Some feedback procedures for control of flows

By Roger Temam[1]

In this article, we present some avenues concerning the application of the mathematical methods of control theory to the difficult and challenging problem of the control of turbulent flows. The effective implementation of these methods for the Burger's and Navier-Stokes equations is in progress at this time, with very satisfactory results already obtained in the case of the stochastic Burgers equation. Effective implementation and discussion of the physical relevance of the results will be presented elsewhere (see Choi, Temam, Moin and Kim, 1991).

1. Introduction

The control of turbulent flows has been identified as an important problem with many potential benefits in science and engineering: aeronautics first of all, but also combustion, laser, fusion, chemistry, etc. At a time where the available computing power is increasing and expected to continue to increase rapidly, the problem of controlling turbulent flows does not seem out of reach any more.

In aeronautics, the main objective is to reduce skin-friction and drag by limiting the counter productive effects of turbulent boundary layers. This can be achieved in fluid mechanics by using passive means (*passive control*) such as riblets or large eddy break-up (LEBU) devices. On the other hand, *active control* of turbulence is achieved by active (mechanical) devices which tend to change the kinematics of the flow. When the physics of the problem is well-known, in particular the appearance of organized patterns, one can think at destroying these patterns or at least impeding their formation by preassigned kinematical modifications. Such a procedure based on the modification of wall velocities has been proposed and studied in Moin, Kim and Choi (1989). When the physics of the phenomenon is not known or is too complicated, we are tempted to appeal to the more systematic but less intuitive methods of *control theory*.

We show in Sec. 2 how to cast the problem of controlling turbulence for a channel flow into a problem in optimal control theory and on this occasion we introduce the formalism and language of control theory. Similarly several physical problems of fluid mechanics and thermodynamics have been formulated in Abergel and Temam (1990) into problems of control and studied with the methods of optimal control theory.

As we recall hereafter, the methods of control theory presented in Abergel and Temam (1990) are not practical, in some sense because these methods are *too good*, i.e., we try to find the best control producing, for example,the *maximal reduction of drag*. For engineering applications, we would be, however, satisfied with less

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perfection; for example, some reduction of drag may have significant practical effects. This is the object of the methods proposed here. They pertain to suboptimal control and feedback theory. As we will see below, they consist of restricting the class of controls to some specific (simple) subclasses and replacing the global in time procedures as in Abergel and Temam (1990) by time evolutive procedures.

2. Introduction to control theory: Some model problems in control of flows

Although we keep in mind that turbulent flows are time dependent, we will distinguish between stationary and time dependent flows and start with the somehow academic but instructive case of stationary flows.

2.1 Stationary channel flow

Consider the stationary channel flow. The streamwise direction is the x direction, the spanwise direction is the z direction, and the walls are at $y = \pm 1$. The mass flux is prescribed equal to M. Periodicity of velocities and pressure is assumed in the z direction; periodicity of velocities with (unknown) drop of pressure is assumed in the x direction. Let $u = (u_1, u_2, u_3)$ denote the velocity vector in the fluid and assume that we control the flow through the wall boundary of the normal velocity

$$\phi = u_2|_w. \tag{2.1}$$

It can be shown that the stationary Navier-Stokes equations reduce to a functional equation for u (see e.g. Temam (1984, 1991)) involving ϕ:

$$\nu A u + R(u, \phi) = 0. \tag{2.2}$$

Here $\nu > 0$ is the kinematic viscosity, A is the so-called Stokes operator, and R corresponds to inertial and boundary terms; in particular R depends on M although the dependence is not made explicit.

A typical optimal control problem for (2.2) is the following (see J. L. Lions (1969)): to find the best ϕ such that some observation $z = Cu$ of u achieves some desired value z_d or is at least as close as possible from z_d. In the language of control theory,

u describes (is) the *state* of the system, and (2.2) is the *state equation.*

ϕ is the *control.*

z is the *observation.*

The *cost function* could be, for instance, the function $J = J(\phi)$ [1]

$$J(\phi) = \frac{m}{2}||\phi||^2 + \frac{1}{2}||Cu - z_d||^2. \tag{2.3}$$

Here some norm of $Cu - z_d$, $||Cu - z_d||$, accounts for the *cost* of z being different from z_d; $m/2$ $||\phi||^2$ ($m > 0$, $||\phi||$ = some appropriate norm of ϕ) accounts for the

[1] u is a function of ϕ through (2.1), $u = u(\phi)$. Hence J is a function of ϕ. Note that u is the traditional notation for the control in control theory, and it is also commonly used for the velocity in fluid mechanics!

cost of the control itself: $m = 0$ or small for cheap controls, m large for expensive controls. For example, this term will account for the price of achieving high values of the velocity ϕ or, for time dependent problems, the price of realizing fast responses demanding fast electronic chips. We keep in mind, howeve,r that (2.2) is an academic problem which, due to the absence of turbulence, would only make sense physically for very viscous fluids.

