Turbulence and vortex structures

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1. Motivations

One of the most sought after goal in turbulence physics is to understand exactly what features of the incompressible Navier–Stokes equations

$$\partial_t \vec{\Omega} + (\vec{u} \cdot \nabla)\vec{\Omega} = (\vec{\Omega} \cdot \nabla)\vec{u} + \nu \nabla^2 \vec{\Omega} \qquad \vec{\Omega} = \nabla \times \vec{u}$$

(here written in vorticity form) lead to the well-known Kolmogorov energy spectra $E(k) \propto k^{-5/3}$ for intermediate wavenumbers $1/L \ll k \ll 1/\eta$ well separated from both large scales L associated with boundary conditions and small scales η associated with viscous effects. If the two non-linear terms $(\vec{u} \cdot \nabla)\vec{\Omega}$ (the large-scale sweeping of vorticity by the mean flow) and $(\vec{\Omega} \cdot \nabla)\vec{u}$ (the small-scale stretching of vorticity by velocity gradients) were dynamically unimportant, the Navier–Stokes equation would show no inertial range; the energy spectrum would be as for fluid particles undergoing random walks: $E(k) \propto k^2 \exp(-k^2)$ for all wavenumbers k. Clearly, some dynamical feature of the two non-linear terms then must be responsible for changing the relatively more cutoff "inertial range" spectrum $k^2 \exp(-k^2)$ of the hypothetical random walker turbulence to the relatively less cutoff $k^{-5/3}$ observed in real fluid turbulence. For this to be the case, the non-linear terms must generate energy on small scales relative to random walker dynamics and, moreover, must do so in a scale invariant way. A precise dynamical understanding of these two non-linear terms using both theoretical (mathematical) and experimental (in this case primarily numerical) tools as necessary is — even at the relatively low Reynolds numbers of direct numerical simulations — at least a good starting point towards a precise understanding of the critical behavior of the Navier–Stokes equation in inertial range turbulence.

One of the outstanding features of direct numerical simulations of turbulence and, presumably, of naturally occurring turbulence is the presence of coherent highly localized strands of strong vorticity. In light of the above argument, these "vortex structures," derived (as will be outlined below) mostly but not completely from the vortex stretching non-linear term $(\vec{\Omega} \cdot \nabla)\vec{u}$, must be in some way involved in the energy transmission process from large to small scales. The presence of vortex structures makes it tempting to view high Reynolds numbers velocity fields as some sort of spatial and temporal superposition of structured and random regions (She, Jackson, Orszag 1990). In this view, random regions contain most of the turbulent kinetic energy of the fluid; structured regions, on the other hand, contain little kinetic energy but do contain most of the flow's viscous energy dissipation. Thus, vortex structures are skeletons of non-linear dynamical activity which *separate* regions of relatively strong kinetic energy from regions of relatively strong viscous dissipation and dynamically moderate energy transfers from large to small scales. Seen in this way, the traditional picture of a spatially fractal energy cascade, in which eddies interact with smaller eddies which interact with smaller eddies and so on until being dissipated at viscous scales, is likely wrong. Energy can certainly be transmitted directly from large to small scales, presumably via vortex structures and their dynamical interactions, without necessarily being moderated by a recursive hierarchy of intermediate scales. If this is the case, then what is it about the non-linear interactions which make turbulence self-similar?

As a starting point for these more general issues, the following questions need to be answered. What are vortex structures? How are they born? How do they die? What Navier–Stokes dynamics generate them? These questions have been the subject of previous work (Dresselhaus and Tabor, 1991) and are the subject of present investigations whose progress will be briefly described below. Questions beyond these are even more fundamental towards a basic understanding of inertial range turbulent self-similarity. How do vortex structures interact and what is spatially and/or temporally self-similar about these interactions? If vortex structures are primarily the precise dynamical result of the vortex stretching term $(\vec{\Omega}\cdot\nabla)\vec{u}$, what is the precise role of the sweeping term $(\vec{u}\cdot\nabla)\vec{\Omega}$? Is there a dynamical connection between stretching and sweeping terms? These further questions, not adequately answered here, are the subject of future investigation.

