Scaling analysis of energy transfer in the inertial range

By Ye Zhou

1. Motivation and objectives

The classical Kolmogorov phenomenology of energy transfer still remains an important element in turbulence theory (Monin and Yaglom, 1975; Tennekes and Lumley, 1972). Since all turbulence theories and models rely on assumptions about the energy transfer process, attempts are being made to verify the underlying assumptions. Recently, direct numerical simulation (DNS) measurements (Domaradzki and Rogallo, 1990; Yeung and Brasseur, 1991) have suggested that energy is largely transferred downscale locally, supporting a basic concept of the Kolmogorov phenomenology that leads to the universal inertial subrange. However, these authors concluded that the local energy transfer results from triad interactions that are nonlocal in the spectral k space.

The claim that local energy transfer results from nonlocal triadic interactions has important consequences for turbulence theory. Indeed, it questions the validity of the assumption of the statistical independence of the large- and small-scale motions in the Kolmogorov universal theory of turbulence (Ohkitani and Kida, 1991). In a low Reynolds number DNS, Yeung and Brasseur (1991) observed that anisotropy is induced in the small scales by forcing in the large scale. Moreover, they argued that such interactions will persist at high Reynolds numbers. The consequences of their argument are clearly at variance with the classical hypothesis of a universal isotropic structure at the small scales independent of the large-scale structure. In a computation with an extended period of forcing, Yeung et al. (1991) recently found that the small scale anisotropy eventually decreases at later time.

While we have no disagreement with these studies concerning the actual measurement of the raw interaction statistics— the triad nonlinear transfer $T(k,p,q)$— we believe that $T(k,p,q)$ is not the appropriate quantity one should use to determine whether the nonlinear interactions are local or not. Rather, we argue that these raw interaction statistics should be viewed only as a mathematical building block in the energy transfer process, and their physical interpretation requires further summation, during which much cancellation occurs. Following a suggestion by Kraichnan (1971), we have summed the measured raw band-band transfer interactions in a way that directly indicates the scale disparity of contributions to the *net* energy flux across the spectrum. We found that the net flux results primarily from interactions in which the ratio of largest to smallest scale is less than 10. Similar results have been found from the analysis of the net energy transfer. As a result, we conclude that DNS measurements, in fact, lend support to the classical Kolmogorov phenomenology of local interactions and local transfer in an inertial range.

2. Accomplishments

2.1. Preliminary

We restrict our attention to the velocity field $u_\alpha(\mathbf{k},t)$ that is a solution of the incompressible Navier-Stokes equation

$$[\frac{\partial}{\partial t}+\nu_0 k^2]u_\alpha(\mathbf{k},t)=-\frac{i}{2}P_{\alpha\beta\gamma}(k)\int d\mathbf{j}u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t), \qquad (1)$$

where $P_{\alpha\beta\gamma}(k)=k_\beta D_{\alpha\gamma}(k)+k_\gamma D_{\alpha\beta}(k)$, $D_{\alpha\beta}(k)=\delta_{\alpha\beta}-k_\alpha k_\beta/k^2$, f is the external force ($f=0$ for the decaying case), and ν_0 is the molecular viscosity.

The energy equation $E(\mathbf{k})=\frac{1}{2}|u(\mathbf{k})|^2$ is formulated as

$$[\frac{\partial}{\partial t}+2\nu k^2]E(\mathbf{k})=T(\mathbf{k})+F(\mathbf{k}), \qquad (2)$$

where $F(\mathbf{k})$ is the forcing spectrum and $T(k)$ is the the energy transfer function. The contribution to $T(k)$ resulting from nonlinear interactions between wavenumbers in band k and wavenumbers in bands p and q is denoted by $T(k,p,q)$. The triad energy transfer function $T(k,p,q)$ is given by

$$T(k,p,q)=\frac{1}{2}\sum Im[u^*_\alpha(\mathbf{k})P_{\alpha\beta\gamma}(k)u_\beta(\mathbf{p})u_\gamma(\mathbf{q})], \qquad (3)$$

where the asterisk denotes a complex conjugate and $\sum$ denotes a summation over shells in $\mathbf{k},\mathbf{p},\mathbf{q}$ subject to the triangle constraint. In turn, the triad energy transfer function $T(k,p,q)$ is related to $T(k)$ as:

