec{x}) = \sum_{\vec{k}} \vec{u}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

where $\vec{k} = (m,n,l)\,2\pi/L$, with $m,n,l = 0,\pm1,\pm2,\ldots$. In Fourier space the continuity equation requires that $\vec{k}\cdot\vec{u}(\vec{k}) = 0$, and thus there are only two degrees of freedom per wavevector. Here, the two degrees of freedom are chosen as the *maximum* and *minimum* helicity modes,

$$\vec{u}(\vec{k}) = a_+\vec{h} + a_-\vec{h}^* \tag{1}$$

where

$$\vec{h} = \vec{\nu}\times\vec{\kappa} + i\,\vec{\nu}. \tag{2}$$

The * superscript denotes a complex conjugate, $\vec{\kappa} = \vec{k}/k$ is the unit vector in the direction of $\vec{k}$, and $\vec{\nu}$ is a unit vector orthogonal to $\vec{k}$, $\vec{k}\cdot\vec{\nu} = 0$. One can choose, for instance, $\vec{\nu} = (\vec{z}\times\vec{\kappa})/||\vec{z}\times\vec{\kappa}||$, where $\vec{z}$ is an arbitrary vector. The vectors $\vec{h}$, $\vec{h}^*$ are the eigenmodes of the curl operator,

$$i\vec{k}\times\vec{h} = k\,\vec{h}. \tag{3}$$

The modal kinetic energy and helicity are given by, respectively:

$$\begin{aligned} e(\vec{k}) &= \frac{1}{2}\vec{u}(\vec{k})\cdot\vec{u}^*(\vec{k}) = a_+a_+^* + a_-a_-^* \\ h(\vec{k}) &= \frac{1}{2}\vec{u}(\vec{k})\cdot\vec{\omega}^*(\vec{k}) = k\,(a_+a_+^* - a_-a_-^*) \end{aligned} \tag{4}$$

where $\vec{\omega} = i\vec{k}\times\vec{u}$ is the vorticity. It is clear that the + mode corresponds to maximum helicity and the − mode to minimum helicity (normalized by the energy). It is said that the two modes have opposite *polarities*.

The quadratic non-linearity of the Navier-Stokes equations induces interactions between triads of wavevectors $\vec{k}+\vec{p}+\vec{q}=0$ only. There are eight fundamental interactions corresponding to each value of the triplet (s_k, s_p, s_q), where s_k, s_p, s_q are sign coefficients equal to ± 1 which identify the helical mode involved for $\vec{k}, \vec{p}, \vec{q}$, respectively. The eight possible interactions will be denoted by the integer $i = 1, \ldots, 8$, following a binary ordering: $1 \equiv (+,+,+)$, $2 \equiv (+,+,-)$, $3 \equiv (+,-,+)$, $4 \equiv (+,-,-)$, ..., $8 \equiv (-,-,-)$. As a result of these interactions, the modal energy and helicity evolve according to equations of the form:

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e(\vec{k}) = \frac{1}{2} \sum_{\vec{k}+\vec{p}+\vec{q}=0} \sum_{i=1}^{8} t^{(i)}(\vec{k}, \vec{p}, \vec{q}) \tag{5}$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) h(\vec{k}) = \frac{1}{2} \sum_{\vec{k}+\vec{p}+\vec{q}=0} \sum_{i=1}^{8} s_k k\, t^{(i)}(\vec{k}, \vec{p}, \vec{q}) \tag{6}$$

The factor 1/2 comes from a symmetrization of the energy transfers, $t^{(i)}(\vec{k}, \vec{p}, \vec{q}) = t^{(i)}(\vec{k}, \vec{q}, \vec{p})$.

In an inviscid fluid, total energy and helicity must be conserved at all times. A single triad of helical modes constitutes a kinematically acceptable initial state which must conserve energy and helicity at time zero. Therefore, the transfer functions $t^{(i)}(\vec{k}, \vec{p}, \vec{q})$ must satisfy

$$t^{(i)}(\vec{k}, \vec{p}, \vec{q}) + t^{(i)}(\vec{p}, \vec{q}, \vec{k}) + t^{(i)}(\vec{q}, \vec{k}, \vec{p}) = 0$$

$$s_k k\, t^{(i)}(\vec{k}, \vec{p}, \vec{q}) + s_p p\, t^{(i)}(\vec{p}, \vec{q}, \vec{k}) + s_q q\, t^{(i)}(\vec{q}, \vec{k}, \vec{p}) = 0$$

which imply that

$$\frac{t^{(i)}(\vec{k}, \vec{p}, \vec{q})}{s_p p - s_q q} = \frac{t^{(i)}(\vec{p}, \vec{q}, \vec{k})}{s_q q - s_k k} = \frac{t^{(i)}(\vec{q}, \vec{k}, \vec{p})}{s_k k - s_p p} \tag{7}$$