The mathematical formulation of the problem is the following:

To find ϕ which minimizes J subject to (2.2):

$$Inf_{\phi} J(\phi). \tag{2.4}$$

The control ϕ can be unrestricted or restricted to some admissible set of controls $\mathcal{U}_{ad}$ taking into account some physical and technological restrictions.

The methods of calculus of variations tell us that a problem such as (2.4) possesses at least one solution, and they give us some characterizations of the best ϕ through the adjoint state and some algorithms to reach the best (optimal) control.

Feedback theory consists in looking for ϕ as a function of u or of some observation of u. Although feedback problems are mainly relevant to time dependent problems, we can formulate such a problem here.

For instance, if we look for a feedback control, then, E being a scalar and F a vector, we would look for

$$\phi = E + Fu. \tag{2.5}$$

Now problem (2.4) with (2.5) substituted into (2.2) becomes:

To find E, F which minimize $J(\phi) = \tilde{J}(E, F)$ subject to (2.2), (2.5):

$$Inf_{E,F} \tilde{J}(E, F). \tag{2.6}$$

More general shape functions could be considered $\theta_1(u), ..., \theta_r(u)$ with

$$\phi = \sum_{i=1}^{r} E_i \theta_i(u). \tag{2.7}$$

2.2 Time dependent channel flow

The *state equation* is the Navier-Stokes equation including the boundary condition (2.1) and the other boundary conditions. It is classical that all these conditions/equations amount to an evolution equation in infinite dimension for the velocity field $u = u(x, t)$. It reads (compare to (2.2))

$$\frac{du}{dt}(t) + \nu A u(t) + R(u(t), \phi(t)) = 0. \tag{2.8}$$

Here $u(t)$ is the vector field $\{x \to u(x,t)\}$; again R accounts for the inertial and boundary terms and depends on the constant mass flux M, although the dependence on M is not made explicit.

We are interested in properties concerning the statistical solution of (2.8). This solution is described by a measure which is not a solution of a simple ***state equation*** (in fact, it is a stationary solution of the Hopf equation !). Instead of considering the equation for the measure, we will consider the time averaged solution of (2.8) on a long interval of time $(0,T)$ with the hope that this average accurately represents the statistical solution.

The drag is essentially measured in average by $D = D(u)$:

$$D = \frac{1}{T}\int_0^T \int\int \left[\frac{\partial u_1}{\partial x_2}\Big|_{x_2=-1} - \frac{\partial u_1}{\partial x_2}\Big|_{x_2=1}\right] dx_1 dx_3 dt. \tag{2.9}$$

Here $x_1 = x, x_2 = y, x_3 = z$, and $x_2 = y = \pm 1$ is the wall.

The choice of the cost function is at our disposal, depending on the costs that we want to reduce. If we choose to reduce the drag as expressed by (2.9), then the cost function could be

$$J(\phi) = \frac{m}{2}\int_0^T \int\int_w |\phi|^2 dx_1 dx_3 dt + \frac{1}{2}|D|^2, \tag{2.10}$$

where D is a function of ϕ through u which is itself function of ϕ.

A control problem like (2.4) can be set:

To find $\phi = \phi(x_1, x_3, t)$ which minimizes J subject to (2.8) and (2.9)

$$Inf_\phi J(\phi). \tag{2.11}$$

The method of control theory and calculus of variation (J. L. Lions (1969)) as developed in Abergel and Temam (1990) yield the existence of an optimal control (the best ϕ) and produce algorithms for its determination. However, these classical methods and algorithms necessitate the resolution iteratively (i.e., several times) of the Navier-Stokes equation in (2.8) and its adjoint (see below) on the whole, large interval $(0,T)$; such computations are out of reach at this time. Furthermore, the optimal control depends on the initial distribution of velocities $u|_{t=0}$, although one can hope that the effect of initial velocities dissipates as T becomes large.

