Modeling turbulent boundary layers in adverse pressure gradients

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1. Motivation and objectives

The phenomenon of separation of a turbulent boundary layer has important implications in practical applications but remains little understood. The overall aims of this research are to gain theoretical understanding of the physical processes that are important in governing the separation and thence to develop closure models to predict these flows.

The model problem that is considered is an incompressible turbulent boundary layer on a flat plate that is subjected to a prescribed, external, pressure gradient.

The structure of attached turbulent boundary layers is more complex than that of laminar layers. The zero-pressure-gradient (hereafter ZPG) turbulent boundary layer has a well known two layer structure with a logarithmic velocity profile at the common overlap. Many of the existing turbulence models that are used for boundary layer calculations were conceived and calibrated using data from the ZPG boundary layer. However, the application of an adverse pressure gradient (hereafter APG) to a turbulent boundary layer leads to a very different structure because the adverse pressure gradient alters both the mean flow and turbulent transport properties. In order to develop improved models for boundary layers, it is, therefore, natural to start by examining the fundamental differences between the APG and ZPG turbulent boundary layers.

When taken with reasonable physical assumptions, asymptotic methods provide a systematic framework that can contribute to our understanding of turbulent boundary layers. Hence much of the effort to date has been in using asymptotic methods to study the structure of a turbulent boundary layer that is approaching a separation point, thereby leading to some conception of the important length and velocity scales that determine the nature of the separation.

2. Accomplishments

The asymptotic structure of a turbulent boundary layer that is subjected to a strong APG has been investigated in the context of eddy viscosity closure. The results, which are described briefly in §2.1, show that the APG turbulent boundary layer is very different from the classical, ZPG boundary layer structure.

The mean point of separation of a turbulent boundary layer may by defined as the point where the boundary layer approximation of the Reynolds averaged momentum equations ceases to apply. The asymptotic theory has been extended to consider whether or not the separation point coincides with the point of zero skin friction. The results, described in §2.2, suggest that no singularity occurs in the boundary layer equations at the point of zero skin friction. This is in strong contrast to the laminar APG case, where the classical square root singularity of Goldstein terminates the validity of the boundary layer approximation at zero skin friction.

The findings of this study lead to conclusions that are rather different than the results of two other recent investigations. In §2.3, an attempt is made to reconcile these differences and to highlight the areas of conflict.

2.1 The scaling of adverse pressure gradient turbulent boundary layers

A brief overview of the asymptotic scaling of an APG turbulent boundary layer is now given. Full details may be found in Durbin & Belcher (1991). The reader is also referred to the article by Durbin in this volume.

The free stream velocity is $U_\infty(x)$ and α is the prescribed, kinematic pressure gradient $((1/\rho)(dP/dx))$. The streamwise length scale is then $L = U_\infty^2/\alpha$. If δ is a measure of the boundary layer thickness, then one small parameter is δ/L. The second is the reciprocal of the Reynolds number, $Re_L^{-1} = \nu/U_\infty L$, and it emerges that the small parameter in the wall region is $\epsilon = Re_L^{-1/3}$. For the purposes of the formal asymptotic ordering it is sufficient to let $\delta/L \sim \epsilon$. This ordering implies that the flow is slowly varying, *i.e.* the eddy turnover time scale δ/u' (where u' is a measure of the fluctuating velocity) is of the same order as the mean flow advection time scale, L/U_∞. The APG turbulent boundary layer is then composed of three distinguishable asymptotic regions.

In the outer region, the nondimensional variables are distinguished by a tilde:

$$\tilde{y} = y/\delta, \quad d\tilde{x} = dx/L, \quad \tilde{U} = U/U_\infty, \quad \tilde{\nu}_T = \nu_T/(U_\infty\delta^2/L). \tag{1}$$

Here ν_T is the eddy viscosity. It is observed experimentally that the velocity deficit across the outer region is large so that the flow is governed by the full nonlinear boundary layer equations (*cf.* the ZPG boundary layer where the velocity deficit in the wake region is small and the equations can be linearized).