The full expression for $t^{(i)}(\vec{k}, \vec{p}, \vec{q})$ is derived in [Waleffe 1991]; it reads:

$$t^{(i)}(\vec{k}, \vec{p}, \vec{q}) = \frac{1}{2}\,(s_p p - s_q q)\,[s_k k + s_p p + s_q q]\,\frac{Q}{2kpq} \times s_{kpq}\, e^{i\beta}(a_{s_k} a_{s_p} a_{s_q}) + c.c. \tag{8}$$

where $s_{kpq} = s_k s_p s_q = \pm 1$, $\exp(i\beta)$ is a phase factor representing the orientation of the triad with respect to some reference frame, and

$$\frac{Q}{2kpq} = \frac{\sin\alpha_k}{k} = \frac{\sin\alpha_p}{p} = \frac{\sin\alpha_q}{q}$$

with $Q^2 = 2k^2p^2 + 2p^2q^2 + 2q^2k^2 - k^4 - p^4 - q^4 \geq 0$.

2.1.2 Energy transfers and the instability principle

The three coefficients $s_p p - s_q q$, $s_q q - s_k k$, $s_k k - s_p p$ sum up to zero, so in general when k, p, q are distinct, one coefficient has a sign opposite to the other two. It is clear that this can never be the coefficient of the longest leg. If q is the longest leg for instance, then $s_p p - s_q q$ has the sign of $-s_q q$ and $s_q q - s_k k$ has the sign of $s_q q$, thus one of these must necessarily be the coefficient whose sign is opposite to those of the other two. The relations (7) then show that there are only two types of triadic energy transfers depending on whether the helical modes associated to the two longest legs have helicities of the same or of the opposite sign as illustrated in fig.1.

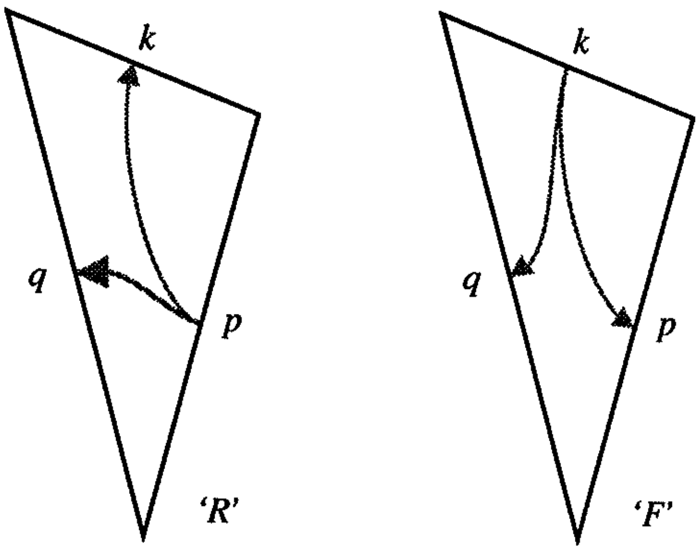

FIGURE 1. The two types of energy transfer for $k < p < q$: 'B' interactions when $s_p = s_q$, energy flows out of middle wavenumber; 'E' interactions when $s_p = -s_q$, energy flows out of smallest wavenumber.

The (inviscid) equations are reversible and the transfer of energy can occur in both directions in any particular realization (e.g. when $k < p < q$ the $s_p = -s_q$ transfers can be either to or from $\vec{k}$). However, it is proposed here that *statistically* each triad interaction extracts energy from the mode whose coefficient has a sign opposite to those of the other two. This says that interactions involving small scale helical modes of opposite polarities will draw energy out of the large scale, while interactions where the small scale helical modes have the same polarity draw energy out of the medium scale. This is called the *instability principle.* It is inspired by the stability properties of the elementary triad interactions described below. The instability principle is consistent with the TFM model and DNS [Waleffe 1991]. Kraichnan (1967) used an equivalent assumption in his analysis of 2D turbulence, where interactions are of the 'B' type only, stating that one intuitively expects a "statistical spreading of the excitation in wave space". He also showed that assumption to be consistent with the early time development of an initially Gaussian distribution.