$$T(k)=\sum_{p,q}T(k,p,q). \qquad (4)$$

Another important measurement for the energy transfer process is the energy flux

$$\Pi(k)=\int_k^\infty dk'\int_\Delta dpdqT(k',p,q). \qquad (5)$$

In order to separate the local and nonlocal interactions, we introduce the parameter

$$s=\frac{max(k',p,q)}{min(k',p,q)} \qquad (6)$$

which indicates directly the disparity of the interacting scales. This parameter has been used to classify the interactions as local ($s\leq 2$) and nonlocal ($s>2$) by Lesieur (1987). Kraichnan (1971) introduced a different set of parameters (v,w) where v ($v\leq 1$) is the ratio of the shortest to the middle leg and w is defined as k'/p $(1<w<1+v)$. The pair (v,w) completely determines a unique triangle shape. Using the test field model, Kraichnan (1971) calculated the energy transfer locality function that gives the fraction of energy transfer due to triangles whose smallest

leg is larger than v times the middle leg. Analysis of this function indicates that 65% of the transfer involves wavenumber triads in which the smallest wavenumber is less than one-half of the middle wavenumber.

The work of Kraichnan (1971) provided a theoretical framework in which one can determine whether the nonlinear interactions are local or nonlocal. For a given scale k, he argued that all raw interaction statistics must be summed such that physical quantities contain only *one* parameter which indicates the scale disparity of the interaction.

2.2. Analysis of the energy transfer function

We measure the individual contributions to $T(k)$, characterized by the shape parameter s

$$T(k) = \sum_s T(k,s), \tag{7a}$$

where

$$T(k,s) = \sum_{p,q|_s} T(k,p,q) \tag{7b}$$

is the partial sum of $T(k,p,q)$, over all (p,q) at constant s. The key point here is that the summation covers all interaction scales, subject to the triangle constraint, leaving only the dependence on scale disparity. This follows in spirit the procedure described in Kraichnan (1971). This measure has several properties that aid in its interpretation. First, as can be seen in Fig. 2, the contributions for all s are of the same sign; there is no further cancellation in the sum (7a). Second, $\int dk T(k,s) = 0$; this follows immediately from the detailed energy balance of $T(k,p,q)$ and the invariance of $s(k,q,p)$ under permutation of its arguments. Note that $\int dk T(k,p,q) \neq 0$. Fig. 1A shows the contributions $T(k,s)$ of each octave of s to the total energy transfer $T(k)$ for a simulated inertial range. The database was generated by a large–eddy simulation (LES) with an eddy viscosity derived from a stochastic equation that is consistent with EDQNM (Chasnov, 1990). Energy is dissipated at high k by the eddy viscosity, and the numerically resolved transfer $T(k)$ is non-zero there as a result of this artifact. The energy is injected with a forcing spectrum $F(\mathbf{k})$ peaked about a wavenumber $k_0 \approx 2$, and a stationary state above $k = 8$ is maintained in the Kolmogorov inertial range. Note that the energy is removed from the energy containing region primarily by nonlocal interactions $(s > 4)$.

The $30 < k < 65$ region of Fig. 1A is enlarged in Fig. 1B to determine whether local or nonlocal interactions are responsible for the increase of $T(k)$ at a large k. The major contributions to $T(k)$ are those of local interactions $(s \leq 4)$, consistent with classical phenomenology. The nonlocal interactions become nonzero at about $k = 55$, eventually exceeding the local interactions at a very high k. This is an artifact of the sharp numerical spectral truncation.