2. Past and present

2.1 Vortex structures and the intermediate strain

A number of direct numerical simulations of incompressible turbulence (e.g. Ashurst et al. 1987) have observed a strong tendency for regions of strong vorticity to be aligned with the direction corresponding to the intermediate strain. That is, vorticity, once stretched, is strongly perpendicular to the plane in which the most local straining is occurring. Theoretical arguments (Betchov 1956) suggest that — at least for isotropic turbulence — the intermediate strain which is primarily responsible for stretching vorticity is on average positive. This result is initially a bit surprising: one expects that since vorticity is stretched by strain

$$\frac{d}{dt}\vec{\Omega} = \mathbf{S}(t)\vec{\Omega} + \nu\nabla^2\vec{\Omega},$$

vorticity should align and stretch along the principal direction of strain $\mathbf{S}$ with largest positive eigenvalue. (Here and in what follows, $\mathbf{S}(\vec{x},t) = (\partial\vec{u}/\partial\vec{x}+\partial\vec{u}^\dagger/\partial\vec{x})/2$ is the strain tensor, $\mathbf{\Omega}(\vec{x},t) = (\partial\vec{u}/\partial\vec{x} - \partial\vec{u}^\dagger/\partial\vec{x})/2$, $\dagger$ denotes transpose, $\mathbf{P}$ is the second derivative of the scalar pressure $p(\vec{x},t)$, and d/dt is a material derivative so that all equations are thought of in Lagrangian coordinates.) Although stretching in the largest positive strain direction is observed for short times, it is not the predominant effect. We will see shortly that this preferred alignment between vorticity and the intermediate strain is precisely what makes "coherent structures" coherent.

To understand why vorticity stretches normal to the plane of largest strain, it seems natural to transform the evolution equations for strain and vorticity

$$\frac{d}{dt}\mathbf{S} = -\mathbf{S}^2 - \mathbf{\Omega}^2 - \mathbf{P} + \nu\nabla^2\mathbf{S} \qquad \frac{d}{dt}\mathbf{\Omega} = \mathbf{S}\mathbf{\Omega} + \nu\nabla^2\mathbf{\Omega} \tag{1}$$

into the moving orthonormal basis (with eigenvectors $\{\hat{\xi}_1(t), \hat{\xi}_2(t), \hat{\xi}_3(t)\}$) in which the strain $S(t)$ is diagonal (with eigenvalues $s_1(t) \geq s_2(t) \geq s_3(t)$). Incompressibility ensures that $s_1(t)$ is always positive and that $s_3(t)$ is always negative; Navier–Stokes dynamics ensure that $s_2(t)$ is on average positive at least for idealized turbulence (Betchov 1956) and for numerically simulated turbulence (Ashurst et al. 1987). The pressure second derivative can be written by inverting the Poisson equation for pressure $\nabla^2 p = \sum_\alpha \Omega_\alpha^2 - s_\alpha^2$ to get

$$P_{ij}(\vec{x}, t) = \frac{1}{4\pi}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\int \frac{\sum_\alpha s_\alpha^2(\vec{x}') - \Omega_\alpha^2(\vec{x}')}{\|\vec{x} - \vec{x}'\|} d^3\vec{x}'. \tag{2}$$

Pressure strongly couples in distant strain and vorticity into what would otherwise be local dynamical equations for strain and vorticity. When strain and vorticity have accumulated either locally or globally, pressure effects will be important in the dynamics of (1).

In order to write the evolution equations for strain and vorticity in the moving strain basis, we must know how the strain axes rotate as strain and vorticity evolve. The strain axes instantaneously rotate about a vector $\vec{\Omega}'$ given in strain coordinates by the expression

$$\Omega_k' = \vec{\Omega} \cdot \hat{\xi}_k = \frac{1}{2}\sum_{ij} \epsilon_{ijk} \frac{d\mathbf{S}/dt_{ij}}{s_i - s_j} \tag{3}.$$

The rotation of the strain axes depends on the Navier–Stokes evolution of strain (equation (1)) via $d\mathbf{S}/dt$, and hence the rotation of the strain axes are coupled to the evolution of strain and vorticity. This coupling differentiates the stretching of passive vectors such as magnetic field (in the ideal limit) from active vectors such as vorticity. With the use of (1), the strain axis rotation vector splits into three parts

$$\Omega_k'(t) = \frac{1}{2}\sum_{ij} \epsilon_{ijk} \left(\frac{-\Omega_i\Omega_j - P_{ij} + \nu(\nabla^2 S)_{ij}}{s_i - s_j}\right),$$

respectively due to vorticity ($\vec{\Omega}'_{\vec{\Omega}}$), off-diagonal pressure second derivatives ($\vec{\Omega}'_P$), and viscosity ($\vec{\Omega}'_\nu$). Alignment with the intermediate strain will shortly be seen as a consequence of the special dynamical role played by the strain axis rotation due to vorticity $\vec{\Omega}'_{\vec{\Omega}}$.