In the middle region, the turbulent transport processes make a transition from their near wall behavior (where the mixing length ideas are expected to be valid) to the outer region behavior (where the Clauser, constant eddy viscosity model is adopted). The nondimensional variables are denoted with an overbar:

$$\bar{y} = y/(\delta\gamma), \quad \bar{U} = U/\sqrt{\alpha\delta\gamma}, \quad \bar{\nu}_T = \nu_T/(U_\infty\delta^2/L). \tag{2}$$

The velocity scale is determined by the pressure gradient and the boundary layer thickness—*not* the free stream velocity. Matching to the constant eddy viscosity that is adopted in the outer region determines that $\gamma = (\delta/L)^{1/3}$. With these scalings, the x-momentum equation reduces to a balance between the Reynolds stress and the pressure gradients.

In the wall region, care must be exercised in defining the velocity scale. The appropriate choice is the viscous, pressure gradient velocity, defined by

$$u_p = (\alpha\nu)^{1/3}, \tag{3}$$

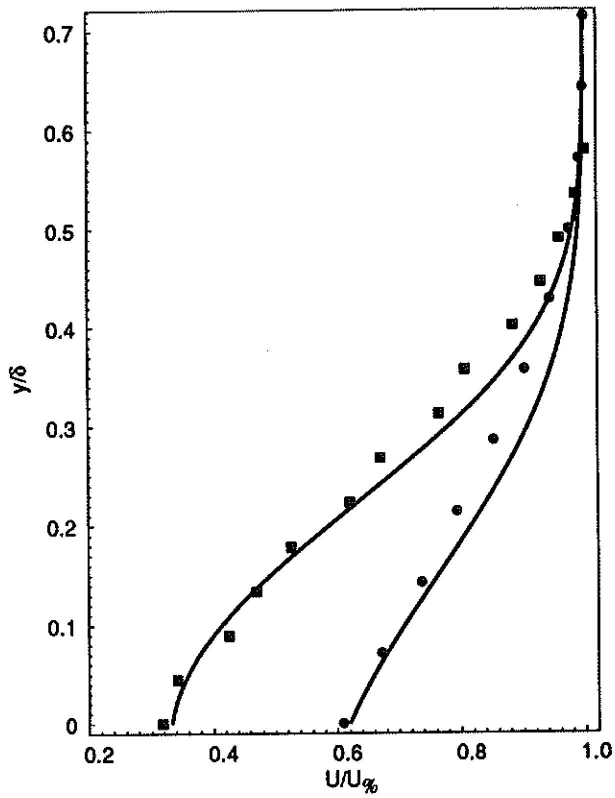

FIGURE 1A. Comparison of computations with experimental data on self-similar APG boundary layers. Data tabulated in Coles & Hirst (1968): circles Bradshaw & Ferris; squares Stratford.

where ν is the molecular viscosity. It is inappropriate to use the local friction velocity, u_*, because we are concerned with flows that include those near separation, when the friction velocity approaches zero. Similarly, the classical, 'wall-layer' length-scale based on the friction velocity, ν/u_*, becomes infinite at zero wall shear stress and so is clearly inappropriate. The appropriate choice is ν/u_p, which is well-behaved when the wall stress is zero. The nondimensional variables, which are denoted by a hat, then become

$$\hat{y} = yu_p/\nu, \quad \hat{U} = U/u_p, \quad \hat{\nu}_T = \nu_T/\nu. \tag{4}$$

The mean x-momentum then expresses a balance between the gradients of viscous and turbulent stress and the pressure gradient.