If equation (2.8) were linear, the optimal control would be given by a linear feedback law:

$$\phi = Pu + E, \tag{2.12}$$

where P is solution of a Riccati type equation (J. L. Lions (1969)), and E is easily determined. When equation (2.8) is nonlinear, there is no satisfactory feedback control theory even for finite and small dimensions (as in flight control), not mentioning high or infinite dimensional problems.

We describe hereafter some empirical and not yet fully mathematically justified procedures proposed to address this problem.

3. Suboptimal control and feedback procedures

We start by considering the stationary case.

3.1 Stationary problem

Equation (2.2) is considered as an abstract equation, totally independent of the original Navier-Stokes equation. Then (2.2), (2.3), (2.4) is an optimal control problem which can be satisfactorily resolved by a gradient algorithm (conjugate gradient would be better, but we restrict ourselves to a gradient algorithm).

The gradient algorithm consists in computing the Fréchet derivative

$$\frac{\mathcal{D}J}{\mathcal{D}\phi}(\phi) \tag{3.1}$$

and looking for a sequence of controls ϕ^n recursively defined by

$$\phi^{n+1} - \phi^n = -\rho \frac{DJ}{D\phi}(\phi^n). \tag{3.2}$$

By Taylor's formula and (3.2),

$$\begin{aligned} J(\phi^{n+1}) &\approx J(\phi^n) + \frac{\mathcal{D}J}{\mathcal{D}\phi}(\phi^n)\ (\phi^{n+1} - \phi^n), \\ J(\phi^{n+1}) &\approx J(\phi^n) - \rho\ \left|\frac{\mathcal{D}J}{\mathcal{D}\phi}(\phi^n)\right|^2 \end{aligned} \tag{3.3}$$

so that the sequence $J(\phi^n)$ is clearly decreasing. As in Abergel and Temam (1990), we infer from optimization theory that the sequence ϕ^n will converge to an optimal control for suitable ρ's and if the initial value ϕ^o is chosen sufficiently close from this optimal state.

Furthermore, the introduction of the adjoint state and adjoint state equation produces a convenient way to compute the Fréchet differential (3.1).

Indeed, define first

$$\eta = \frac{\mathcal{D}u}{\mathcal{D}\phi} \cdot \hat{\phi}, \tag{3.4}$$

where the right hand side of (3.4) is the Fréchet differential of u with respect to ϕ applied to a test function $\hat{\phi}$ (of the same type as ϕ). Then by linearization of (2.2), we promptly see that η is solution of equation

$$\nu A\eta + \frac{\mathcal{D}R}{\mathcal{D}\phi}(u,\phi) \cdot \hat{\phi} + \frac{\mathcal{D}R}{\mathcal{D}u}(u,\phi) \cdot \eta = 0. \tag{3.5}$$

We do not discuss here the fact that the solution of (2.2) may not be unique or the fact that (3.5) may have no (or many) solution: this difficulty will not be encountered in the case of interest for us (see below).

Now, by Fréchet differentiation of (2.3), using (3.4) and (3.5), we obtain

$$\frac{\mathcal{D}J}{\mathcal{D}\phi}(\phi) \cdot \hat{\phi} = m\ ((\ \phi, \hat{\phi}\ )) + ((\ Cu(\phi) - z_d, C\eta\ )).$$

Define then the adjoint state ζ through the following equation called the adjoint state equation

$$\nu A^*\zeta + \big(\frac{\mathcal{D}R}{\mathcal{D}u}(u,\phi)\big)^*\zeta = C^*\big(Cu(\phi) - z_d\big). \tag{3.6}$$

In (3.6) and hereafter, stars $*$ indicate adjoint operators (with respect to the scalar product under consideration $((\cdot,\cdot))$). Then

$$\begin{aligned}
((\ Cu(\phi) - z_d, C\eta\ )) &= ((\ C^*(Cu(\phi) - z_d), \eta\ )) \\
&= ((\ \nu A^*\zeta + \big(\frac{\mathcal{D}R}{\mathcal{D}u}(u,\phi)\big)^*\zeta, \eta\ )) \\
&= ((\ \zeta, \nu A\eta + \big(\frac{\mathcal{D}R}{\mathcal{D}u}(u,\phi)\big)\eta\ )) \\
&= (\ by\ (3.5)\ ) \\
&= -((\ \zeta, \frac{\mathcal{D}R}{\mathcal{D}\phi}(u,\phi)\hat{\phi}\ )) \\
&= -((\ \big(\frac{\mathcal{D}R}{\mathcal{D}\phi}(u,\phi)\big)^*\zeta, \hat{\phi}\ )).
\end{aligned}$$

