Derivation of the $\kappa - \varepsilon$ model equations using the renormalization group method

By L. M. Smith[1]

1. Motivation and objectives

We have made a critical study of the renormalization group (RNG) theory of turbulence proposed by Yakhot and Orszag (YO, 1986). The results of that study were reported in *CTR Annual Research Briefs - 1990* and in full detail elsewhere (Smith and Reynolds (SR), 1991). Our independent study led to confirmation of YO's basic theory of the Navier-Stokes equations, but errors were found in their derivations of the velocity-derivative skewness and the model transport equation for the mean dissipation rate of energy $\mathcal{E}$. The most consequential changes over what was reported by YO were in the $\mathcal{E}$ model equation. As will be explained, our efforts have led to a reformulation by Yakhot and Smith (YS, 1991) of the RNG method for derivation of model transport equations.

2. Review of the basic theory

The RNG model is isotropic turbulence in an unbounded domain, driven by a Gaussian random force $\mathbf{f}$,

$$\frac{\partial v_i}{\partial t} + v_j \nabla_j v_i = f_i - \nabla_i p + \nu_o \nabla^2 v_i \tag{1}$$

where the velocity $\mathbf{v}$ is divergence-free ($\nabla_j \equiv \partial/\partial x_j$), ν_o is the molecular viscosity, and the constant density ρ has been absorbed into the pressure p. The force $\mathbf{f}$ must satisfy incompressibility, homogeneity in space and time, and isotropy in space. It is designed to produce a scale-invariant field with energy spectrum given by power-law decay in wavenumber space. If $\mathbf{f}$ is further assumed to be white-noise in time, the most general form of its two-point correlation function in time is (Leslie, 1973)

$$< \hat{f}_i(\hat{\mathbf{k}})\hat{f}_j(\hat{\mathbf{k}}') >= 2D_o(2\pi)^{d+1}k^{-y}P_{ij}[\mathbf{k}]\delta[\hat{\mathbf{k}}+\hat{\mathbf{k}}'], \quad \Lambda_L \leq k \leq \Lambda_o \tag{2}$$

where $\hat{\mathbf{k}} = (\mathbf{k}, \omega)$ is the wavevector-frequency vector and the dimension $d = 3$. Herein square brackets [...] are used to denote the arguments of a function or variable. The delta function guarantees homogeneity and the projection operator $P_{ij}[\mathbf{k}] = \delta_{ij} - k_i k_j/k^2$ guarantees isotropy and incompressibility. The wavenumber Λ_o is an ultraviolet cutoff above which the viscosity is the molecular viscosity ν_o, and $\Lambda_L \to 0$ is the low-wavenumber end of the scaling regime. The exponent $y = 3$

1 Current address: Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544

leads to a Kolmogorov inertial-range energy spectrum $E(k) =\propto k^{-5/3}$ (YO). The amplitude D_o has units of $k^{y-d-2}\omega^3$. One sees that for $y = 3$, D_o has the dimensions of the mean dissipation rate of energy $\mathcal{E}$, and thus in this case, the parameters of (2) are k and $\mathcal{E}$. Dannevik, Yakhot and Orszag (1987) found the relation between D_o and $\mathcal{E}$ when $y = 3$ by requiring overall energy conservation of the renormalized Navier-Stokes equation after removal of all scales above k in the inertial range (see (4) and (6) with $\Lambda = k$),

$$\frac{D_o S_d}{(2\pi)^d} = 1.59\mathcal{E} \tag{3}$$

where S_d is the area of a unit sphere in d-dimensions.

The RNG procedure to eliminate small scales from the equations of motion (1) has been described in many papers in the literature, for example, Forster, Nelson, and Stephen (1977), De Dominicis and Martin (1979), Fournier and Frisch (1983) and YO (1986), and will not be repeated here for brevity. SR also provides a detailed discussion of the scale removal procedure and the approximations involved. The removal procedure is carried out in wavenumber space and uses a perturbation series for the velocity modes in a thin shell of high wavenumbers in powers of the local Reynolds number. Substitution of the perturbation solution for the velocity at high wavenumbers, into the equation for the velocity at low wavenumbers, leads to a modified equation for the velocity at low wavenumbers. Among the modifications is a correction to the viscosity. The corrections accumulate as more thin shells are removed.