The middle region is required formally because the wall and outer regions do not have a common overlap. Hence it is not possible to adapt Millikan's overlap argument and deduce the skin friction relation. Instead, the skin friction law for

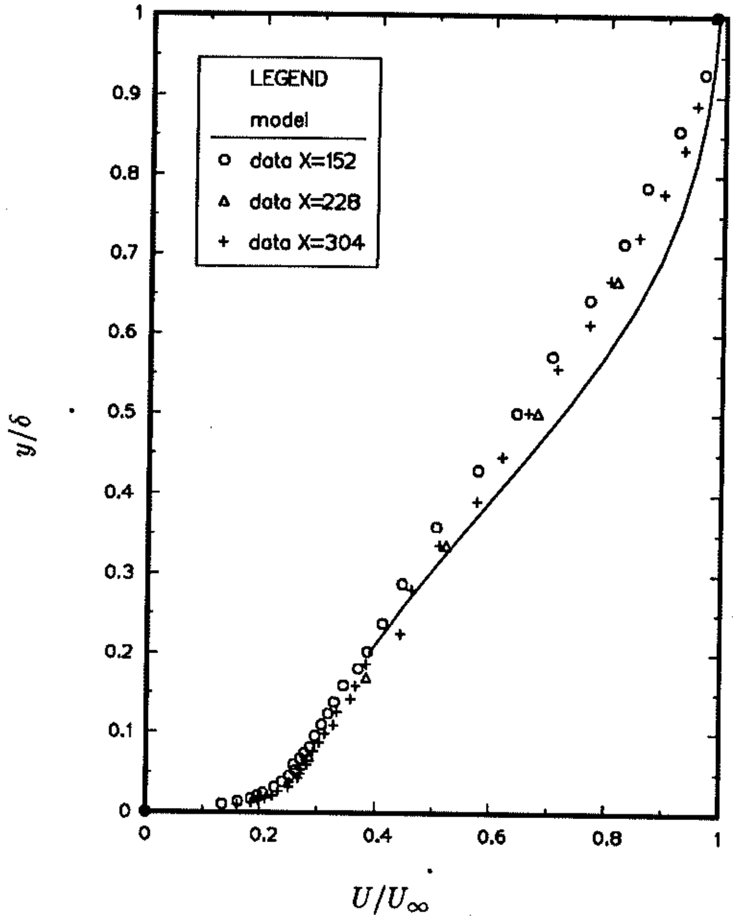

FIGURE 1B. Comparison of model with experimental data of Driver (1991). (The experimental data sets are from different x-stations measured in mm from a reference point.)

the APG boundary layer is obtained by solving for the nonlinear flow in the outer region, with boundary conditions imposed by matching through the middle and wall regions.

In this preliminary part of the study, self-similar flows were considered explicitly. In the outer region, where the Clauser eddy-viscosity model is used, the x-momentum equation becomes the Falkner-Skan equation. The boundary conditions were determined from the matching of the wall and middle layers with the outer region. Figures 1*a* and 1*b* show profiles of the self-similar velocity computed using the present model and comparisons with experimental data.

2.2 *On the singularity at separation*

The asymptotic analysis of Durbin & Belcher (1991) has been extended to investigate how a turbulent boundary layer behaves at a point of zero skin friction. This analysis is now briefly described.

Physically, the separation of a boundary layer is marked by mean streamlines moving abruptly away from the bounding surface. Hence separation is associated with a mean velocity component normal to the wall that is comparable with the streamwise velocity component. This signifies that the boundary layer approximation has broken down and the full, elliptic, Navier-Stokes equations govern the flow. Mathematically, a singularity occurs in the boundary layer equations. The separation point is, therefore, defined as the point at which the boundary layer approximation ceases to be valid. It is important to make this definition precise, because it is not clear *a priori* that this definition of the separation point coincides with the point of zero skin friction. Indeed the present results indicate that the boundary layer approximation holds at, and slightly beyond, the zero skin friction point.

In the laminar flow, the classical analysis of Goldstein (1948) shows that the boundary layer equations have a singularity at x_s, where the wall shear stress varies as $\tau_w \propto \sqrt{x_s - x}$, which does coincide with the point of zero skin friction. The scaling described in §2.1 is now extended to examine any singularity in the turbulent boundary layer equations.