From this calculation, we conclude that

$$\frac{\mathcal{D}J}{\mathcal{D}\phi}(u,\phi)\ \hat{\phi} = -\big(\frac{\mathcal{D}R}{\mathcal{D}\phi}(u,\phi)\big)^*\zeta \tag{3.7}$$

and we are in position to implement the gradient algorithm (3.2):

Once ϕ^n is known, we compute u^n by solving the state equation (2.2) with $\phi = \phi^n$. Then we compute the adjoint state ζ^n by solving equation (3.6) with $\phi = \phi^n, u = u^n$. We obtain ϕ^{n+1} from (3.2), and we can continue.

<u>Suboptimal feedback laws</u>

Suboptimal feedback laws can be implemented in the same way. For example, for a linear feedback law as (2.5),

$$\phi = E + Fu,$$

J becomes a function $\tilde{J}$ of E, F through (2.5) and (2.2). The analog of the gradient algorithm (3.2) consists in constructing two sequences E^n, F^n, recursively defined by

$$\begin{aligned}
E^{n+1} - E^n &= -\rho_1\ \frac{\mathcal{D}J}{\mathcal{D}E}(E^n, F^n), \\
F^{n+1} - F^n &= -\rho_2\ \frac{\mathcal{D}J}{\mathcal{D}F}(E^n, F^n).
\end{aligned} \tag{3.8}$$

Note that the relaxation parameters $\rho > 0$ are chosen differently in the two equations (3.8). One can define an adjoint state ζ through an equation similar to (3.6) and compute in a convenient way the Fréchet derivatives $\mathcal{D}J/\mathcal{D}E$ and $\mathcal{D}J/\mathcal{D}F$. Details will be given elsewhere (see Choi, Temam, Moin and Kim (1991), hereafter referred to as [CTMK]).

3.2 Time dependent problems

The suboptimal procedure that we propose consists of the following:

(1) Time discretization of the state equation
(2) At each instant of time, the discretized equation is a stationary one to which we apply the procedure above.

Of course, there is no reason which guarantees that the controls will be optimal, but at least (3.3) shows that the cost function tends to decrease. Numerical experiments conducted in the case of the stochastic Burgers' equation shows that indeed the cost function decreases significantly (see Sec. 4). A mathematical analysis of the procedure will be conducted elsewhere.

Consider the evolution state equation (2.8): again this could be the original Navier-Stokes equation for the channel flow or an abstract equation originating from a totally different problem.

For step (1) we consider here a simple time discretization scheme, the implicit Euler one. More accurate and more involved schemes will be considered in [CTMK]. Hence

$$\frac{u^n - u^{n-1}}{\Delta t} + \nu A u^n + R(u^n, \phi^n) = 0, \tag{3.9}$$

which we rewrite as

$$\mathcal{A}u + \mathcal{R}(u, \phi) = 0, \tag{3.10}$$

with $u = u^n, \phi = \phi^n$,

$$\begin{aligned} \mathcal{A}u^n &= u^n + \nu \Delta t A u^n, \\ \mathcal{R}(u^n, \phi^n) &= -u^{n-1} + \Delta t R(u^n, \phi^n). \end{aligned} \tag{3.11}$$

At each step n, the cost function J is still given by (2.3)

$$J(\phi^n) = \frac{m}{2}||\phi^n||^2 + \frac{1}{2}||Cu^n - z_d||^2, \tag{3.12}$$

with u^n function of $\phi^n, u^n = u^n(\phi^n)$ through (3.9) - (3.11); hence J actually depends on n, $J = J^n$ because equation (3.9) depends on n due to the term u^{n-1}. Note that for Δt sufficiently small, there exists a unique solution u^n to (3.9). Therefore, the difficulty of non-uniqueness of solution for (2.2) does not arise anymore for (3.10).

