An exact turbulent closure for the hydromagnetic dynamo

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1. Motivation and objectives

The problem of turbulence in an electrically-conducting fluid, although central to controlled fusion and many geo- and astrophysical processes, is in its infancy compared to (nonconducting) hydrodynamics. "The understanding and manipulation of magnetohydrodynamic (MHD) turbulence is more in need of theory than is the case for Navier-Stokes fluids..." (Montgomery 1989). As pointed out by Montgomery, there is a surprising, almost embarrassing gap in our understanding of what the elementary or equilibrium states of a driven, dissipative MHD fluid are. Much work has been done in the past couple of decades (as reviewed by e.g., Moffatt 1978, Soward & Childress 1986, Roberts & Soward 1992) on the closely- related "dynamo" problem, namely, how fluid motion can overcome Ohmic dissipation to induce and maintain magnetic fields in geophysical and astrophysical contexts. Dynamo models, however fruitful and illuminating, are often constrained to rely upon phenomenological, statistical or perturbative assumptions or to limit themselves to the so-called "kinematic" dynamo problem, in which one ignores the back- reaction of the generated magnetic field upon the velocity. Work on the self-consistent and fully-developed "hydromagnetic dynamo, though difficult and fraught with uncertainties, needs to be extended. Dynamo theory is far from reaching its final form." (Cowling 1981)

In what follows we consider the hydromagnetic dynamo from the Hopf functional point of view. This has recently been developed in the context of Navier-Stokes turbulence (Shen & Wray 1991); here we incorporate buoyancy, rotation, electrical conductivity, scalar diffusion, and source. No perturbative, phenomenological, or statistical assumptions or variational arguments are invoked; we seek an exact turbulent closure of the MHD equations. This leads to closed-form analytic expressions for correlation functions (such as the mean electromotive force (emf)) and moment-generating functionals for the velocity, magnetic field, and scalar which generalize the usual ideal, static, nonstatistical solutions. Equilibrium, stationary nonequilibrium and time-dependent solutions are proposed. The incorporation of compressibility and realizability are outlined. This method of testing the validity of so-called alpha models (Moffatt 1978) for the generation of large-scale fields should also be applicable to testing the validity of analogous non-MHD models (Frisch et al. 1987 and references therein) for large-scale turbulent structure generation in (non-magnetic) anisotropic, helical, or compressible flows. No claim is made for uniqueness or completeness; however, the fact that (1) exact statistical solutions can be obtained at all, and that (2) in fact more than one class of solutions appears to emerge from this approach, seems sufficiently promising to warrant further investigation.

2. Accomplishments

2.1. Functional MHD equations

The equations which we consider here are the MHD equations consisting of the Navier-Stokes equations with Lorentz force and buoyancy

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\mathbf{u}(\mathbf{x},t) = -\mathbf{u}\cdot\nabla\mathbf{u}(\mathbf{x},t) - \frac{\nabla p}{\rho} + \hat{g}c(\mathbf{x},t) + \frac{\mathbf{J}(\mathbf{x},t)\times\mathbf{B}}{\rho} \tag{1.1}$$

the induction equation in the usual nonrelativistic low-frequency regime

$$\left(\frac{\partial}{\partial t} - \frac{1}{\mu\sigma}\nabla^2\right)\mathbf{B}(\mathbf{x},t) = \nabla\times(\mathbf{u}\times\mathbf{B}) \tag{1.2}$$

and the scalar (temperature) equation with diffusion and source term

$$\frac{\partial c}{\partial t} + \nabla\cdot(\mathbf{u}c) - \nabla\cdot(D\nabla c) = Q \tag{1.3}$$

supplemented by incompressibility, Ohm's law, the definition of the Lorentz force, and the "pre-Maxwell" equations:

$$\nabla\cdot\mathbf{u}(\mathbf{x},t) = 0 \qquad \mathbf{J} = \sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \qquad \mathbf{F} = (q\mathbf{E}+\mathbf{J}\times\mathbf{B}) \tag{1.4}$$

$$\nabla\times\mathbf{B} = \mu\mathbf{J} \qquad \nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\cdot\mathbf{E} = \frac{q}{\epsilon} \tag{1.5}$$

Coriolis and baroclinic effects will also be briefly discussed. The pure thermal convection problem (in the Boussinesq approximation) is, of course, recovered by setting the magnetic field and electrical conductivity to zero and going to a potential temperature formulation (Busse 1981).