The possibility of a singularity at separation is made apparent by an argument due to Terrill (1960). Consider the x-momentum equation in the boundary layer approximation (x is the streamwise direction measured such that $x = 0$ is the point of zero skin friction). Then, taking $\partial^2/\partial y^2$ of this equation and setting $y = 0$ shows that

$$\frac{d}{dx}\left(\frac{1}{2}\tau_w^2\right) = \left[\frac{\partial^3}{\partial y^3}\left\{(\nu + \nu_T)\frac{\partial U}{\partial y}\right\}\right]_{y=0} \tag{5a}$$

$$\approx a_0 + a_1 x + \cdots \tag{5b}$$

Where the a_i are the coefficients of a Taylor expansion in x of the right hand side of (5a). Integration of (5b) over x shows that the wall shear stress varies approximately as

$$\tau_w \approx \sqrt{-2a_0 x + \cdots} \tag{6}$$

Hence there is a singularity in the boundary layer equations if a_0 is non-zero.

In equation (5*a*), the term $d/dx(\tau_w^2/2)$ arises from the nonlinear, advection term, $U\partial U/\partial x$. The key role played by the nonlinear term in producing the singularity is to be expected since only nonlinear differential equations have solutions with moveable singularities.

2.2.1 Asymptotic arguments

With the wall-region asymptotic scaling of §2.1, the x-momentum equation becomes

$$\frac{\partial}{\partial \hat{y}}\left(\{1 + \hat{\nu}_T\}\frac{\partial \hat{U}}{\partial \hat{y}}\right) = \alpha + O(\epsilon^2) \tag{7}$$

so that, according to the asymptotic theory, the advection terms in the wall region are of $O(\epsilon^2)$. The heuristic argument for the origin of the singularity presented above makes it clear that, in order for the singularity to occur in the wall region, the nonlinear inertial term, $U\partial U/\partial x$, must become of the order of α. This can occur only if the streamwise velocity varies on a short length scale $\hat{x} = \epsilon^2 x$, *i.e.* $\partial\hat{U}/\partial\hat{x} = O(1)$. We shall demonstrate that there is no such rapid variation of $\hat{U}$.

On the $\hat{x}$ length scale, the pressure gradient, which by definition varies on the $x = O(1)$ length scale, is constant so that, in this study, it is sufficient to consider a constant APG, *i.e.* $\alpha =$ constant.

The solution from Durbin & Belcher (1991) for the leading order contribution to the streamwise velocity in the wall region may be written

$$\hat{U} = \int_0^{\hat{y}} \frac{\hat{y} + u_*^2/u_p^2}{1 + \hat{\nu}_T} d\hat{y}. \tag{8}$$

For an adverse pressure gradient of constant strength (where u_p is constant), the numerical results described below show that the wall shear stress varies approximately linearly. Hence, differentiation of equation (8) shows that, in order for the velocity in the wall region to vary by order one on the $\hat{x}$ length scale,

$$\frac{\partial\hat{\nu}_T}{\partial\hat{x}} = O(1). \tag{9}$$

According to the reasoning of §2.1, the key velocity and length scales in the wall region are u_p and ν/u_p. Hence the eddy viscosity might be expected to depend on these parameters. Furthermore, it is reasonable that $\hat{\nu}_T$ depends on u_*^2. However, provided the functional dependence of the eddy viscosity on these parameters remains analytic as $u_*^2 \to 0$ (as it does for the mixing length and $k - \varepsilon$ models), none of these terms leads to a singular behavior in $\partial\hat{\nu}_T/\partial\hat{x}$ as $u_*^2 \to 0$. If, however, the eddy viscosity is *erroneously* modeled in terms of u_* (as in, for example, the Van Driest damping function), then a singularity does occur. As described above, ν/u_* becomes infinite at the zero skin friction point so that the use of u_* as a velocity scale is unphysical.