After iterative removal of many thin wavenumber shells, the local Reynolds number, based on the modified viscosity, was shown (YO) to be proportional to $\epsilon^{1/2}$ where $\epsilon = 4 + y - d$. Thus for $\epsilon \to 0$, the solution for the high-wavenumber modes is given by the lowest-order term in the series expansion, and in this case, the only modifications to the equations for the low wavenumbers and low frequencies (in the limit $k \to \Lambda_L \to 0$ and $\omega \to 0$) are the modified viscosity and an induced force $\mathbf{F}$. Taking inverse transforms, the equations for the long times and large scales are

$$\frac{\partial v_i}{\partial t} + v_j \nabla_j v_i = f_i + F_i - \nabla_i p + \nu_T[\Lambda]\nabla^2 v_i \tag{4}$$

where $\nu_T[\Lambda]$ is the effective viscosity acting at large scales after removal of wavenumbers $\Lambda < k < \Lambda_o$. To lowest order in an expansion in powers of ϵ,

$$\nu_T[\Lambda] = \left(\frac{3}{10\epsilon}\right)^{1/3} \left(\frac{2D_o S_d}{(2\pi)^d}\right)^{1/3} \Lambda^{-\epsilon/3}. \tag{5}$$

The induced force $\mathbf{F}$ is Gaussian at lowest order in ϵ (Forster *et al.*, 1977) and given by its two-point correlation function

$$< \hat{F}_i[\hat{\mathbf{k}}]\hat{F}_j[\hat{\mathbf{k}}'] >= 2D_o D'(2\pi)^{d+1} k^2 P_{ij}[\mathbf{k}]\delta[\hat{\mathbf{k}} + \hat{\mathbf{k}}'] \tag{6}$$

where the amplitude D' is also found at lowest order in ϵ. The "backscatter" force $\mathbf{F}$, with correlation function proportional to k^2, is negligible compared to the bare force $\mathbf{f}$ in $\Lambda_L < k < \Lambda_o$. However, it is important in $0 < k < \Lambda_L$ and leads to an induced energy spectrum $E[k] \propto k^2$ in $0 < k < \Lambda_L$ (Forster *et al.*, 1977). The induced energy spectrum is important for the RNG derivation of the $\mathcal{E}$ transport equation (section 3.4).

The Yakhot-Orszag theory of turbulence is the evaluation of the results (4)-(6) at lowest order in an expansion in powers of ϵ with $\epsilon = 4$, which gives Kolmogorov scaling. Using relation (3), the effective viscosity becomes

$$\nu_T[\Lambda] = 0.49\mathcal{E}^{1/3}\Lambda^{-4/3}. \tag{7}$$

3. Accomplishments

We shall now describe the evolution of the RNG derivation of the model $\mathcal{E}$ equation. The exact transport equation and "standard" model transport equation are first reviewed to remind the reader of what might be expected from the RNG analysis.

3.1 The exact transport equation for $\mathcal{E}$

We consider the velocity field u_i of incompressible flow with constant viscosity ν_o. The total velocity u_i may be written $u_i = U_i + v_i$, where $U_i =< u_i >$ and v_i is the zero-averaged fluctuations from the mean.

The dissipation rate of the kinetic energy of the fluctuations in homogeneous turbulence is $\mathcal{E} \equiv \nu_o < (\nabla_j v_i)^2 >$. The Navier-Stokes equations may be used to derive an equation for the time rate of change of $\mathcal{E}$ in homogeneous flow,

$$\frac{\partial \mathcal{E}}{\partial t} = +\overbrace{\{-2\nu_o^2 < (\nabla_j\nabla_m v_i)^2 >\}}^{T_2} + \overbrace{\{-2\nu_o\nabla_j U_m < (\nabla_j v_i)(\nabla_m v_i) >\}}^{T_3}$$

$$+\overbrace{\{-2\nu_o\nabla_m U_i < (\nabla_j v_i)(\nabla_j v_m) >\}}^{T_4} + \overbrace{\{-2\nu_o < (\nabla_j v_i)(\nabla_j v_m)(\nabla_m v_i) >\}}^{T_1}. \tag{8}$$

Intuitive scaling analysis (Tennekes and Lumley, 1972) shows that the dominant balance is between T_1 and T_2 and that these terms scale as $O[R_T^{1/2}]$, where the turbulence Reynolds number $R_T = \mathcal{K}^2/(\nu_o\mathcal{E})$.