We define the moment-generating functional

$$\phi \equiv \left\langle e^{\int_{-\infty}^{\infty} d\mathbf{x}\,[\mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})+\mathbf{g}(\mathbf{x})\cdot\omega(\mathbf{x})+\mathbf{h}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})+\mathbf{l}(\mathbf{x})\cdot\mu\mathbf{J}(\mathbf{x})+z(\mathbf{x})c(\mathbf{x})+q(\mathbf{x})Q(\mathbf{x})]}\right\rangle \tag{1.6}$$

where the brackets indicate ensemble average over all realizations of $\mathbf{u}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $c(\mathbf{x})$ and $Q(\mathbf{x})$. $\phi[\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{x}), \mathbf{l}(\mathbf{x}), z(\mathbf{x}), q(\mathbf{x})]$ is the functional Fourier transform of the joint probability density. (The customary factor of "i" in the exponent has been absorbed into the dummy functions $\mathbf{f}$, $\mathbf{g}$, etc. for notational simplicity.) ϕ evolves under the above dynamics to a stationary state given by the Hopf equations:

$$\nabla\times\left[\left(\frac{\delta}{\delta\mathbf{f}(\mathbf{x})} + \nu\nabla\right)\times\frac{\delta\phi}{\delta\mathbf{g}(\mathbf{x})} + \frac{\delta}{\delta\mathbf{l}(\mathbf{x})}\times\frac{1}{\mu\rho}\frac{\delta\phi}{\delta\mathbf{h}(\mathbf{x})} + \hat{g}\frac{\delta\phi}{\delta z(\mathbf{x})}\right] = 0 \tag{1.7}$$

$$\nabla \times \left[\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} + \frac{1}{\mu\sigma}\nabla + \frac{\lambda\nabla}{\nabla^2}\right] \times \frac{\delta\phi}{\delta \mathbf{h}(\mathbf{x})} = 0 \tag{1.8}$$

$$\nabla \cdot \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} - D\nabla\right) \frac{\delta\phi}{\delta z(\mathbf{x})} = \frac{\delta\phi}{\delta q(\mathbf{x})} \tag{1.9}$$

where $\lambda \equiv$ steady-state frequency or growth rate of $\mathbf{B}$. Solutions have been obtained for the special case of isotropic flow neglecting buoyancy, the scalar equation, and either all nonlinear or all dissipative terms (Stanisic 1985). We impose no such restrictions; we first solve the vorticity and magnetic functional equations without buoyancy and then incorporate buoyancy and the scalar equation.

2.2. Equilibrium solutions

In order to find a solution, we add and subtract "ghost" torques

$$\nabla \times (\mathbf{u} \times \alpha_4 \mathbf{B}) + \nabla \times \nabla \times \alpha_5 \mathbf{B}/\mu\sigma \tag{2.1}$$

whose purpose is to "interpolate" between terms in the Hopf equations, so that the resulting adjacent terms in the equations differ from each other by changing only one ordinary or functional derivative. We recognize this type of equation as essentially wavelike or convective in nature, thereby enabling us to write down a general functional ansatz for its solution. Portions of each term are balanced pairwise by portions of other terms to achieve an overall statistical steady-state. (Those who prefer to visualize in topological terms may view this method as analogous to the familiar decomposition of a knotted vortex tube into two or more linked tubes by the insertion of equal and opposite flux tube elements between two points. The condition of detailed balance corresponds to solenoidality of vorticity, i.e., the condition that the knot form a closed loop.)

Balancing a fraction α_1 of the dissipative term against a fraction $(1 - \alpha_2)$ of the transfer term yields

$$\nabla \times \nabla \times \nu\alpha_1 \frac{\delta\phi}{\delta \mathbf{g}(\mathbf{x})} = \nabla \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times (1-\alpha_2)\frac{\delta\phi}{\delta \mathbf{g}(\mathbf{x})}\right) \tag{2.2}$$