Equation (9) can be rewritten in terms of dimensional variables using the definition $\epsilon = Re_L^{1/3}$, the ordering $\epsilon \sim \delta/L$, and equation (4). The condition (9) then requires that in the wall region

$$\frac{\partial\nu_T}{\partial x} \sim \frac{U_\infty\delta}{L}, \tag{10}$$

Whilst the experimental data is not entirely conclusive, it does suggest that, near the wall, the eddy viscosity varies only slowly close to the point of zero skin friction (Driver, 1991; Simpson *et al.* 1981). None of the data show a variation of the magnitude needed to satisfy equation (10). We recognize, however, that the 'elliptic effects', which must become significant at separation in the real flow, may cause the eddy viscosity to adjust more smoothly than it would in a strict boundary layer approximation.

This argument suggests that, in the wall region, the advection terms remain small at the point of zero skin friction. Similarly, they are expected to remain small in the middle region. By contrast, in the outer region, the nonlinear terms are of leading order. The boundary conditions at the lower limit of the outer region flow are

$$U \sim \epsilon^{2/3} U_s, \quad U' \sim \epsilon \frac{u_*^2}{u_p^2} \frac{1}{\nu_{T_\infty}}, \quad V \sim \epsilon V_s, \quad \text{as } \tilde{y} \to 0, \tag{11},$$

where the 'slip velocities', U_s and V_s, are determined by the wall and middle region solutions. Melnik (1989) showed, by a similar argument as that leading to equation (6), that a singularity can occur when $U_s = 0$. The analysis leading to (11) indicates no reason for U_s and u_*^2 to vanish simultaneously. The precise form of the variation of U_s depends sensitively on the model used for the viscous sublayer. Experimental data (Driver, 1991; Simpson *et al.* 1981) shows that, very close to the surface, the mean shear at the zero-skin-friction point is large and that the mean velocity, just above the viscous sublayer, is non-zero. Hence the experimental evidence strongly suggests that the slip velocity is non-zero as $u_*^2 \to 0$.

The conclusion is that, according to the asymptotic theory, a turbulent boundary layer might pass through a point of zero skin friction without the boundary layer approximation breaking down. This implies two possibilities: (i) the separation can remain confined to a small separation bubble without the boundary layer approximation breaking down; or (ii), if large scale separation occurs, the mean streamlines break away from the surface some small distance downstream of the zero skin friction point.

2.2.2 Numerical solution

The full nonlinear boundary layer equations have been integrated numerically using an algebraic eddy viscosity model developed from the asymptotic scaling in order to check the deductions from the asymptotic analysis.

The eddy viscosity is given by

$$\nu_T^2 = \nu_\infty^2 \left(1 - \exp\left\{-\ell^2/\nu_\infty^2 \left(\alpha y + u_*^2\right)\right\}\right). \tag{12}$$

When $y/\delta \sim 1$, this becomes the Clauser viscosity $\nu_\infty = C_c(\delta/L)U_\infty \delta_*$, and when $yu_p/\nu \sim 1$, $\nu_T \sim \ell\sqrt{u_*^2 + \alpha y}$, the mixing length formula. The mixing length, ℓ, is damped on the ν/u_p length scale near the surface:

$$\ell = \kappa y \left(1 - \exp(-yu_p/26\nu)\right). \tag{13}$$

In order to have confidence that the numerical solution would capture any singularity, it was important to keep the errors associated with the numerical solution procedure to a minimum. An error analysis, based on the assumption that a singularity did occur, was performed on numerical scheme; the streamwise step length was then adjusted to keep this error fixed. The smallest step length was of the order $\epsilon^2/10$.