In strain coordinates, the Navier–Stokes evolution for strain and vorticity become

$$\frac{d}{dt}s_j = -s_j^2 + (\vec{\Omega}^2 - \Omega_j^2) - P_{jj} + \nu\nabla^2 s_j \qquad j = 1, 2 \tag{4}$$

and

$$\frac{d}{dt}\Omega_i = (s_i + s_i')\Omega_i - \left(\vec{\Omega}'_P \times \vec{\Omega}\right)_i + \nu\nabla^2\Omega_i \qquad i = 1,2,3. \tag{5}$$

The rotation of the strain axes due to vorticity itself has appeared as a non-linear self-stretching term in (5) with stretching rate s_i' defined by

$$s_i' = \sum_{j\neq i} \frac{\Omega_j^2}{s_j - s_i}.$$

These self-stretching rates will be seen to be primarily responsible for vorticity stretching in the intermediate strain. The rotation of the strain axes due to viscous effects $\vec{\Omega}'_\nu$ cancels with other terms arising from projecting (1) into the strain basis and is not present in (5). The rotation effects due to pressure second derivatives play a complex non-local role in the evolution of strain and vorticity; a discussion of the role of **P** in general will be deferred to the next section. These five coupled ordinary integro-differential equations form the basis for the current analysis.

The sign and relative magnitude of the self-stretching rates s_i' compared to the strains s_i are of great dynamical significance. The quadratic dependence of s_i' on vorticity suggests that the self-stretching rate s_i' should dominate the strain s_i as long as vorticity is sufficiently large (for example, in a vortex structure in turbulence). Using incompressibility and the ordering $s_1 \geq s_2 \geq s_3$, one can see that $s_1'(t) \leq 0$ and that $s_3'(t) \geq 0$ for all times t. A similar argument shows that s_2' can be either positive or negative depending on the relative alignment of vorticity with the $\hat{\xi}_1$ and $\hat{\xi}_3$ directions. In either case, the non-linear stretching terms oppose the production of vorticity — at least in $\hat{\xi}_1$ and $\hat{\xi}_3$ directions. This opposition is particularly striking in the $\hat{\xi}_1$ direction.

The above discussion of the self-stretching rates s_i' suggests the following Lagrangian description vortex stretching. Suppose that at some time t, the intermediate shear is approximately zero, i.e. $s_2(t) \approx 0$, and that the vorticity is weak, i.e. $\vec{\Omega}(t) \approx 0$. Since strain and vorticity are both by hypothesis small, the components of P_{ij} are also negligible. At this instant, the non-linear stretching rates s_i' are small, and vorticity will begin to stretch in the $\hat{\xi}_1$ direction. From equation (4), it follows that since $s_2 \approx 0$, the stretched $\hat{\xi}_1$ component of vorticity will cause s_2 to grow and become positive resulting in an increase in Ω_2. As a consequence of this sequence of events, s_1' will become large and negative and will quickly act to nullify the previous growth in Ω_1 (see equation (5)). Since s_2 is still small compared to s_1, the Ω_2 stretching will proceed for a longer time than the initial stretching of Ω_1 due to s_1. This relative long-time persistent alignment with $\hat{\xi}_2$ survives until Ω_2 itself becomes large at which point contraction of vorticity in the $\hat{\xi}_3$ direction becomes significant (due to the quadratic contribution of Ω_2 to the always positive non-linear stretching rate s_3'). The net result is that now s_2' becomes negative and large. This suppresses the previous stretching of the intermediate vorticity Ω_2. The alignment with the strong contracting direction causes a quick but not total destruction of the previous vorticity build up. It should be emphasized that the relatively small magnitude of the intermediate shear s_2 implies that the vortex stretching in the intermediate direction is the most long-lived of the stretchings described in this scenario.