3.2 The "standard" model equation for $\mathcal{E}$

The time rate of change of $\mathcal{E}$ is usually modeled by the sum of $O[1]$ terms, which assumes the exact cancelation at leading order of T_1 and T_2. There has been no solid theoretical justification for this exact cancelation, and the only "proof" has been model performance. The most widely used model is

$$\frac{\partial \mathcal{E}}{\partial t} = C_{\mathcal{E}1}\frac{\mathcal{E}}{\mathcal{K}}P_{\mathcal{K}} - C_{\mathcal{E}2}\frac{\mathcal{E}^2}{\mathcal{K}} \tag{9}$$

where $P_{\mathcal{K}} \equiv -\nabla_j U_i < v_i v_j >$ is the production of $\mathcal{K}$. The Reynolds stress is usually modeled by

$$< v_i v_j >= \frac{2\mathcal{K}\delta_{ij}}{3} - 2\nu_e S_{ij} \tag{10a}$$

$$S_{ij} = \frac{1}{2}(\nabla_j U_i + \nabla_i U_j) \tag{10b}$$

where ν_e is an eddy viscosity and S_{ij} is the rate of strain tensor. The model coefficients $C_{\mathcal{E}1}$ and $C_{\mathcal{E}2}$ are dimensionless. These coefficients are independent of R_T at high R_T, and typical values are $C_{\mathcal{E}1} = 1.4$ and $C_{\mathcal{E}2} = 1.9$ (Patel, Rodi and Scheurer, 1985).

3.3 The $\mathcal{E}$ model equation found using the original YO method

The starting point for the RNG derivation of the model $\mathcal{E}$ equation is the transport equation for the instantaneous quantity $\phi = \nu_o(\nabla_j v_i)^2$. Taking the time derivative of ϕ followed by substitution of the (unforced) Navier Stokes equations leads to

$$\frac{\partial \phi}{\partial t} = -v_j \nabla_j \phi + \chi_o \nabla_j \nabla_j \phi - \overbrace{2\nu_o(\nabla_j \nabla_m v_i)^2}^{T_{2i}}$$

$$-2\nu_o[\nabla_j v_i](\nabla_j \nabla_i p) + \overbrace{\{-2\nu_o(\nabla_j v_i)(\nabla_j v_m)(\nabla_m v_i)\}}^{T_{1i}} \tag{11}$$

where $\chi_o = \nu_o$. The terms labeled T_{1i} and T_{2i} are the instantaneous forms of T_1 and T_2 in equation (8).

Following the RNG scale elimination procedure, the small scales are systematically removed from the equation for $\mathcal{E} = \lim_{\hat{\mathbf{k}} \to 0} \hat{\phi}(\hat{\mathbf{k}})$ (see SR), where $\hat{\phi}(\hat{\mathbf{k}})$ is the four-dimensional Fourier transform of ϕ, and the limit $\hat{\mathbf{k}} \to 0$ means $k \to \Lambda_L \to 0$ and $\omega \to 0$.

In the original YO derivation of the $\mathcal{E}$ equation, the $O[R_T^{1/2}]$ contributions from T_1 and T_2 were *assumed* to cancel. The $O[1]$ contributions from T_1 and T_2 were calculated in $0 < k < \Lambda_L$ after removal of $\Lambda_L < k < \Lambda_o$.