A single helical mode is an exact solution of the Navier-Stokes equation,; this is a direct consequence of the continuity equation $\vec{u} \cdot \vec{k} = 0$. That solution is unstable when perturbed by a smaller scale helical mode of the same polarity and a larger scale mode (of either polarity), such that the three modes form a triad. It is also unstable if perturbed by smaller scale helical modes of mutually opposite polarities. These conclusions are deduced from the equations for a single triad interaction. From (7) they have the form

$$\begin{aligned} \dot{a}_{s_k} &= (s_p p - s_q q)\, C\, a^*_{s_p} a^*_{s_q} \\ \dot{a}_{s_p} &= (s_q q - s_k k)\, C\, a^*_{s_q} a^*_{s_k} \\ \dot{a}_{s_q} &= (s_k k - s_p p)\, C\, a^*_{s_k} a^*_{s_p} \end{aligned} \tag{9}$$

The evolution of small disturbances a_{s_k}, a_{s_q} on the base flow $(a_{s_k}, a_{s_p}, a_{s_q}) = (0, A, 0)$, for instance, is determined by the equation

$$\frac{d^2 a_{s_k}}{dt^2} = (s_p p - s_q q)(s_k k - s_p p)\, CC^*\, AA^*\, a_{s_k}$$

There are exponentially growing solutions if $(s_p p - s_q q)(s_k k - s_p p) > 0$. Hence the unstable mode is that whose coefficient in (7,9) has a sign opposite to the other two. Some justification for the instability principle might be that although there are disturbances both growing and decaying exponentially, the average flow of energy for the unstable mode is outward because $(e^{2t} - e^{-2t})/2 \geq 0$.

2.1.3 Non-local interactions

A non-local interaction is such that one leg of the triad is much smaller than the other two, which are then nearly equal. Choose for instance $|q - k| < p \ll k \simeq q$, then from (7)

$$\begin{aligned} t^{(i)}(\vec{k}, \vec{p}, \vec{q}) &\simeq \frac{-q}{q - (s_k s_q) k}\, t^{(i)}(\vec{p}, \vec{q}, \vec{k}) \\ t^{(i)}(\vec{q}, \vec{k}, \vec{p}) &\simeq \frac{(s_k s_q) k}{q - (s_k s_q) k}\, t^{(i)}(\vec{p}, \vec{q}, \vec{k}) \end{aligned}$$

When $s_k = -s_q$ this reduces to

$$t^{(i)}(\vec{k}, \vec{p}, \vec{q}) \simeq t^{(i)}(\vec{q}, \vec{k}, \vec{p}) \simeq -\frac{1}{2} t^{(i)}(\vec{p}, \vec{q}, \vec{k}) \tag{10}$$

and when $s_k = s_q$,

$$\begin{aligned} t^{(i)}(\vec{k}, \vec{p}, \vec{q}) &\simeq \frac{-q}{q - k}\, t^{(i)}(\vec{p}, \vec{q}, \vec{k}) \\ t^{(i)}(\vec{q}, \vec{k}, \vec{p}) &\simeq \frac{k}{q - k}\, t^{(i)}(\vec{p}, \vec{q}, \vec{k}) \end{aligned} \tag{11}$$

From (7,8),

$$t^{(i)}(\vec{p},\vec{q},\vec{k}) \simeq \frac{1}{2}(q^2-k^2)\,\frac{Q}{2kpq}\,[s_{kpq}\,e^{i\beta}a_{s_k}a_{s_p}a_{s_q}+c.c.]$$

There is no reason to expect the triple correlations $< a_{s_k}a_{s_p}a_{s_q} >$ to vary strongly from one interaction to the other, at least in non-helical turbulence. Thus the transfer $t^{(i)}(\vec{p},\vec{q},\vec{k})$ into the smallest leg should be of about the same magnitude for all interactions, whether $s_k = s_q$ or $s_k = -s_q$, and of order $O(q-k)$. However, (10,11) show that the transfer into the long legs k and q will be strongly dominated by $s_k = s_q$ interactions with a large exchange of energy between the long legs. According to the instability principle, that transfer should be from the medium to the longest leg. This is also what the DNS suggest. This analysis confirms, for a single triad, the results of Domaradzki and Rogallo. The important point for the following is that *the large local transfer from the medium leg to the longest must necessarily be accompanied by a feedback into the small leg* (11).