Thus, vorticity initially grows in the largest positive strain direction. As self-stretching effects take hold, it switches towards alignment with the intermediate strain. Vorticity stretching proceeds in the intermediate direction as long as the intermediate strain is positive and until further self-stretching effects cause it to quickly and violently contract in the largest negative strain direction. This sequence of events is robustly observed in a variety of simulations we have performed — even in the presence of random forms of the pressure terms. The details of these simulations are presented in Dresselhaus 1991.

2.2 The role of pressure

The precise role played by pressure (via **P**) in the above description is sketchy at best. In light of (2), pressure is negligible only when the "charge density" $\rho(\vec{x},t) = \sum_\alpha s_\alpha^2 - \Omega_\alpha^2$ (by analogy with electrostatics) is locally small. This charge density ρ assigns positive charge to regions where strain is larger than vorticity and negative charge where vorticity dominates strain. Also, pressure is only unimportant when other regions of strong vorticity are distant, randomly oriented, and numerous enough to screen each other's charge density. We have implicitly assumed this to be the case in the previous section and will continue to do so here. That is, we will consider here only semi-local effects of pressure, effects due to the self-interaction of the isolated strain and vorticity.

Based on the evolution of strain (4) and vorticity (5), the diagonal and off-diagonal components of **P** play different roles. The diagonal components $P_{11}(t)$ and $P_{22}(t)$, depending on their sign and magnitude, increase or decrease the largest and intermediate strains $s_1(t)$ and $s_2(t)$. The off-diagonal components are involved in rotating the strain axes in the vorticity evolution (5). Strong random off-diagonal $P_{ij}(t)$ will unpredictably upset the alignment of vorticity with the intermediate strain and will likely destroy the delicate emergence of a vortex structure.

The pressure integral (2) can be written explicitly

$$P_{ij}(\vec{x},t) = \frac{1}{4\pi}\int \rho(\vec{x}',t)G_{ij}(\vec{x}-\vec{x}')d^3\vec{x}'$$

$$G_{ij}(\vec{x}-\vec{x}') \equiv \frac{3(x_i - x_i')(x_j - x_j') - \delta_{ij}(\vec{x}-\vec{x}')^2}{\|\vec{x}-\vec{x}'\|^5}.$$

In regions of the fluid where strain and vorticity form no coherent structures, one expects that the tensor P_{ij} will be isotropic and, although likely to be strongly fluctuating, will average to zero. On the other hand, near coherent structures, this isotropy likely breaks down. For a mature growing vortex structure aligned along the intermediate strain, most of the semi-local charge $\rho(\vec{x}',t)$ in the integral is oriented along this intermediate direction (which is here at least locally presumed to be straight). Also, as long as vorticity has significantly stretched, it can be argued that the net semi-local charge density will be vorticity dominated and hence negative. In this case, the kernels G_{11} and G_{22}

$$G_{11}(\vec{\xi}) = 2\xi_1^2 - \xi_2^2 - \xi_3^2 \qquad G_{22}(\vec{\xi}) = -\xi_1^2 + 2\xi_2^2 - \xi_3^2$$

will be dominated by terms involving $\xi = \vec{x} - \vec{x}' \propto \hat{\xi}_2$. Thus, as long as the vortex is fairly straight and aligned along the intermediate direction, P_{11} will be positive and P_{22} will be negative. If these pressure effects are purely local and the vortex structure is perfectly straight, one would expect that $-2P_{11} \approx P_{22}$. The positivity of P_{11} tends to decrease the largest strain s_1; the negativity of P_{22} tends to increase the intermediate strain. Also, the increase in s_2 should be more pronounced than the decrease in s_1. In this view, the semi-local effects of the pressure on vorticity growth act in concert with the self-stretching effects described in the previous section. Both mechanisms favor the growth of vorticity in the intermediate strain direction.