The remainder of the transfer term is in turn balanced by a fraction $(1 - \alpha_3)$ of the magnetic term, mediated by one of the ghost terms:

$$\nabla \times \frac{\delta}{\delta \mathbf{l}(\mathbf{x})} \times \frac{(1-\alpha_3)}{\mu\rho}\frac{\delta\phi}{\delta \mathbf{h}(\mathbf{x})} = -\nabla \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \alpha_4 \frac{\delta\phi}{\delta \mathbf{h}(\mathbf{x})}\right) \tag{2.3}$$

$$\nabla \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \alpha_2 \frac{\delta\phi}{\delta \mathbf{g}(\mathbf{x})}\right) = \nabla \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \alpha_4 \frac{\delta\phi}{\delta \mathbf{h}(\mathbf{x})}\right) \tag{2.4}$$

Finally, the remainder of the magnetic term is balanced by the remainder of the dissipative term, mediated by the other ghost term:

$$\nabla \times \nabla \times \nu(1-\alpha_1)\frac{\delta\phi}{\delta \mathbf{g}(\mathbf{x})} = \nabla \times \nabla \times \frac{\alpha_5}{\mu\sigma}\frac{\delta\phi}{\delta \mathbf{h}(\mathbf{x})} \tag{2.5}$$

$$\nabla \times \frac{\delta}{\delta \mathbf{l}(\mathbf{x})} \times \frac{\alpha_3}{\mu\rho} \frac{\delta\phi}{\delta \mathbf{h}(\mathbf{x})} = -\nabla \times \nabla \times \frac{\alpha_5}{\mu\sigma} \frac{\delta\phi}{\delta \mathbf{h}(\mathbf{x})} \tag{2.6}$$

The fractions α_j are to be determined by imposing self-consistency upon the dynamics, as we will see.

This pairwise balancing procedure enables us to organize previous deterministic or linearized solutions (Taylor 1986, Shercliff 1965) and generalize them to resistive, nonperturbative, and statistical (nonfactoring correlation functions) solutions. For example, the usual magnetostatic equilibria in which velocity vanishes and the Lorentz force balances pressure would correspond to setting $\alpha_5 = 0$. The kinematic dynamo, for which one neglects magnetic backreaction but has in general nonvanishing transfer and dissipation, would correspond to setting $\alpha_4 = \alpha_5 = \alpha_2 = 0, \alpha_1 = 1$. Alfven waves, for which $\mathbf{u}$ is parallel to $\mathbf{B}$, would correspond to setting $\alpha_3 = 1, \alpha_2 = 0$. Hartmann flows or weak-field dynamos, in which magnetic forces balance viscosity, and Eulerized flows (exhibiting depressed nonlinearity) would be generalized by setting $\alpha_1 = \alpha_4 = 0, \alpha_3 = 1$, while Stokes flows would correspond to the case $\alpha_2 = 1, \alpha_5 = \alpha_3 = 0$. Magnetostrophic flows or strong field dynamos (in the absence of buoyancy), in which Coriolis torques balance Lorentz torques, would follow from decomposing $\mathbf{g} \cdot \omega$ in equation (1.6) into mean and fluctuating vorticity (with a separate dummy function for each) and setting $\alpha_1 = 1 - \alpha_2$. Ideal flows (Gilbert & Sulem 1990) would correspond to setting $\alpha_2 = \alpha_3 = 1$; in order to satisfy $\nabla \times (\mathbf{u} \times \mathbf{B}) = 0$, the ghost in equations (2.3) and (2.4) would have to be replaced by a nonvanishing ghost (no pun intended) such as $\pm\nabla \times \left(\frac{\delta}{\delta \mathbf{l}(\mathbf{x})} \times \alpha_4 \frac{\delta\phi}{\delta \mathbf{g}(\mathbf{x})}\right)$.

Recognizing equations (1.8) and (2.2)-(2.6) as equations for characteristic curves (albeit in function space) leads us to immediately write down forms for their solution. For equation (2.2),

$$\frac{\delta\phi}{\delta \mathbf{g}(\mathbf{x})} = \frac{1}{\nu\alpha_1} \mathbf{G} \left(H(\mathbf{x}) + \int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{f}(\mathbf{x}) \cdot \frac{\nu\alpha_1}{1-\alpha_2} \left[\nabla H + \mathbf{Z}^{(1)} \right] \right) \tag{2.7}$$

where the "phase shift" $\mathbf{Z}^{(1)}$ of the traveling wave solution ("propagating" in $\mathbf{f}$ and $\mathbf{x}$ rather than in $\mathbf{x}$ and t) is governed by