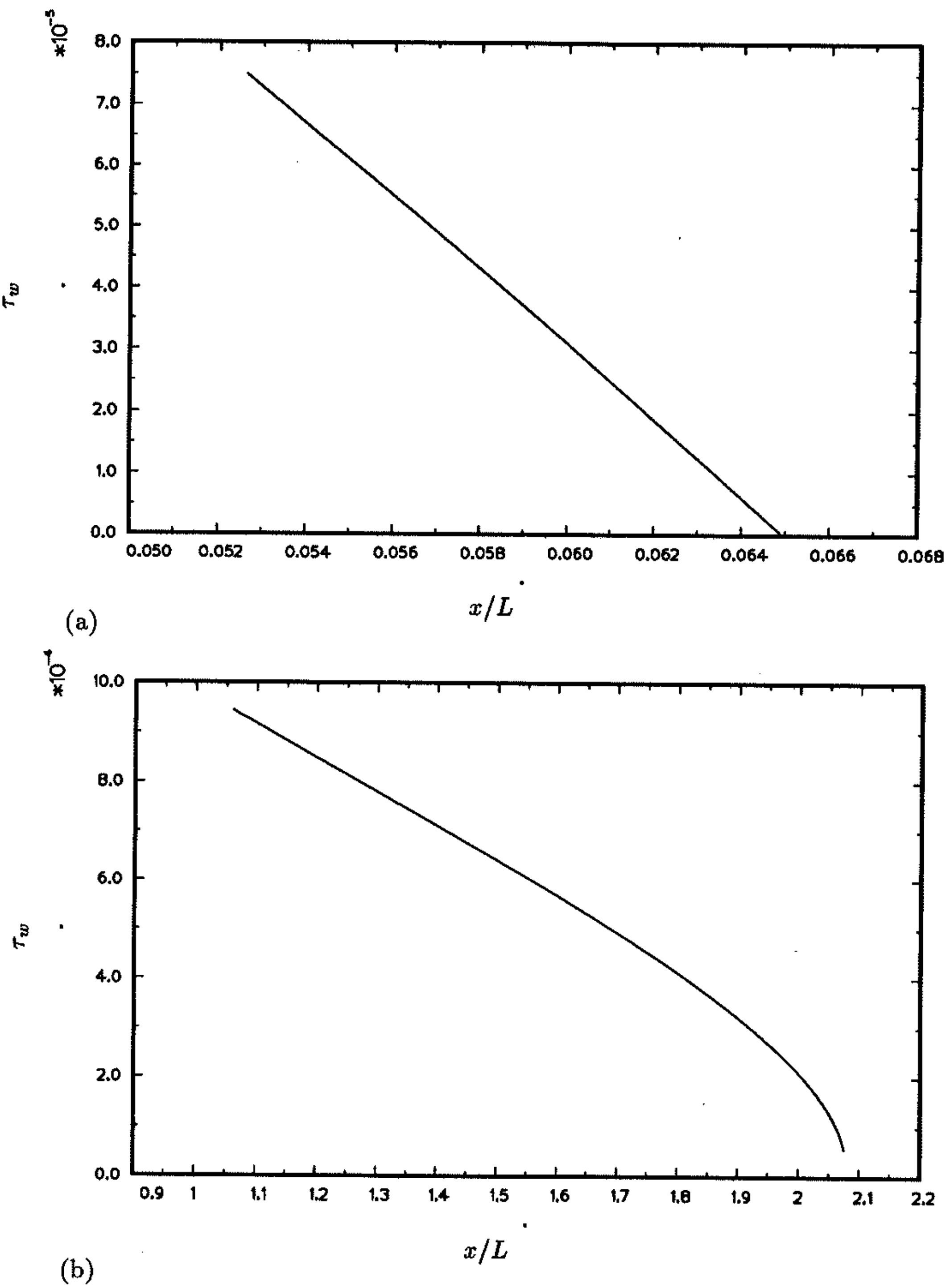

FIGURE 2. Variation of skin friction with downstream distance for a boundary layer in a constant adverse pressure gradient: (a) turbulent layer ($\alpha = 1$, $Re_L = 10^6$ so that $\epsilon = 10^{-2}$); (b) laminar layer ($\alpha = 0.01$, $Re_L = 10^3$).