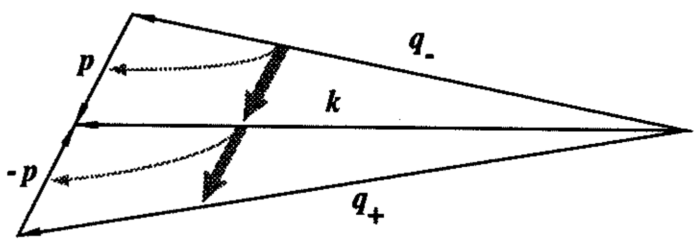

FIGURE 2. The rate of change of mode $\vec{k}$ comes from two triads involving the large scale $\vec{p}$.

Although the transfer between the long legs within a single triad is very large, the net effect on either of the long legs is not necessarily large because of cancellations occurring when summing over several triads. One other triad which must always be considered is that involving the complex conjugate of the large scale (fig.2). In the simplest case where the small scales interact with a single large scale mode $\vec{U}(\vec{p})$, the equation for $\vec{u}(\vec{k})$ is

$$(\frac{\partial}{\partial t}+\nu k^2)\,u_l(\vec{k}) = -i\,P_{lmn}\,[U_m(\vec{p})u_n(\vec{k}-\vec{p})+U_m(-\vec{p})u_n(\vec{k}+\vec{p})] \qquad (12)$$

where $P_{lmn} = k_m P_{ln} + k_n P_{lm}$ with $P_{lm} = \delta_{lm} - k_l k_m/k^2$. Separating the large scale into its real and imaginary part $\vec{U} = \vec{U}^r + i\vec{U}^i$ and assuming that $\vec{u}(\vec{k}\pm\vec{p}) = \vec{u}(\vec{k}) \pm \vec{p}\cdot\vec{\nabla}^k\,\vec{u}(\vec{k})$, one derives the equation for the energy of the small scale,

$$(\frac{\partial}{\partial t}+2\nu k^2)\,e(\vec{k}) = -2(\vec{U}^i\cdot\vec{k})\,\vec{p}\cdot\vec{\nabla}^k\,e(\vec{k}) + \vec{u}^*\cdot\boldsymbol{S}\cdot\vec{u} \qquad (13)$$

where $e(\vec{k}) = 1/2\,\vec{u}^*\cdot\vec{u}$, $\boldsymbol{S} = [\vec{p}\vec{U}^i + \vec{U}^i\vec{p}]$, and $\vec{\nabla}^k = \partial/\partial\vec{k}$ is the gradient in Fourier space. One gets the same equation from the linear evolution of disturbances on an unbounded shear obtained by expanding the large scale flow in the neighborhood of $\vec{p}\cdot\vec{x}=0$, $\vec{U}(\vec{x})\simeq 2\vec{U}^r - 2\vec{U}^i\vec{p}\cdot\vec{x}$. Each term in (13) scales on the strain rate $\vec{p}\vec{U}^i$ of the large scale unless the energy distribution is very sharp in Fourier space. The advection term comes from the difference of two large triadic terms which scale on the amplitude of the large scale. That term advects energy in the direction of $\vec{p}$ (fig.3). Thus it tends to deplete the energy of the small scales in the direction perpendicular to $\vec{p}$. This conclusion is opposite to that of Brasseur and Yeung (1991).

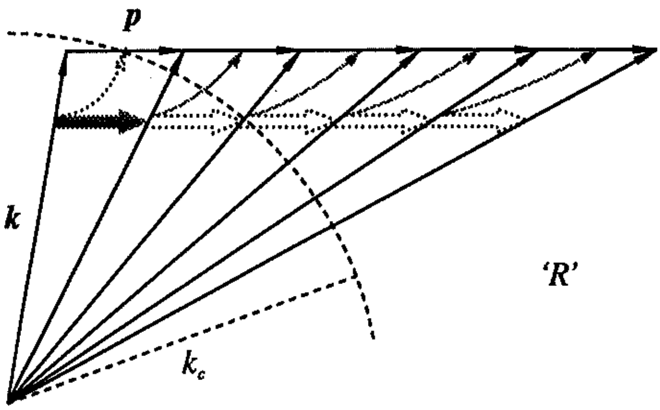

FIGURE 3. The multiple 'B' interactions with the large scale $\vec{p}$ tend to advect energy in the direction of $\vec{p}$, with feedback on the large scale.