We showed (SR) that the final form of YO's model $\mathcal{E}$-equation is in error and we provided the "correct" model *found following YO's original procedure.* In the limit of high Reynolds numbers, a consistent application of that procedure leads to the model

$$\frac{\partial \mathcal{E}}{\partial t} = -U_j \nabla_j \mathcal{E} + \nabla_j \chi_T \nabla_j \mathcal{E} - 5.65\frac{\mathcal{E}^2}{\mathcal{K}} + \Pi^* \tag{12}$$

where $\chi_T = \alpha\nu_T$, $\alpha = 0.77$ and $\nu_T = 0.085\mathcal{K}^2/\mathcal{E}$, and these values are exactly as given by YO. The decay rate for isotropic flow 5.65 is in poor agreement with observations and $\mathcal{K} - \mathcal{E}$ models in current use. YO had previously reported the decay rate 1.7, which agrees well with observations and current models.

The term Π^* (initially thought responsible for the production of $\mathcal{E}$ in anisotropic flow) is $O[\nabla^3 \mathbf{v}^3]$ and cannot have the form $\nabla \mathbf{v}^3$ as reported by YO. Thus the model (12) does not have the form of the production term $P_{\mathcal{K}}\mathcal{E}/\mathcal{K}$ used in current models. YO have now confirmed our results.

3.4 Reformulation of the procedure to derive model transport equations

Applying the RNG scale removal procedure to (11), we found cancelations between terms describing the effect of the small scales on the large scales. For example, terms from T_1 of the form $(\nabla \mathbf{v})^3$ exactly cancel. Reviewing the SR work, V. Yakhot noticed that these cancelations were among $O[R_T^{1/2}]$ terms but that some $O[R_T^{1/2}]$ contributions remained when the RNG scale removal procedure was applied to (11). Subsequently, YS considered the transport equation for ϕ derived by taking the time derivative and substituting the *forced* Navier-Stokes equations.

Since the force $\mathbf{f}$ is assumed to be the result in $\Lambda_L < k < \Lambda_o$ of all turbulence production mechanisms, using the forced Navier Stokes equations is the only procedure consistent with the basic theory reviewed in Section 2. They also introduced a mean flow, and thus they started with the equation

$$\frac{\partial \phi}{\partial t} = \overbrace{-2\nu_o(\nabla_j v_i)(\nabla_j f_i)}^{P_f} - v_j \nabla_j \phi + \chi_o \nabla_j \nabla_j \phi - \overbrace{2\nu_o(\nabla_j \nabla_m v_i)^2}^{T_2^{(i)}}$$
$$-2\nu_o(\nabla_j v_i)(\nabla_j \nabla_i p) + \overbrace{\{-2\nu_o(\nabla_j v_i)(\nabla_j v_m)(\nabla_m v_i)\}}^{T_{1i}}$$
$$+\overbrace{\{-2\nu_o \nabla_j U_m(\nabla_j v_i)(\nabla_m v_i)\}}^{T_{3i}} + \overbrace{\{-2\nu_o \nabla_m U_i(\nabla_j v_i)(\nabla_j v_m)\}}^{T_{4i}} \qquad (13)$$

where $\mathbf{f}$ is given by (2). Notice the inclusion of the instantaneous terms T_{3i} and T_{4i} as well as the random-force contribution to $\mathcal{E}$-production P_f.

The RNG scale removal procedure may be applied to remove wavenumbers $\Lambda_L < k < \Lambda_o$ from (13), thereby deriving the equation for $\mathcal{E} = \lim_{\hat{\mathbf{k}} \to 0} \phi(\hat{\mathbf{k}})$. One finds that all $O[R_T^{1/2}]$ terms cancel at low orders in the ϵ-expansion. Thus RNG provides theoretical support for expressing the equation for $\mathcal{E}$ in terms of $O[1]$ inertial-range parameters. At low order in the ϵ-expansion the equation for $\mathcal{E}$ has a similar form to the equation for ϕ but with modified transport coefficients,

$$\frac{\partial \mathcal{E}}{\partial t} = -U_j \nabla_j \mathcal{E} + \chi_T[\Lambda_L] \nabla_j \nabla_j \mathcal{E} - \overbrace{2\nu_T[\Lambda_L] < (\nabla_j \nabla_m v_i^<)^2 >}^{T_2}$$
$$+\overbrace{\{-2\nu_T[\Lambda_L] \nabla_j U_m < (\nabla_j v_i^<)(\nabla_m v_i^<) >\}}^{T_3}$$
$$+\overbrace{\{-2\nu_T[\Lambda_L] \nabla_m U_i < (\nabla_j v_i^<)(\nabla_j v_m^<) >\}}^{T_4} \qquad (14)$$

where $\nu_T[\Lambda_L]$ is given by (7) and $\chi_T[\Lambda_L] = \alpha \nu_T[\Lambda_L]$ with $\alpha = 0.77$ as before. The fluctuations $\mathbf{v}^<$ are the fluctuations in $0 < x < \pi/\Lambda_L$ and for $t \to \infty$.