The streamwise development of the skin friction in a boundary layer that is subjected to a constant adverse pressure gradient is shown in figure (2*a*). For comparison, a corresponding laminar case is also shown in figure (2*b*). (Note the different values of the pressure gradient between the turbulent and laminar cases—with the same pressure gradient, the model does predict that a laminar layer separates earlier than a turbulent layer!). In the turbulent case, $\tau_w \to 0$ linearly with no indication of a $\sqrt{x/L}$ behavior; by contrast the laminar flow shows clear evidence of $\tau_w \sim \sqrt{x/L}$. Other quantities were also monitored, but none showed a singular behavior at $\tau_w = 0$ for the turbulent boundary layer. The numerical results then concur with the findings from the asymptotic study, namely that, in the turbulent flow, the skin friction has no singularity when $\tau_w \to 0$.

2.3 Comparison with other asymptotic theories

Recently, there have been proposed two other theories for the asymptotic structure of a separating turbulent boundary layer. These analyses differ in fundamental ways from the scaling developed by Durbin & Belcher (1991); therefore, an attempt is briefly made here to reconcile these different viewpoints and to suggest the reasons for the different results of these analyses.

Melnik (1989, 1991) developed an analysis based on a simple two layer eddy viscosity model (see Cebeci & Smith, 1974), The outer part of the eddy viscosity is of the constant, Clauser form. Melnik found asymptotic solutions in the double limit of $C \to 0$ (where C is the multiplier in the Clauser eddy viscosity model) and the Reynolds number $Re \to \infty$. The solution then develops a three layer structure. The outer region, which extends over the outermost bulk of the boundary layer, is equivalent to the outer region described in §2.1, when the parameter C of Melnik's analysis is identified with δ/L in the present scaling. In his middle layer, Melnik suggests that solutions may be found as linear perturbations to the 'slip velocity', U_s (the velocity at the bottom of the outer region). If his middle region solutions are put into dimensional form, it is found that this approach is valid only when $U_s \gg (\delta/L)^{\frac{1}{2}}$. Hence Melnik's analysis is appropriate only to the initial development of a boundary layer in an adverse pressure gradient. The condition $U_s \sim (\delta/L)^{\frac{1}{2}}$ is just that necessary for the Durbin & Belcher scaling to become valid. There is also a significant difference between Melnik's treatment of the wall layer and the scaling of §2.1. Melnik assumes that the logarithmic law of the wall holds very close to the wall. Since the middle and wall regions are linear, the slip velocity is then linearly related to the friction velocity (by a slightly modified form of the ZPG logarithmic overlap). Melnik's analysis then implies that the slip velocity is zero at the same point that the wall shear stress is zero. This is entirely a consequence of using the ZPG drag law across the wall region. Hence Melnik's equation of the point of singularity to the zero-skin-friction point is not necessarily valid; it seems to be more of an assumption than a deduction. It has been shown in §2.2 that, if the middle region is treated as in the Durbin & Belcher scaling, then the slip velocity need not vanish at the point of zero skin friction. There is then no correspondence between the singularity and the zero-skin-friction point.

Neish & Smith (1991) have also analyzed the effect of a pressure gradient on a turbulent boundary layer using asymptotic methods. Like Melnik (1989, 1991), they also adopt the 'Cebeci-Smith' turbulence model. The main difference between this analysis and the theories of Melnik and Durbin & Belcher is in the treatment of the outer region. Neish & Smith treat the Clauser constant (C in the above notation) as an order one constant. The outer region is then inviscid at leading order. An analysis of the magnitude of the terms in the boundary layer equation using the data of Driver (1991) shows that this is not an appropriate approximation: the shear stress gradient is not negligible in the outer region. Furthermore, Neish & Smith assume a logarithmic overlap between the wall and outer regions, which leads to the logarithmic friction law holding asymptotically close to the separation point. This overlap law has no theoretical justification, and the experimental data shows that the skin friction in an APG boundary layer falls more rapidly than predicted by the logarithmic friction law (see data in Coles & Hirst, 1968). Hence we can have little confidence in the development suggested by Neish & Smith (1991).