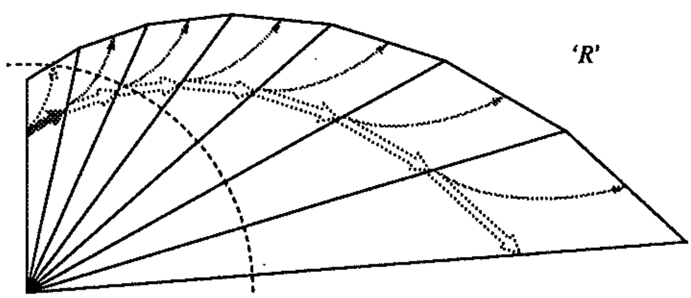

FIGURE 4. Partial sum over triads of a given shape shows the necessary cancellations both for the rate of change of a small scale and for the net cascade across k_c.

Although there is a cancellation of the large local transfers for the net effect on a small scale, one might still think that the large transfers represent a strong flux of energy through a wavenumber k_c and are the essence of the energy cascade from large to small scales. This is not the case because of the feedback of energy into the large scale associated with the transfer from medium to small scale. The net flux of energy across a wavenumber k_c is the sum of contributions from many triads. There is a large local contribution but many small non-local feedbacks into the large scale (fig.3,4). In fact, it can be shown by summing over all triads of a given shape (somewhat as in fig. 4) that, for an infinite inertial range, the net cascade from 'B' interactions which are responsible for the large local transfers is actually *from small to large scales* [Waleffe 1991, sect.5]. The two types of triadic transfers ('E' for eddy-viscosity, and 'B' for backscatter) and their contributions to the energy cascade are shown in fig.5.

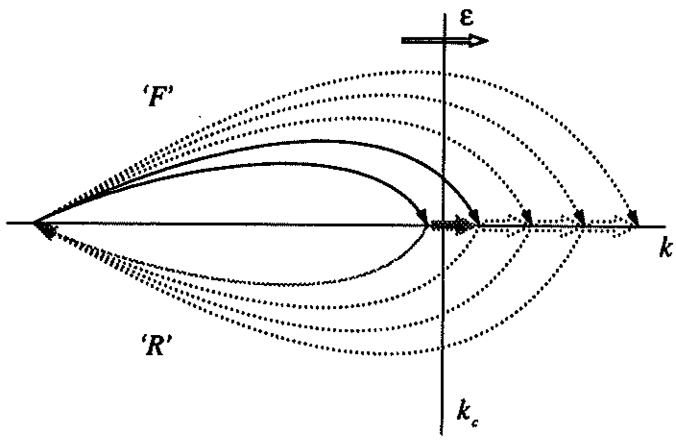

FIGURE 5. The various types of triadic transfers involving a large scale, which contribute to the cascade through k_c. Solid lines represent transfers in a single triad.

The significance of the non-local interactions with local transfer is to be found in the "cusp-up" behavior of eddy-viscosity models near the cut-off wavenumber. This is clearly illustrated in fig. 5. If wavenumbers above k_c are not included, the flux of energy to small scales from the missing 'E' interactions can be modeled by an eddy-viscosity, but there is also a large sink of energy for wavenumbers near the cut-off which requires a cusp-up in the eddy-viscosity. This energy drain near the cut-off is linked to a negative contribution to the eddy-viscosity at small wavenumbers.

2.2. Turbulence under strong background rotation

Linear perturbation of a state of solid body rotation can give rise to a spectrum of inertial waves (Greenspan 1968). These inertial waves have the structure of the helical modes introduced in section 2.1.1. Indeed the linear inviscid equations in the presence of uniform background rotation $\vec{\Omega}$ read

$$\frac{\partial}{\partial t}\vec{u} + 2\vec{\Omega}\times\vec{u} = -\vec{\nabla}p$$

which for a Fourier mode $\vec{u} = \vec{h}\exp(i\vec{k}\cdot\vec{x} + i\omega t)$, where $\vec{k}\cdot\vec{h} = 0$ for continuity, becomes

$$i\omega\vec{h} + 2\vec{\Omega}\times\vec{h} = -i\vec{k}\hat{p}$$

Taking the cross-product of this equation with $i\vec{k}$ gives

$$\omega(i\vec{k}\times\vec{h}) = 2(\vec{k}\cdot\vec{\Omega})\,\vec{h}$$

which shows that the helical modes $\vec{h}_s$, with $i\vec{k}\times\vec{h}_s = sk\,\vec{h}_s$ ($s = \pm 1$), are the eigenmodes of the linear perturbations of rigid rotation. The dispersion relation for those helical eigenmodes follows as

$$\omega = 2s\,\vec{\kappa}\cdot\vec{\Omega}$$

showing that the two eigenmodes have opposite polarities and opposite eigenfrequencies.