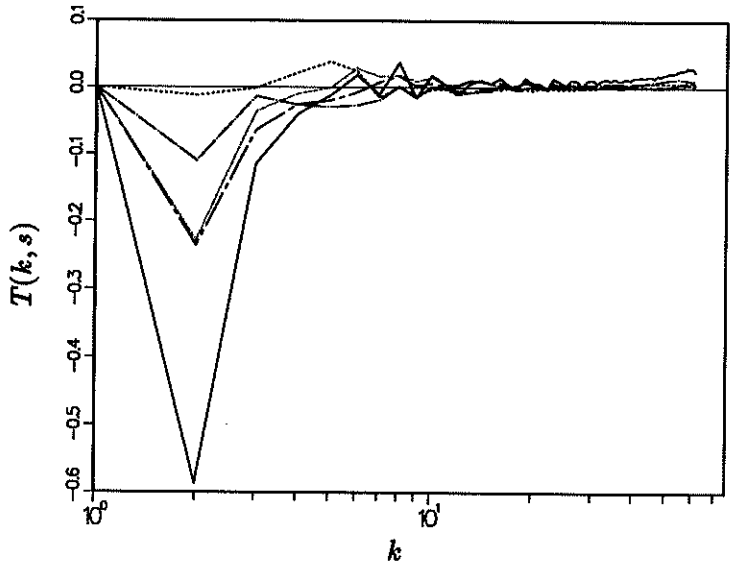

FIGURE 1A. The numerically resolved transfer spectrum. The contributions to total transfer from various interaction disparities are indicated. —— total, ---- $1 < s < 2$, —·— $2 < s < 4$, —·— $4 < s < 8$, ····· $s > 8$.

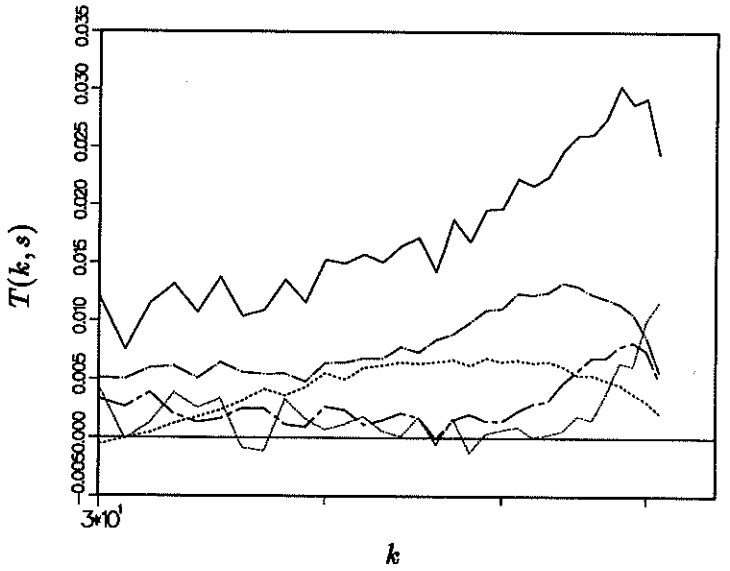

FIGURE 1B. The same as figure 1A but enlarged for $30 < k < 65$.

Recall that $T(k, p, q)$ is a very smooth curve with a pair of positive and negative peaks (Domaradzki and Rogallo, 1990). The results of our summing procedure contain some statistical noise because of the high degree of cancelation among the raw interaction statistics. The noise is most pronounced at small wave number

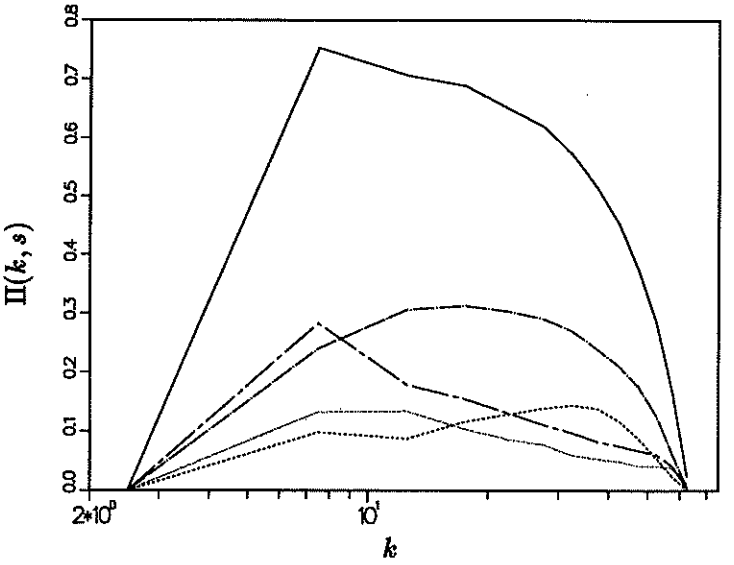

FIGURE 2. The numerically resolved flux spectrum. The contributions to total flux from various interaction disparities are indicated. —— total, - - - - $1 < s < 2$, —·— $2 < s < 4$, —·— $4 < s < 8$, ····· $s > 8$.

where the statistical sample is relatively small.