$$\nabla \times (\mathbf{Z}^{(1)} \times \mathbf{G}') = 0 \tag{2.8}$$

Equations (2.3) and (2.6) imply

$$\frac{\delta\phi}{\delta \mathbf{h}(\mathbf{x})} = \frac{\mu\sigma}{\alpha_5} \mathbf{W} \Big(M(\mathbf{x}) + \int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{l}(\mathbf{x}) \cdot \frac{\alpha_5 \rho}{-\sigma\alpha_3} \left[\nabla M + \mathbf{Z}^{(3)} \right]$$
$$+ \int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{f}(\mathbf{x}) \cdot \frac{(1-\alpha_3)\alpha_5}{\mu\sigma\alpha_3\alpha_4} \left[\nabla M_1 + \mathbf{Z}^{(3)} \right] \Big) \tag{2.9}$$

where

$$\nabla \times (\mathbf{Z}^{(3)} \times \mathbf{W}') = 0 \tag{2.10}$$

Equation (1.8) yields

$$\frac{\delta\phi}{\delta\mathbf{h}(\mathbf{x})} = \mathbf{L}\left(N(\mathbf{x}) + \int_{-\infty}^{\infty} d\mathbf{x}\,\mathbf{f}(\mathbf{x})\cdot\left[\nabla(\frac{N}{\mu\sigma} + \frac{\lambda N}{\nabla^2}) + \mathbf{Z}^{(5)}\right]\right) \tag{2.11}$$

where

$$\nabla\times(\mathbf{Z}^{(5)}\times\mathbf{L}') = 0 \tag{2.12}$$

(The arguments of **G** and **L** in the above equations have additional contributions from $\mathbf{l}(\mathbf{x})$ as in (2.9).) "Prime" denotes derivative with respect to argument. Equation (2.4) implies

$$\alpha_2\frac{\delta\phi}{\delta\mathbf{g}(\mathbf{x})} = \alpha_4\frac{\delta\phi}{\delta\mathbf{h}(\mathbf{x})} + \mathbf{Z}^{(4)} \tag{2.13}$$

where

$$\nabla\times\frac{\delta}{\delta\mathbf{f}(\mathbf{x})}\times\mathbf{Z}^{(4)} = 0 \tag{2.14}$$

while equation (2.5) implies

$$(1-\alpha_1)\frac{\delta\phi}{\delta\mathbf{g}(\mathbf{x})} = \frac{\alpha_5}{\nu\mu\sigma}\frac{\delta\phi}{\delta\mathbf{h}(\mathbf{x})} + \mathbf{Z}^{(2)} \tag{2.15}$$

where

$$\nabla\times\nabla\times\mathbf{Z}^{(2)} = 0 \tag{2.16}$$

Solenoidality of velocity and magnetic field will be guaranteed if

$$\nabla\cdot\frac{\mathbf{G}^{(n)}}{\alpha_1} = \nabla\cdot\frac{\mathbf{W}^{(n)}}{\alpha_5} = 0, \quad n = 0, 1, \cdots \tag{2.17}$$

implying

$$\nabla\cdot\frac{\alpha_1}{1-\alpha_2}\left[\nabla H + \mathbf{Z}^{(1)}\right] = \nabla\cdot\frac{\alpha_5}{\alpha_3}\left[\nabla M + \mathbf{Z}^{(3)}\right] = 0 \tag{2.18}$$

These equations have a solution:

$$\frac{1-\alpha_1}{\alpha_1}\mathbf{G} = \mathbf{W} \qquad \mathbf{L} = \frac{\mu\sigma}{\alpha_5}\mathbf{W} \tag{2.19}$$

$$\nabla\left(\frac{M_1}{\mu\sigma} + \frac{\lambda M_1}{\nabla^2}\right) = \frac{\alpha_5(1-\alpha_3)}{\mu\sigma\alpha_4\alpha_3}\nabla M_1 \tag{2.20}$$

$$H = M_1 = N, \qquad \mathbf{Z}^{(2)} = \mathbf{Z}^{(4)} = 0 \tag{2.21}$$

$$\mathbf{Z}^{(1)} = \mathbf{Z}^{(3)} = \mathbf{Z}^{(5)}\frac{-\sigma\alpha_3}{\rho\alpha_5} \tag{2.22}$$