The adjoint state is defined as in (3.5), (3.6)

$$\mathcal{A}\,\eta^n + \frac{\mathcal{D}\mathcal{R}}{\mathcal{D}\phi}(u^n, \phi^n)\hat{\phi} + \frac{\mathcal{D}\mathcal{R}}{\mathcal{D}u^n}(u^n, \phi^n)\,\eta^n = 0, \tag{3.13}$$

$$\mathcal{A}^*\zeta^n + \Big(\frac{\mathcal{D}\mathcal{R}}{\mathcal{D}u}(u^n, \phi^n)\Big)^*\zeta = C^*(Cu^n - z_d). \tag{3.14}$$

The gradient algorithm (3.2) now reads

$$\phi^{n,k+1} - \phi^{n,k} = -\rho\, \frac{\mathcal{D}J^n}{\mathcal{D}\phi}(\phi^{n,k}). \tag{3.15}$$

By Taylor's formula, as in (3.3)

$$J^n(\phi^{n,k+1}) \leq J^n(\phi^{n,k}),\ \textit{for all } n, k,$$

and as $k \to \infty$, $\phi^{n,k}$ converges to ϕ^n which achieves the minimum of J^n. It is not necessarily true that the minimum of J^n decreases as n increases, i.e.,

$$J^n(\phi^n) \leq J^{n-1}(\phi^{n-1}),\ \textit{for all } n. \tag{3.16}$$

In our computations, we observed that (3.16) is not always true. However, an overall sharp decrease of this infimum occurs (see Sec. 4). The explicit calculation of $\mathcal{D}J/\mathcal{D}\phi$ $(\phi^{n,k})$ using sequences $u^{n,k}, \phi^{n,k}, \zeta^{n,k}, k = 0, 1, ...$ (and n fixed) is straightforward; see [CTMK].

4. Application to the stochastic Burger's equation

The following is a short excerpt from [CTMK].

We consider the randomly forced Burgers' equation with non-zero velocity boundary conditions.

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} = \chi, \qquad 0 < x < L, \tag{4.1}$$

$$u(0,t) = \phi_0,\ u(L,t) = \phi_L. \tag{4.2}$$

If $\chi = 0$, (4.1) is the classical Burgers' equation for the velocity u in the x direction which represents a balance of time dependence, nonlinear convection, and diffusion. The parameter ν represents the viscosity. In the absence of forcing ($\chi = 0, \phi_0 = \phi_L = 0$), the solutions of (4.1) decay to zero from any bounded initial data (and even from any initial data with finite spatial mean-square value).

The forcing function χ is a white noise random process in x with zero mean (see Chambers et al. 1988, Bensoussan and Temam 1972, 1973). The mean-square value of the dimensional forcing, σ^2, defines a velocity scale $U = \sqrt{\sigma L}$ where L is the length of the computational domain. We denote by Re the Reynolds number UL/ν. Burgers' equation in nondimensional form using U and L as the typical velocity and length reads

$$\frac{\partial u}{\partial t} - \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} = \chi, \qquad 0 < x < 1, \tag{4.3}$$

$$u(0,t) = \phi_0,\ u(1,t) = \phi_1, \tag{4.4}$$

where now x, t, u, χ are nondimensional and

$$< \chi >= 0, \; < \chi^2 >= 1. \tag{4.5}$$

We consider the space and time discretization of (4.3) - (4.5) using as in (3.9) an implicit Euler scheme. The space mesh is $\Delta x = 1/N$, N an integer, and the time mesh is Δt. The approximate values of u and χ are

$$u_j^n \approx u(j\Delta x, n\Delta t), \; \chi_j^n \approx \chi(j\Delta x, n\Delta t).$$

The time mesh discretization for χ is Δt_r usually larger than Δt. Actually Δt_r and Δx are chosen first, and Δt is then chosen as large as possible so as to ensure accuracy of the numerical scheme. Hence at each instant of time $n\Delta t$, the $\chi_j^n, j = 2, ..., N-1$, are totally uncorrelated random variables; χ_j^n is constant on a time interval $(k\Delta t_r,\ (k-1)\ \Delta t_r)$, where k is an integer, and if $n\Delta t$ and $n'\Delta t$ belong to two consecutive or different such intervals, all the $\chi_i^{n'}$ are totally independent of all the χ_j^n $(n' > n)$, with $< \chi_j^n >= 0$, $< (\chi_j^n)^2 >= 1$.

The analog of (3.9) reads

$$\begin{aligned} &u_j^n - u_j^{n-1} - \frac{1}{Re}\frac{\Delta t}{\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \frac{1}{4}\frac{\Delta t}{\Delta x}({u_{j+1}^n}^2 - {u_{j-1}^n}^2) \\ &= \chi_j^n \Delta t, \qquad 1 \le j \le N-1, \end{aligned} \tag{4.6}$$

$$u_0^n = \phi_0^n, \; u_N^n = \phi_1^n. \tag{4.7}$$

We easily write (4.6), (4.7) in the form (3.10) - (3.11).