In the presence of background rotation $\vec{\Omega} = \Omega\vec{z}$, the helical formulation of the Navier-Stokes equations reads, from (9),

$$\left(\frac{\partial}{\partial t} - i\omega_{s_k} + \nu k^2\right) a_{s_k} = Ro \sum (s_p p - s_q q)\, C\, a^*_{s_p} a^*_{s_q} \tag{14}$$

The Coriolis force contributes the linear term $i\omega_{s_k} a_{s_k}$, with $\omega_{s_k} = s_k \cos\theta_k$, $\cos\theta_k = \vec{\kappa}\cdot\vec{z}$, $\vec{\kappa} = k^{-1}\vec{k}$. The symbol $\sum$ represents a sum over all triads $\vec{k}+\vec{p}+\vec{q} = 0$ and all interactions (s_k, s_p, s_q). The equations have been non-dimensionalized using $(2\Omega)^{-1}$ as the time scale and V and L as the characteristic velocity and length scales, respectively. The parameter $Ro = V/(2\Omega L)$ is the Rossby number, and $\nu = \nu^*/(2\Omega L^2)$ is the Ekman number with ν^* as the dimensional viscosity. Our interest here is in small Ekman and Rossby numbers (large Ω). This is a multiple time scale problem. On the time scale of the rotation, the amplitude a_{s_k} behave as

$$a_{s_k} = b_{s_k} e^{i\omega_{s_k} t} \tag{15}$$

where the b_{s_k} are essentially constant. The rate of change of the b_{s_k} is found by substituting (15) in (14),

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) b_{s_k} = Ro \sum (s_p p - s_q q)\, C\, b^*_{s_p} b^*_{s_q}\, e^{-i(\omega_{s_k}+\omega_{s_p}+\omega_{s_q})t} \tag{16}$$

Clearly the b_{s_k} evolve on the slow time scale Rot from non-linear interactions. Interactions such that $\omega_{s_k} + \omega_{s_p} + \omega_{s_q} \neq 0$ will tend to average out over the long time scale so that the approximate equation for b_{s_k} is

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) b_{s_k} = Ro \sum_{s_p, s_q} \sum_{\substack{\vec{k}+\vec{p}+\vec{q}=0 \\ \omega_{s_k}+\omega_{s_p}+\omega_{s_q}=0}} (s_p p - s_q q)\, C\, b^*_{s_p} b^*_{s_q} \tag{17}$$

This equation is identical to that arising in homogeneous turbulence except that the sum has been restricted to *resonant* triads. The only acting triads are those which satisfy:

$$\begin{aligned} k\cos\theta_k + p\cos\theta_p + q\cos\theta_q &= 0 \\ s_k\cos\theta_k + s_p\cos\theta_p + s_q\cos\theta_q &= 0 \end{aligned} \tag{18}$$

which requires that

$$\frac{\cos\theta_k}{s_p q - s_q p} = \frac{\cos\theta_p}{s_q k - s_k q} = \frac{\cos\theta_q}{s_k p - s_p k}. \tag{19}$$

The stability characteristics of a single triad (9) are unchanged, and the instability principle can be used here as well. For the purpose of the discussion, let again $k < p < q$ as in fig.1. The instability principle states that interactions where $s_p = -s_q$ transfer energy from the small leg k to p and q, while interactions where $s_p = s_q$ transfer energy from p to k and q. From (19) when $s_p = -s_q$ one finds

$$\begin{aligned}\cos\theta_p &= \frac{k-(s_k s_q)q}{-(q+p)}\cos\theta_k\\ \cos\theta_q &= \frac{k+(s_k s_q)p}{-(q+p)}\cos\theta_k\end{aligned} \tag{20}$$