Self-consistency of (2.7) with (2.9) and (2.13) with (2.15) respectively constrains the fractions α_j to satisfy

$$\alpha_1\alpha_2\alpha_3 = (1-\alpha_1)(1-\alpha_2)(1-\alpha_3) \tag{2.23}$$

$$\nu\mu\sigma\frac{\alpha_4}{\alpha_5} = \frac{\alpha_2}{1-\alpha_1} \tag{2.24}$$

If $\mathbf{B}$ is stationary ($\lambda = 0$, DC dynamo), we have the further condition

$$\frac{\alpha_4}{\alpha_5} = \frac{1-\alpha_3}{\alpha_3} \tag{2.25}$$

2.3. Effect of scalar equation

2.3.1 No scalar diffusion or explicit source

In the absence of scalar diffusion and source, the stationary scalar Hopf equation (1.9) becomes

$$\frac{\delta}{\delta \mathbf{f}(\mathbf{x})}\cdot\nabla\frac{\delta\phi}{\delta z(\mathbf{x})} = 0 \tag{3.1}$$

Hence the buoyancy term in the vorticity Hopf equation becomes

$$\nabla\times\hat{g}\frac{\delta\phi}{\delta z(\mathbf{x})} = -\hat{g}\times\left(\frac{\delta}{\delta\mathbf{f}(\mathbf{x})}\times\mathbf{n}\right) \tag{3.2}$$

for some functional $\mathbf{n}$ where

$$\nabla\times\frac{\delta}{\delta\mathbf{f}(\mathbf{x})}\times\mathbf{n} = 0 \tag{3.3}$$

in order for $\frac{\delta}{\delta\mathbf{f}(\mathbf{x})}\times\mathbf{n}$ to represent the gradient $\nabla\frac{\delta\phi}{\delta z(\mathbf{x})}$. Add and subtract a "ghost" torque

$$\alpha_6\hat{g}\times\left(\frac{\delta}{\delta\mathbf{f}(\mathbf{x})}\times\frac{\delta\phi}{\delta\mathbf{g}(\mathbf{x})}\right) \tag{3.4}$$

where α_6 depends upon $\mathbf{x}$ and replace $(1-\alpha_2)$ by $(2-\alpha_2)$ above. Then we may obtain a solution by imposing the condition that buoyancy is balanced by part of the transfer term (mediated by the ghost term), yielding

$$\nabla\times\left(\frac{\delta}{\delta\mathbf{f}(\mathbf{x})}\times\frac{\delta\phi}{\delta\mathbf{g}(\mathbf{x})}\right) = -\alpha_6\hat{g}\times\left(\frac{\delta}{\delta\mathbf{f}(\mathbf{x})}\times\frac{\delta\phi}{\delta\mathbf{g}(\mathbf{x})}\right) \tag{3.5}$$

$$\hat{g}\times\left(\frac{\delta}{\delta\mathbf{f}(\mathbf{x})}\times\mathbf{n}\right) = \alpha_6\hat{g}\times\left(\frac{\delta}{\delta\mathbf{f}(\mathbf{x})}\times\frac{\delta\phi}{\delta\mathbf{g}(\mathbf{x})}\right) \tag{3.6}$$

This will be satisfied if

$$(\nabla H + \mathbf{Z}^{(1)})\times\mathbf{G}' = \nabla\psi_1\,\exp(-\int^{\mathbf{x}}\alpha_6\hat{g}\cdot d\mathbf{l}) \tag{3.7}$$

$$\hat{g} \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \mathbf{n}\right) = \hat{g} \times \nabla\psi_1 \exp(-\int^{\mathbf{x}} \alpha_6 \hat{g} \cdot d\mathbf{l}) \tag{3.8}$$

Equation (3.8) is readily satisfied by **n** of the form

$$\mathbf{n} \equiv -\frac{1}{3}\int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{f}(\mathbf{x}) \times \left[\nabla\psi_1 \exp(-\int^{\mathbf{x}} \alpha_6 \hat{g} \cdot d\mathbf{l}) + \hat{g}\psi_2(\mathbf{x})\right] \tag{3.8a}$$

The three scalar functions in the above expression are determined by the three equations (3.3).