2.3. Analysis of the energy flux function

It is desirable to measure the relative contribution of local and nonlocal interactions to the energy flux as a function of k. Following (5), this is equivalent to rewriting (7) in terms of the shape parameter s as:

$$\Pi(k) = \sum_s \Pi(k, s). \tag{7}$$

In the classical Kolmogorov inertial range, where injection is absent and dissipation is negligible, energy conservation implies that the energy flux is a constant.

Fig. 2 displays the energy flux and the contributions $\Pi(k, s)$ of the various scale disparities. While the Kolmogorov phenomenology implies a constant energy flux in the inertial range, our computed energy flux decreases at high values of k because we included only the numerically resolved-scales and omitted the flux due to the subgrid-eddy viscosity. It is clear that the first and second octaves of s play a much more important role than the higher octaves.

Fig. 3 illustrates that the normalized energy flux $\Pi(k, s)/\Pi(k)$ is dominated by local interactions (small shape parameter s) for all scales k. This closely resembles the classical picture of the energy transfer process described in detail by Tennekes and Lumley (1971). Moreover, the dependence upon the shape parameter is the same for all inertial range scales, that is, beyond the forced scales, the normalized

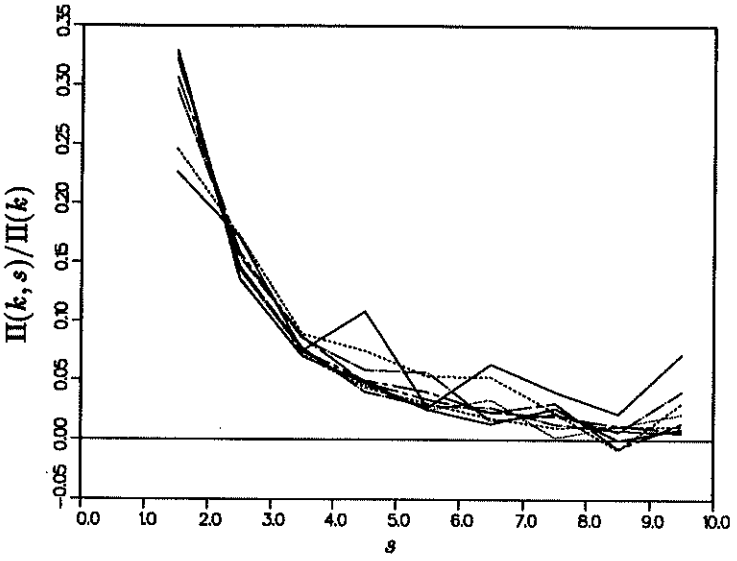

FIGURE 3. Dependence of energy flux upon the scale disparity of contributing interactions. The various curves are for $k = 15, 20, ..., 50$. The statistical noise is associated with the lower wavenumbers.

individual energy flux contributions $\Pi(k,s)/\Pi(k)$ are essentially independent of k as would be expected in a scale–similar inertial range.

Using the detailed conservation property of $T(k,p,q)$, the energy flux through scale k can be divided into two parts:

$$\Pi(k) = \Pi^s(k) + \Pi^e(k) \tag{9}$$

where

$$\Pi^s(k) = \int_k^\infty dk' \int_0^k dp \int_0^k dq T(k',p,q), \tag{10}$$

and

$$\Pi^e(k) = 2\int_k^\infty dk' \int_0^k dp \int_k^\infty dq T(k',p,q). \tag{11}$$

It should noted that $\Pi^s(k)$ and $\Pi^e(k)$ correspond to $\Pi^+(k)$ and $-\Pi^-(k)$, respectively, in Kraichnan (1971).