These interactions then transfer energy to modes p and q such that $|\cos\theta_p|, |\cos\theta_q| < |\cos\theta_k|$, because $k < p < q$. Likewise, when $s_p = s_q$, (19) gives

$$\begin{aligned}\cos\theta_k &= \frac{q-p}{k-(s_k s_p)q}\cos\theta_p\\ \cos\theta_q &= \frac{(s_k s_q)p-k}{k-(s_k s_p)q}\cos\theta_p\end{aligned} \tag{21}$$

and these interactions transfer energy from p to modes k and q with $|\cos\theta_k|, |\cos\theta_q| < |\cos\theta_p|$. Thus *all interactions transfer energy towards smaller values of* $\cos\theta$, i.e. towards wavenumbers perpendicular to the rotation axis. However, resonant interactions can not transfer energy *directly* to wavenumbers perpendicular to the rotation axis. If $\cos\theta_q = 0$ for instance, then from (19) $(s_k p - s_p k)\cos\theta_k = (s_k p - s_p k)\cos\theta_p = 0$, which requires $s_k = s_p$ and $k = p$. The resonance condition (19) then imply that $\cos\theta_k = -\cos\theta_p$, but the rate of transfer of energy (7) into mode q is proportional to $s_k k - s_p p$ and thus vanishes in this case. This last result has been obtained by Greenspan (1969) for eigenmodes in a bounded container. Resonant interactions transfer energy towards smaller values of $|\cos\theta| = |k_z/k|$ but not to $\cos\theta = 0$. This transfer of energy towards smaller $|\cos\theta|$ is verified numerically in the DNS of Mansour (1990). It is also predicted by an EDQNM model of Cambon and Jacquin (1989). The average over a random distribution of modes with $\cos\theta \simeq 0$ shows that $2 < w^2 > / < u^2 + v^2 > \simeq 2$, while this ratio would be equal to 1 in isotropic turbulence. This tendency is observed in the experiments of Veeravalli (1991) (fig.10 with u and w interchanged).

3. Future plans

3.1. Maximum instability principle

The instability principle described in 2.1.2 determines the sign of the triadic energy transfers. Its mathematical formulation is that $s_{kpq} \exp i\beta < a_{s_k} a_{s_p} a_{s_q} >$ $+c.c.$ has the sign of

$$-(s_p p - s_q q)(s_q q - s_k k)(s_k k - s_p p)(s_k k + s_p p + s_q q). \tag{22}$$

This assumption gives qualitatively correct results for the direction of the energy transfers. In order to obtain quantitative information, it is necessary to strengthen the assumption. As a first example, consider the *maximum instability principle*:

$$s_{kpq} \exp i\beta < a_{s_k} a_{s_p} a_{s_q} >= \sigma |a_{s_k}| \, |a_{s_p}| \, |a_{s_q}| \tag{23}$$

where σ is the sign of expression (22). With this assumption, each interaction drains the maximum amount of energy permitted by the respective energy of each mode from the unstable mode for that particular interaction. This assumption of maximum correlation is likely to be too strong, but it satisfies all detailed conservation properties and gives a realizable model. The rate of change of the modal energy is proportional to the square root of the energy and vanishes when the energy is zero, so energies should not become negative. In isotropic turbulence, this leads to a model of the form

$$(\frac{\partial}{\partial t} + 2\nu k^2) E(k) = \sum C_{kpq} [E(k) E(p) E(q)]^{1/2}.$$

3.2. Cusp-up behavior of the eddy-viscosity

The cusp-up in eddy-viscosity models is linked to an advection process in wave space. When a sharp cut-off is introduced, energy can not be advected beyond the cut-off and needs to be removed by an increased viscosity. Instead it might be possible to design some "free-outflow" boundary condition at the cut-off. Note that the '3/2-rule' used for dealiasing spectral computations perhaps acts as a "soft" boundary condition which does not require a cusp in the eddy-viscosity. In real space, one needs to study how numerical methods deal with the "squishing" of structures beyond the grid resolution. If small or narrow eddies can be squished below the grid size, a cusp is not necessary in the eddy viscosity. The evolution of a small Fourier component on a large scale stagnation point flow should be a good test.

3.2. Effect of a wall in rotating turbulence

As shown above, the instability principle predicts a transfer of energy towards smaller values of $\vec{\Omega} \cdot \vec{k}/k$, but resonant interactions can not transfer energy to wavevectors orthogonal to the rotation axis. Thus there is only a tendency towards two-dimensionality. The experiments of Hopfinger *et al.* (1982) show a much more dramatic two-dimensionalization of the flow.