If Coriolis forces are present, we obtain instead that

$$\nabla \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \Omega\phi\right) = -\alpha_6 \hat{g} \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \Omega\phi\right) \tag{3.9}$$

$$\hat{g} \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \mathbf{n}\right) = \alpha_6 \hat{g} \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \Omega\phi\right) \tag{3.10}$$

This will be satisfied if

$$-\Omega \times \mathbf{G}^{(-1)} = \nabla\psi_1 \exp(-\int^{\mathbf{x}} \alpha_6 \hat{g} \cdot d\mathbf{l}) \tag{3.11}$$

$$\hat{g} \times \left(\frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \mathbf{n}\right) = \hat{g} \times \nabla\psi_1 \exp(-\int^{\mathbf{x}} \alpha_6 \hat{g} \cdot d\mathbf{l}) \tag{3.12}$$

where $\mathbf{G} = \nabla \times \mathbf{G}^{(-1)}$ defines $\mathbf{G}^{(-1)}$.

2.3.2 Scalar diffusion and source

In the presence of scalar diffusion, we let part of the diffusion be balanced by convection and part by the scalar source. The former condition may be expressed as

$$D\alpha_6 \nabla \frac{\delta\phi}{\delta z(\mathbf{x})} = \frac{\delta}{\delta z(\mathbf{x})} \frac{\delta\phi}{\delta \mathbf{f}(\mathbf{x})} + \nabla \times \psi \tag{3.13}$$

Letting $\psi = 0$ for simplicity yields a buoyancy term

$$\nabla \times \hat{g} \frac{\delta\phi}{\delta z(\mathbf{x})} = \frac{\hat{g}}{D\alpha_6} \frac{\delta}{\delta z(\mathbf{x})} \times \frac{\nabla}{\nabla^2} \times \frac{\delta\phi}{\delta \mathbf{g}(\mathbf{x})} \tag{3.14}$$

Replacing $(1 - \alpha_1)$ by $(2 - \alpha_1)$ and letting the buoyancy be balanced by part of viscosity yields

$$\frac{\delta}{\delta z(\mathbf{x})} \sim \frac{D\alpha_6 \nu}{|\hat{g}|^2} \hat{g} \cdot \nabla\nabla^2 \tag{3.15a}$$

The first moment is given by

$$\hat{g} \frac{\delta\phi}{\delta z(\mathbf{x})} = \nu\nabla^2 \frac{\delta\phi}{\delta \mathbf{f}(\mathbf{x})} \tag{3.16a}$$

In conjunction with (3.15) and (2.2), this allows us to determine α_6. Alternatively, if Coriolis forces are present to balance the buoyancy (geostrophic balance), we have

$$\frac{\delta}{\delta z(\mathbf{x})} = \frac{\hat{g}}{|\hat{g}|^2} \cdot \Omega \times \frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \tag{3.15b}$$

This gives us a prescription for the scalar moments. Compare with the prescription implied by (3.2) and (3.6):

$$\nabla \frac{\delta \phi}{\delta z(\mathbf{x})} = \frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \mathbf{n} \tag{3.16b}$$

$$\nabla \frac{\delta}{\delta z(\mathbf{x})} = \alpha_6 \frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \frac{\delta}{\delta \mathbf{g}(\mathbf{x})} \tag{3.15c}$$

Similarly, taking into account the source term yields

$$\frac{\delta}{\delta q(\mathbf{x})} \sim \nabla \cdot \left(D(1-\alpha_6) \nabla \frac{\delta}{\delta z(\mathbf{x})} \right) \tag{3.17}$$

which provides us with a prescription for the source moments. If there is no explicit source, we may let $\alpha_6 = 1$. The scenario in which buoyancy is balanced by Lorentz forces to yield a guiding-center drift current which is perpendicular to gravity and the magnetic field is already implicit in equations (2.3, 2.4, 3.5, 3.6).