3. Future plans

The asymptotic studies described in this report are being used to develop closure models for separated turbulent boundary layers.

The investigation of the nature of the singularity in the boundary layer equations near a point of zero skin friction suggests that it is important to model correctly the Reynolds stresses in the wall layer. A model that addresses this issue but without using 'damping functions' has been developed by Durbin (1991). This model uses an eddy viscosity hypothesis, together with a transport equation for $\overline{v^2}$ and the standard k and ε equations. This model is currently being generalized to a full Reynolds stress closure. In order to do gain theoretical insight into how this might be done, the rapid distortion calculations of Hunt & Graham (1978) are being extended to the case of initially axisymmetric turbulence, thereby showing how the kinematic blocking effect is affected by anisotropy in the initial turbulence. The Reynolds stress model will be used to compute separated turbulent flows.

The $k-\varepsilon-\overline{v^2}$ turbulence model of Durbin (1991) is also being used to investigate the flow in a channel that is subjected to spanwise rotation. Below a critical value of the rotation rate, the mean flow is two- dimensional, and I have shown that the main features are captured by the $k-\varepsilon-\overline{v^2}$ model. Experimental measurements (Johnston *et al.* 1972) show that when the rotation rate increases beyond a critical value the mean flow becomes three-dimensional, with vortical rolls appearing in the streamwise direction. The full Reynolds stress version of the model will be used to study the bifurcation to the three dimensional flow.

REFERENCES

Cebeci, T. & Smith, A. M. O. 1974 *Analysis of Turbulent Boundary layers.* Academic Press.

Coles, D. E. & Hirst, E. A. (Eds.) 1968 Computation of turbulent boundary layers. *Proc. AFOSR-IFP-Stanford conference.*

DRIVER, D. M. 1991 Reynolds shear stress measurements in a separated boundary layer flow. *AIAA paper 91-1787.*

DURBIN, P. A. 1990 Turbulence closure modeling near rigid boundaries. *CTR Ann. Res. Briefs.* 3-10; also, *Theoret. Comp. Fluid Dynamics*, **3**, 1-13.

DURBIN. P. A. & BELCHER, S. E. 1991 Scaling of adverse pressure gradient turbulent boundary layers. *CTR manuscript 123;* to appear *J. Fluid Mech.*

GOLDSTEIN, S. 1948 On laminar boundary layer flow near a position of separation. *Quart. J. Mech. Appl. Math.* **1**, 43-69.

HUNT, J .C. R. & GRAHAM, J. M. R. 1978 Free-stream turbulence near plane boundaries. *J. Fluid Mech.* **84**, 209-235.

JOHNSTON, J. P., HALLEEN, R. M. & LEZIUS, D. K. 1972 Effects of spanwise rotation on the structure of two-dimensional, fully developed turbulent channel flow. *J. Fluid Mech.* **56**, 533-557.

MELNIK, R. E. 1989 An asymptotic theory of turbulent separation. *Comp. and Fluids.* **17**, 165-184.

MELNIK, R. E. 1991 Some applications of asymptotic theory to turbulent flow. *AIAA paper 91-0220.*

NEISH, A. & SMITH, F. T. 1991 On turbulent separation in the flow past a bluff body. *Manuscript.*

SIMPSON, R. L., CHEW, Y.-T., & SHIVAPRASAD, B. G. 1981 The structure of a separating turbulent boundary layer. Part 1 mean flow and Reynolds stresses. *J. Fluid Mech.* **113**, 23-51.

TERRILL, R. M. 1960 On the laminar boundary layer flow near separation with and without suction. *Phil. Trans.* **253**, 55-100.