At each instant of time, the cost function considered here is

$$\begin{aligned} J(\phi_0^n, \phi_1^n) = &\frac{m_1}{2}\{|\phi_0|^2 + |\phi_1^n|^2\} + \frac{m_2}{2}\{|\frac{u_1^n - u_0^n}{\Delta x}|^2 + |\frac{u_N^n - u_{N-1}^n}{\Delta x}|^2\} \\ &+ \frac{m_3}{2\Delta x}\sum_{j=1}^{N}|u_j^n - u_{j-1}^n|^2, \end{aligned} \tag{4.8}$$

with $m_i \ge 0$ and in most cases $m_1 > 0, m_2 > 0$.

Algorithm (3.15) can be implemented with an appropriate descent parameter ρ. Although quite involved, the computation of $\mathcal{D}J/\mathcal{D}\phi$ follows from (3.13) and (3.14).

The following figures correspond to the case where $Re = 1500, N = 1/\Delta x = 2048, \Delta t = 0.001, \Delta t_r = 0.01, m_1 = 1, m_2 = \Delta x, m_3 = 0$. Figure 1 shows the evolution of the cost function J in the above control case. The velocity gradient at the wall $\partial u/\partial x(x = 0)$ is shown in Fig. 2.

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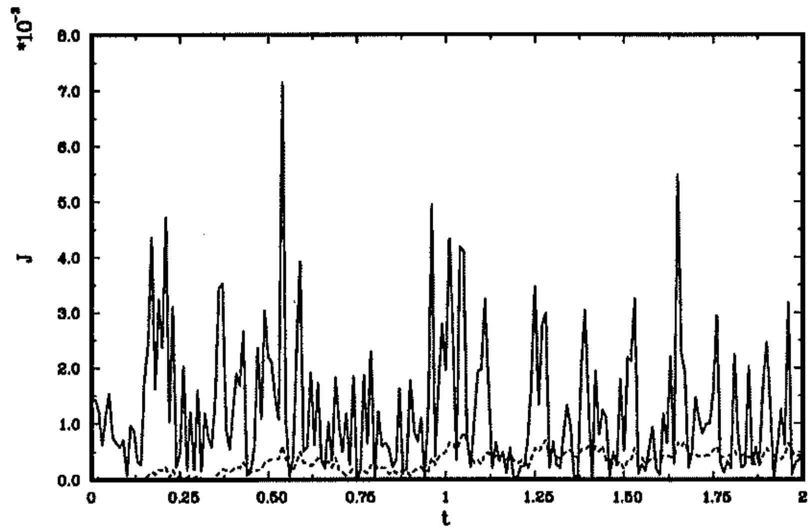

FIGURE 1. Time evolution of the cost function J (Eq. (4.8)). ——— , no control; - - - - , boundary control.

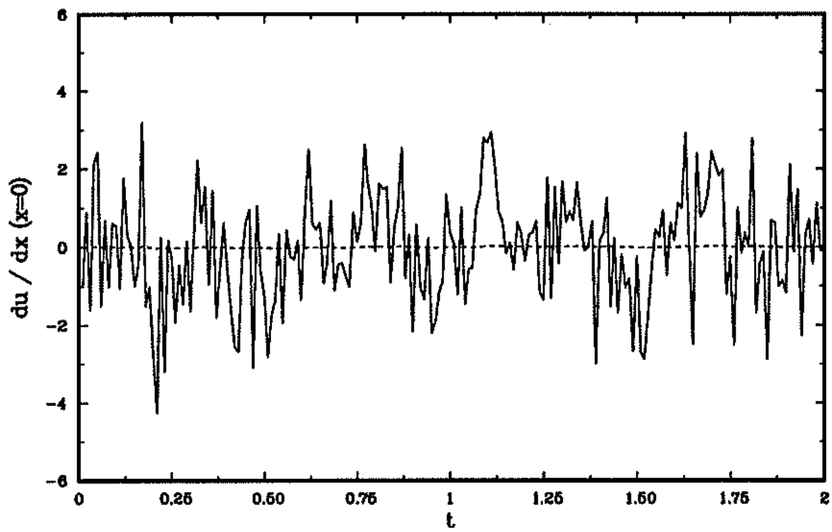

FIGURE 2. Time evolution of the velocity gradient at the wall $\partial u/\partial x$ $(x = 0)$. ——— , no control; - - - - , boundary control.