There are two types of non-local contributions to the energy flux resulting from distinct physical mechanisms: (1) when one of the wave vectors [say p] in $\Pi^s(k)$ is at very low wavenumber while the other [say q] $\approx k$, $\Pi^s(k)$ is closely related to the classical energy transfer closure model proposed by Obukhov (Monin and Yaglom, 1975; Batchelor, 1953) where the strain due to the large scales causes local energy transfer among the small scales, (2) when k' is very small while $p, q \gg k'$, $\Pi^e(k)$ is closely related to the classical eddy viscosity closure model proposed by Heisenberg (Monin and Yaglom, 1975; Batchelor, 1953). We calculate $\Pi^e(k,s)$, $\Pi^s(k,s)$ by

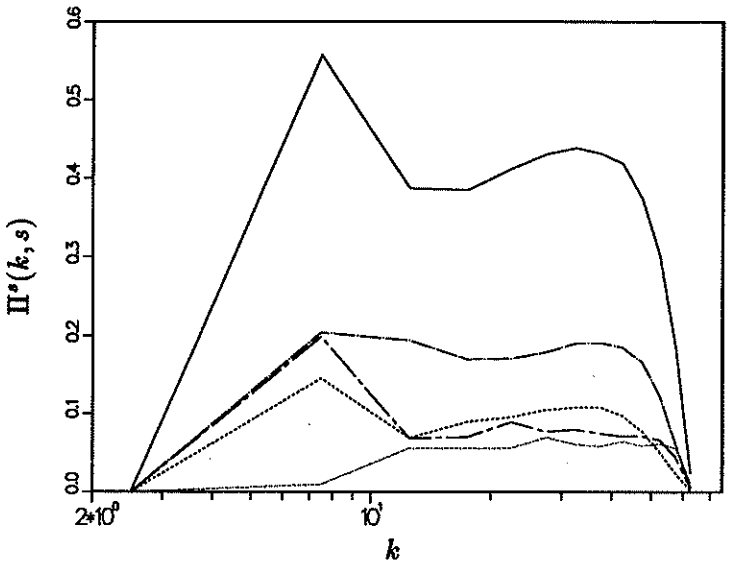

FIGURE 4A. The energy flux of the straining interactions. —— total, ---- $1 < s < 2$, —·— $2 < s < 4$, —·— $4 < s < 8$, ····· $s > 8$.

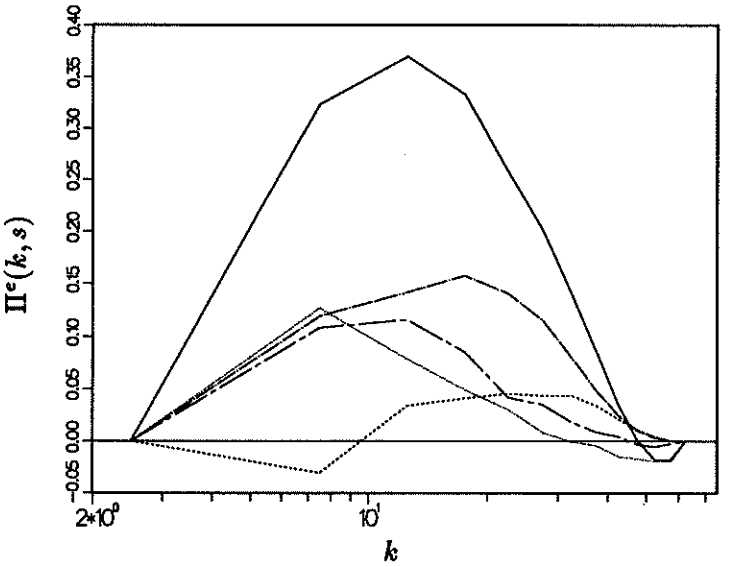

FIGURE 4B. The energy flux of the eddy-viscosity interactions. —— total, ---- $1 < s < 2$, —·— $2 < s < 4$, —·— $4 < s < 8$, ····· $s > 8$.

partial summation of (10) and (11) in the same manner as before. From Figs. 4A and 4B, we see that for both terms in (9), the local interactions are more important than the nonlocal ones. This is in agreement with the classical phenomenology of the inertial subrange and our $T(k,s)$ measurements.