The terms T_2, T_3 and T_4 in (14) must be closed in terms of $\mathcal{E}$, $\mathcal{K}$, and $\mathbf{U}$, and to do this, YS first rewrite (14) in terms of the energy spectrum tensor $E_{ij}[\mathbf{k}]$ of the wavenumbers below Λ_L,

$$\frac{\partial \mathcal{E}}{\partial t} = -U_j \nabla_j \mathcal{E} + \chi_T[\Lambda_L]\nabla_j \nabla_j \mathcal{E}$$

$$\overbrace{-\,4\nu_T[\Lambda_L]\int_0^{\Lambda_L} k^4 E(k)dk}^{T_2}$$

$$\overbrace{+\,\{-2\nu_T[\Lambda_L]\nabla_j U_m \int_0^{\Lambda_L} k_i k_j E[k]dk\}}^{T_3}$$

$$\overbrace{+\,\{-2\nu_T[\Lambda_L]\nabla_m U_i \int_0^{\Lambda_L} k^2 E_{ij}[\mathbf{k}]d\mathbf{k}.\}}^{T_4}\,. \tag{15}$$

YS use the form of the RNG induced spectrum $E_{ij}[\mathbf{k}] \propto k^2$. YS assume that the value of $\mathcal{E}$ is determined by the large scales and that the expression for $\mathcal{E}$ is invariant in the inertial range up to $k = \Lambda_L$,

$$\mathcal{E} \equiv 2\nu_o \int_0^{\infty} k^2 E[k]dk$$

$$= 2\nu_T[\Lambda] \int_0^{\Lambda} k^2 E[k]dk = 2\nu_T[\Lambda_L] \int_0^{\Lambda_L} k^2 E[k]dk. \tag{16}$$

They also assume that the production of kinetic energy $P = -\nabla_j U_i < v_i v_j >$ is determined by the large scales,

$$P = -\nabla_j U_i \int_0^{\Lambda_L} E_{ij}[\mathbf{k}]d\mathbf{k}. \tag{17}$$

The last equality in (16) allows YS to close T_2 in terms of $\mathcal{E}$, and (17) allows YS to close T_4 in terms of P. Using $E_{ij}[\mathbf{k}] \propto k^2$ as suggested by the RNG analysis of the Navier Stokes equations, they find

$$\frac{\partial \mathcal{E}}{\partial t} = -U_j \nabla_j \mathcal{E} + \chi_T[\Lambda_L]\nabla_j \nabla_j \mathcal{E}$$

$$-1.68\frac{\mathcal{E}^2}{\mathcal{K}} + 1.44P\frac{\mathcal{E}}{\mathcal{K}} + \overbrace{\{-2\nu_T[\Lambda_L]\nabla_j U_m \int_0^{\Lambda_L} k_i k_j E[k]dk\}}^{T_3} \tag{18}$$

where T_3 remains unclosed. We note that only the *form* of E_{ij} is used to derive (18), and not its *amplitude*. Furthermore, isotropy of the large scales, imposed by P_{ij} in (6), has been relaxed. Note also that $\mathcal{K}$ which appears in (18) is the kinetic energy *under the inertial range* and not the full kinetic energy.

It is not apparent how to close the term arising from T_3, which we shall denote R, within the framework of RNG. However, analysis shows that R is small in weakly strained turbulence and large in the rapid distortion limit $\eta \to \infty$, where $\eta \equiv S\mathcal{K}/\mathcal{E}$ and $S = (S_{ij}S_{ij})^{1/2}$ is the *rms* mean rate of strain. The nondimensional parameter η is the ratio of the turbulence time scale $\mathcal{K}/\mathcal{E}$ and the time scale of the mean $1/S$.