2.3.3 Compressibility

If there is a baroclinic term $\nabla P \times \nabla \rho$ in the vorticity equation, the statistical description must be augmented to include the pressure and density explicitly. Let us add the term $[n_1(\mathbf{x})P(\mathbf{x}) + n_2(\mathbf{x})\rho(\mathbf{x})]$ to the integrand in the exponent in equation (1.6). Suppose that the equation of state prescribes P as a position-dependent functional $P_1[\mathbf{u}, c, \rho, \mathbf{x}]$ of velocity, scalar and density. Then the argument of G in equation (2.7) has the additional integrals

$$\int_{-\infty}^{\infty} d\mathbf{x} \left\{ z(\mathbf{x})q(\mathbf{x}) + n_2(\mathbf{x})P_2(\mathbf{x}) + n_1(\mathbf{x})P_1 \left[\frac{\nu\alpha_1}{1-\alpha_2}[\nabla H + \mathbf{Z}^{(1)}], q, P_2, \mathbf{x} \right] \right\} \tag{3.18}$$

P_2 is related to the first (vector) argument of P_1 in the same way that density is related to velocity in the steady-state continuity equation $\nabla \cdot (\rho \mathbf{u}) = 0$ (which holds here in a statistical sense):

$$\nabla \cdot \left(P_2[\nabla H + \mathbf{Z}^{(1)}] \right) = 0 \tag{3.19}$$

(see equations (5.10, 5.17-5.19) for one explicit formal solution.) Balancing the baroclinic term solely against buoyancy $\hat{g} \times \nabla \rho$ would yield

$$\nabla P_1 = -\hat{g} + n_3(\mathbf{x}) \nabla P_2 \tag{3.20}$$

This gives us three additional equations for three additional unknowns n_3, q (appearing in equation (3.20) through P_1), and H (appearing in P_1 and implicitly in P_2, through equation (3.19)). The incompressibility condition (2.18) used previously to determine H and the Boussinesq equation of state $\rho \sim T$ used previously to determine q through equations (3.15) or (3.16) no longer apply, of course. More generally, buoyancy is balanced against a combination of baroclinic, transfer, Coriolis, and viscous forces, leading to more unknowns and equations via the procedure outlined above. The adiabatic and isothermal subcases, in which the steady-state equations of state take the form $\mathbf{u} \cdot \nabla(P/\rho^{\gamma}) = 0$ or $\mathbf{u} \cdot \nabla(P/\rho) = 0$ respectively, may be treated by a procedure analogous to §2.3.1.

2.3.4 Arbitrary explicit force

If the dynamo is driven by an arbitrary explicit force (Braginsky 1964) instead of the Boussinesq-type buoyancy forces described above, other approaches may be useful. For example, if the force is solenoidal, the external torque may be written as $\nabla \times \nabla \times \psi$ for some vector function ψ. If this force is balanced by viscosity, we obtain that the ψ moments are proportional to the corresponding moments of $\nu\omega$ (within a function whose curl vanishes) where ω is the vorticity. If the force is nonsolenoidal, we may let the dummy field which is conjugate to the force (in the definition of the generating functional) be a pseudovector "angle" $\vec{\theta}(x)$. Then using the angular momentum identity

$$\frac{\delta}{\delta\vec{\theta}(\mathbf{x})} = \frac{\delta}{\delta\vec{\theta}(\mathbf{x})} \times \frac{\delta}{\delta\vec{\theta}(\mathbf{x})} \tag{3.21}$$

and balancing the force against part of transfer by adding and subtracting a ghost torque

$$\nabla \times \frac{\delta}{\delta\vec{\theta}(\mathbf{x})} \times \frac{\delta}{\delta\mathbf{g}(\mathbf{x})} \tag{3.22}$$

yields

$$\frac{\nu\alpha_1}{1-\alpha_2}\nabla M_1 + \mathbf{Z}^{(1)} = \text{Beltrami} \tag{3.23}$$

This particular choice of dummy field is awkward to implement in practice because of the noncommutativity of the functional derivative $\frac{\delta}{\delta\vec{\theta}(\mathbf{x})}$. However, it may be viewed as justification for the use of Beltrami flows for the velocity in kinematic dynamo models.