Preliminary analysis suggests that the presence of walls induce new interactions. For each wave incident on a wall, there must be a reflected wave in order to satisfy the boundary conditions. This sets up some strong correlations between some Fourier components independently of the non-linear effects. Also, in the reasoning leading to equation (17), the resonance condition should in fact be relaxed to $\omega_{s_k} + \omega_{s_p} + \omega_{s_q} = O(Ro)$. Thus all modes nearly orthogonal to the rotation axis are essentially 'resonant', or in other words, the "phase scrambling" (Mansour *et al.*, 1991 (b)) due to linear effects does not affect low frequency modes. This relaxes the constraints that $s_k = s_p$ and $k = p$ when $\cos\theta_q = 0$. The $s_k = -s_p$ interactions are now allowed, and those transfer energy to mode q. One anticipates that these two effects, correlation between incident and reflected wave and near-resonance of all modes nearly orthogonal to the rotation axis, will lead to a stronger two-dimensionalization of the flow.

REFERENCES

BRASSEUR, J. G. 1991a Comments on the Kolmogorov Hypothesis of Isotropy in the Small Scales. *AIAA 91-0230.*

BRASSEUR, J. G. & YEUNG, P. K. 1991b Large and Small-scale Coupling in Homogeneous Turbulence: Analysis of the Navier-Stokes Equation in the Asymptotic Limit. *Proceedings of the Eight Symposium on Turbulent Shear Flows, Technical University of Munich, Sept. 9-11.*

CAMBON, C. & JACQUIN, L. 1989 Spectral approach to non-isotropic turbulence subjected to rotation. *J. Fluid Mech.* **202,** 295-317.

DOMARADZKI, A. J. & ROGALLO, R. S. 1988 Energy Transfer in Isotropic Turbulence at Low Reynolds Numbers. *Proceedings of the 1988 CTR Summer Program.* Center for Turbulence Research, Stanford University/NASA Ames Research Center. 169-177.

DOMARADZKI, A. J. & ROGALLO, R. S. 1990 Local energy transfer and nonlocal interactions in homogeneous, isotropic turbulence . *Phys. Fluids A.* **2,** 413-426.

DOMARADZKI, A. J., ROGALLO, R. S. & WRAY, A. A. 1990 Interscale energy transfer in numerically simulated turbulence. *Proceedings of the 1990 CTR Summer Program.* Center for Turbulence Research, Stanford University/NASA Ames Research Center. 319-329.

GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids.* Cambridge University Press, UK (New edition: 1990, Breukelen Press, Brookline, Mass., USA)

GREENSPAN, H. P. 1969 On the non-linear interaction of inertial waves. *J. Fluid Mech.* **36,** 257-264.

HOPFINGER, E. J., BROWAND, F. K. & GAGNE, Y. 1982 Turbulence and waves in a rotating tank. *J. Fluid Mech.* **125,** 505-534.

KRAICHNAN, R. H. 1967 Inertial Ranges in Two-dimensional Turbulence. *Phys. Fluids.* **10,** 1417-1423.

MANSOUR, N. N., CAMBON, C. & SPEZIALE, C. G. 1991a *Studies in Turbulence*, edited by T. B. Gatski, S. Sarkar and C. G. Speziale, Springer-Verlag, Heidelberg, 1991.

MANSOUR, N. N., SHIH, T.-H. & REYNOLDS, W. C. 1991b The effects of rotation on initially anisotropic homogeneous flows. *Phys. Fluids A.* **3**, 2421-2425.

SPEZIALE, C. G., MANSOUR, N. N. & ROGALLO, R. S. 1987 The decay of isotropic turbulence in a rapidly rotating frame. *Proceedings of the 1987 CTR Summer Program.* Center for Turbulence Research, Stanford University/NASA Ames Research Center. 205-211.

VEERAVALLI, S. V. 1991 An experimental study of the effects of rapid rotation on turbulence. *CTR Annual Research Briefs-1990* . Center for Turbulence Research, Stanford University/NASA Ames Research Center. 203-220.

WALEFFE, F. 1991 The Nature of Triad interactions in Homogeneous Turbulence. *to appear in Phys. Fluids A (Feb. 1992)*

YEUNG, P. K. & BRASSEUR, J. G. 1991c The response of isotropic turbulence to isotropic and anisotropic forcing at the large scales. *Phys.Fluids A.* **3**, 884-897.