As long as vorticity remains aligned with and stretches along the intermediate strain, similar arguments can be made for the semi-local effects of the off-diagonal components of the pressure second derivative. Based on relative differences in the kernel G_{ij}, one can conclude that typical magnitudes of P_{ij} for $i \neq j$ will be ordered

$$|P_{13}| < |P_{23}| < |P_{12}|.$$

The corresponding denominators $s_i - s_j$ in the expression for $\vec{\Omega}'_P$ are similarly ordered

$$s_1 - s_3 > s_2 - s_3 > s_1 - s_2$$

(we are assuming here that the intermediate strain s_2 is positive as it must be for vorticity to grow in the intermediate direction). Thus, the strain axis rotation due to semi-local pressure effects should have a slightly preferential direction. That is, rotations taking vorticity between $\hat{\xi}_1$ and $\hat{\xi}_2$ will be strongest and more likely; rotations taking vorticity between $\hat{\xi}_2$ and $\hat{\xi}_3$ will be second most strong; rotations between $\hat{\xi}_1$ and $\hat{\xi}_3$ will be least strong and less likely. Thus, when vorticity is strongly aligned with $\hat{\xi}_1$ (as it is for short times), rotations will tend to carry it into the $\hat{\xi}_2$ direction in concert with the self-stretching mechanism described above. When vorticity is subsequently aligned with the intermediate direction, rotations will tend to carry it into the $\hat{\xi}_3$ direction. It is likely that these semi-local rotation effects are thus responsible for the ultimate destruction of vortex structures in the strongly contracting $\hat{\xi}_3$ direction.

2.3 Numerical verifications

In current and future work, these speculations concerning the role of pressure as well as the entire strain basis description of vortex structures will be tested on a variety of direct numerical simulation data bases. The goal of these studies will be to verify and sharpen the intuitions gained by this approach. The dynamical variables of interest will always be strain basis quantities, such as the principal strains s_1 and s_2, the strain basis vorticity components Ω_1, Ω_2, and Ω_3, the strain axis rotation vector due to pressure $\vec{\Omega}'_P$, and the pressure second derivative tensor P_{ij}. The interpretive tools used in these studies will be both statistical and visual. Probability densities for these quantities will be studied conditioned on the strength of vorticity. In this way, we can statistically separate vortex structures from the background flow and contrast strain basis dynamics — seen from this statistical perspective — in these structured regions from that of the background flow. Beyond statistics, I would like to perform visualization studies of a single vortex structure using these strain basis quantities to follow the birth, evolution, and death of a vortex structure. These studies could verify speculations concerning the strain basis evolution of strain and vorticity made in the two preceding sections.

3. Future work

Beyond the near future, one would like to address the more fundamental questions posed at the beginning of this brief. The research outlined thus far aims to understand a single vortex structure in isolation. One would like to go beyond this and understand the consequences of having many mutually interacting vortex structures. In particular, one would like to study whether or not the strain axes (particularly the intermediate strain axis) themselves have global alignment tendencies with respect to fixed axes. In other words, do vortices themselves tend to have mutual alignment properties? Such alignment must exist for anisotropic flows such as turbulent shear flows and boundary layers but may or may not exist in isotropic turbulence.

Along similar lines, it would be interesting to determine what the typical vortex screening distance is in turbulence. This corresponds to the fluid dynamic analogy of Debye screening in plasmas where in complex distributions of positive and negative charge (ions and electrons in the case of a plasma) the Coulomb interaction becomes screened. Individual charges are only dynamically felt over a typical distance — the Debye length. Beyond this distance, the plasma can be thought of as electrically neutral. In a turbulent fluid filled with many dynamically appearing and disappearing vortex structures, one would like to investigate the analogous typical interaction length.

Further future work seeks to understand whether or not Navier–Stokes dynamics is *more* than just vortex stretching and dissipation. In other words, is any relevant Navier–Stokes physics lost by considering the vorticity equation as a Lagrangian (rather than Eulerian) evolution equation. If no physics were lost then, just as the Kolmogorov theory characterizes dissipation by a single number ϵ, it seems plausible that vortex stretching could also be characterized by a single number: the average of the intermediate strain $\langle s_2 \rangle$. Such an approach may provide a more physical, more dynamical characterization of intermittency effects in turbulence than the traditional intermittency corrections to the Kolmogorov theory. If physics is lost in the transformation from Eulerian to Lagrangian coordinates, as I believe it must be, then the precise dynamical role of the sweeping term in vortex interactions needs to be further understood.

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