Yakhot, Orszag, Thangam, Gatski, and Speziale (1991) have proposed a closure for R, which is an approximation to its infinite series in powers of η. The approximation is a partial sum to all orders in η rather than a finite truncation and is constructed to satisfy certain consistency conditions. For example, the approximate expression for R approaches zero faster than $T_4 = 1.44P\mathcal{K}/\mathcal{E}$ in the limit of weak strain $\eta \to 0$.

4. Future plans

The new aspects of the YS procedure are as follows:

1. the transport equation for any instantaneous turbulence quantity (the mean of which is the desired result of RNG) must be derived using the *forced* Navier Stokes equations;
2. the dynamical terms coupling the mean velocity to the fluctuations must be retained.

The first rule insures that production *at the small scales* is properly accounted for, and the second rule insures that production *at the large scales* is properly accounted for.

The YS method can be used to derive turbulence models at any level of complexity, for example, a full Reynolds stress model. The RNG model for the fast pressure strain term in the transport equation for the Reynolds stress is currently being explored.

All the terms in the Launder, Reece, and Rodi (1975) model for the fast pressure strain are predicted, as well as some new terms, analogous to the case of the $\mathcal{E}$-transport equation. It is expected that some of these new terms should be closed by approximation to all orders of their power series in η, following the method of Yakhot *et al.*, (1991).

RNG may also be used to derive models which include important, more complicated physics such as rotation and compressibility. Flows which include such effects are described by several dimensionless parameters and are not easy to model using heuristic methods. RNG provides a systematic procedure to derive models for complex flow systems. The testing of such models may help to improve the theory itself.

Acknowledgements

The detailed critique of YO was carried out in collaboration with Prof. W. C. Reynolds. V. Yakhot provided numerous helpful explanations, saw the critical importance of the corrections to YO found by SR, and is currently collaborating with the author who is now working at Princeton.

REFERENCES

DANNEVIK, W. P., YAKHOT, V. & ORSZAG, S. A. 1987 Analytical theories of turbulence and the ϵ-expansion. *Phys. Fluids.* **30**, 2021.

DE DOMINICIS, C. & MARTIN, P. C. 1979 Energy spectra of certain randomly-stirred fluids. *Phys. Rev. A.* **19**, 419.

FORSTER, D., NELSON, D. R.& STEPHEN, M. J. 1977 Large-distance and long-time properties of a randomly stirred fluid. *Phys. Rev. A.* **16**, 732.

FOURNIER, D., & FRISCH, U. 1983 Remarks on the renormalization group in statistical fluid dynamics. *Phys. Rev. A.* **28**, 1000.

LAUNDER, B. E., REECE G. J.& RODI, W. 1975 Progress in the development of a Reynolds-stress turbulence closure. *J. Fluid Mech.* **68**, 537.

LESLIE, D. C. 1973 *Developments in the theory of turbulence.* Clarendon Press, Oxford.

PATEL, V. C., RODI, W. & SCHEURER, G. 1985 Turbulence models for near-wall and low Reynolds number flows: a review. *AIAA J.* **23**, 1308.

SMITH, L. M. & REYNOLDS, W. C. 1991 On the Yakhot-Orszag renormalization group method for deriving turbulence statistics and models. To appear in *Phys. Fluids A*, February, 1992.

TENNEKES, H. & LUMLEY, J. L. 1972 *A First Course in Turbulence.* MIT Press, Cambridge, MA.

YAKHOT, V. & ORSZAG, S. A. 1986 Renormalization group analysis of turbulence. I. basic theory. *J. Sci. Comput.* **1**, 3.

YAKHOT, V. & SMITH, L. M. 1991 The renormalization group, the ϵ-expansion and derivation of turbulence models. *to appear in Phys. Fluids A.*

YAKHOT, V., THANGAM, S., GATSKI, T. B., ORSZAG, S. A. & SPEZIALE, C. G. 1991 Development of turbulence models for shear flows by a double expansion technique. *Submitted*