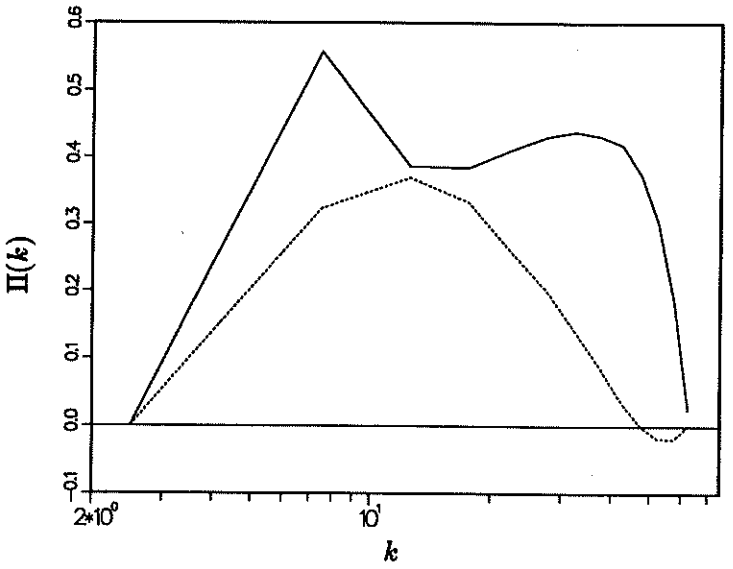

FIGURE 5. Comparison of the straining and eddy viscosity types of interactions. —— straining type, ····· eddy viscosity type.

The relative importance of the straining and eddy-viscosity interactions is compared in Fig. 5. While the two types of interactions are of similar magnitude at low wavenumbers, the straining interactions dominate at high wavenumber. The recursive renormalization group analysis (Zhou et al., 1988; 1989), and numerical measurements (Zhou, 1991) identified these interactions as the source of the cusp in the spectral eddy viscosity in Kraichnan's (1976) formulation. However, we must stress that the dominance of the straining interactions at high k is an artifact of the sharp spectral-cutoff that is used analytically in RNG and numerically in our present measurement. Indeed, one would expect that the relative physical contributions of the eddy-viscosity and straining interactions in an inertial range would be invariant with k, as we found for the disparity contributions (see Fig. 3).

2.4. Summary

This work addresses a fundamental question regarding the energy transfer process. At issue is the appropriate choice of a statistical quantity to indicate the nature of energy transfer across the spectrum. Basically the problem is that the system is conservative, with $T(k) < 0$ for small k and $T(k) > 0$ for large k, but since we cannot "tag" energy, we can not follow its "flow". Our investigation indicates that although the quantity measured by cited papers is a mathematical building block in the energy transfer process, it is not the appropriate physical quantity one should use to determine whether the nonlinear interactions are local or nonlocal. With the use of an existing DNS flow database, we have clarified these issues by making measurements of quantities that directly reflect the actual scale disparity of the interactions contributing to the energy transfer process. We found that the net flux results primarily from interactions in which the ratio of largest to smallest scale is

less than 10.

3. Future plans

3.1 DNS analysis of the scaling and statistics

We now have a fairly good indication that local energy transfer results from relatively local interactions. The next step would be to refine the analysis using larger interaction samples (256^3 or larger flow fields). We also will use the interaction count, which we have already computed, to account for the loss of interacting triads near the sharp spectral cutoff. We believe that this would aid in understanding the energy transfer process by reducing the statistical noise and increasing the range of scales. We would like to also use the larger fields for other measurements of turbulence statistics.

3.2 Numerical RNG procedure

The RNG theory results from an attempt to solve the forced Navier-Stokes equations in frequency-wavenumber space by an analytical iterative scheme. Due to the analytical complexity, the iteration process is carried out for only one step, and some terms that arise are discarded. The goal of this part of the plan is to attempt to use a numerical rather than an analytic evaluation of the convolution integrals, thus retaining all of the terms, and to evaluate the errors in the analytical theory. This is an ambitious project because it discards the advantage of time marching and treats the time dimension in frequency space in a manner analogous to the previous treatment of the spatial dimensions in wavenumber space. This has the obvious disadvantage of requiring an additional dimension of storage and forces us to use relatively coarse meshes, but offers the compensating advantage of providing the entire time history for post-processing. To our knowledge, the explicit use of frequency space in fluid dynamics simulations has never been attempted before. This work will not only extend our current DNS of the Navier-Stokes equation into a new dimension but also make a careful test of the RNG theory (Yakhot and Orszag, 1986; Zhou et al., 1989) possible.

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