2.4. Closure

From the above solution, one can write down exact expressions for correlation functions. For example, the mean emf is given by

$$\langle \mathbf{u}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) \rangle = -\frac{1-\alpha_3}{\alpha_3\alpha_4}\left[\nabla M_1 + \mathbf{Z}^{(1)}\right] \times \mathbf{W}' \tag{4.1}$$

$$= [\mathbf{u_0} - \frac{(1-\alpha_3)\alpha_5}{\mu\sigma\alpha_4\alpha_3}\nabla] \times \langle \mathbf{B}(\mathbf{x})\rangle \tag{4.2}$$

since $\mathbf{W}'$ is parallel to $\mathbf{W}$ by solenoidality. (Equation (4.2) may be taken as the definition of $\mathbf{u_0}$.) This may be viewed as an anisotropic inhomogeneous α effect (e.g., Krause & Rädler 1980). One may compare this exact result with the conventional MFE prediction:

$$-\frac{\tau}{3}\left[\langle \mathbf{u}\cdot\omega\rangle\langle \mathbf{B}\rangle + \langle u^2\rangle\langle \nabla\times\mathbf{B}\rangle\right] =$$

$$-\frac{\tau\mu\sigma}{3\alpha_5(1-\alpha_2)}\left[\nabla H + \mathbf{Z}^{(1)}\right]\cdot\left[\mathbf{G}'\mathbf{W} + \mathbf{G}^{(-1)'}\nabla\times\mathbf{W}\right] \tag{4.3}$$

where $\tau \equiv$ phenomenological correlation time of turbulent velocity. If we superpose solutions with different (nonparallel) $\mathbf{W}$, then $\mathbf{u_0}\times\langle \mathbf{B}(\mathbf{x})\rangle$ will be replaced by a term which is in general not perpendicular to $\langle \mathbf{B}(\mathbf{x})\rangle$, permitting generation of toroidal current (and hence poloidal field) from toroidal field, as desired for dynamo action.

The size of the magnetic fluctuations

$$\frac{\langle B^2\rangle}{\langle B\rangle^2} = \frac{\alpha_2\alpha_5^2(1-\alpha_3)}{\alpha_3(\alpha_4\mu\sigma)^2}\nabla\times\mathbf{Z}^{(1)}\cdot\frac{\mathbf{W}'}{W^2} \tag{4.4}$$

Whether this ratio is $\gg 1$ or $\ll 1$ determines the regimes of validity of Ohmic diffusion and first-order smoothing, respectively. Other correlation functions (e.g., $\langle \mathbf{B}\cdot\nabla\omega\rangle$ and $\langle \mathbf{B}(\mathbf{x})\mathbf{B}(\mathbf{x}')\rangle$) may also be computed to shed light upon questions of quenching (e.g., Malkus & Proctor 1975, Kraichnan 1979), inverse cascade (e.g., Frisch et al. 1975), or the formation of current sheets (e.g., Parker 1989).

2.5. Steady state without detailed balance

Consider again the vorticity Hopf equation without buoyancy. We rewrite it schematically as:

$$\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta} + \frac{\partial}{\partial\zeta}\right)\phi = 0 \tag{5.1}$$

where the three terms correspond to the viscous, transfer, and Lorentz force terms respectively (equations (6.6) - (6.8)). This has the general solution

$$\phi(\xi,\eta,\zeta) = \int db_1\,db_2\;\rho(b_1,b_2)\;\phi\big(b_1\xi + b_2\eta - (b_1+b_2)\zeta\big) \tag{5.2}$$

However, one may gain much more insight by decomposing equation (5.1) into three simultaneous equations:

$$\left(\alpha_1\frac{\partial}{\partial\xi} + (1-\alpha_2)\frac{\partial}{\partial\eta}\right)\phi = F_1[\phi] \tag{5.3}$$

$$\left(\alpha_2\frac{\partial}{\partial\eta} + (1-\alpha_3)\frac{\partial}{\partial\zeta}\right)\phi = F_2[\phi] \tag{5.4}$$

$$\left(\alpha_3 \frac{\partial}{\partial \zeta} + (1-\alpha_1)\frac{\partial}{\partial \xi}\right)\phi = F_3[\phi] \tag{5.5}$$

where the $F_j[\phi]$ are in general operators on ϕ satisfying the stationarity condition

$$\sum_{j=1}^{3} F_j[\phi] = 0 \tag{5.6}$$

For

$$F_j[\phi] = 0 \tag{5.7}$$

this reduces to the earlier, detailed balance case. If, in addition, we restrict ourselves to

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial \zeta} = 0 \tag{5.8}$$

we recover the ideal magnetostatic deterministic solution, which is most often used as a starting point for stability studies but whose fundamental inadequacy has been discussed at length (Montgomery & Phillips 1989). Hence we see that the sought-after generalization of this static solution to resistive, turbulent, nonequilibrium solutions can be achieved within the framework of the decomposition (5.3-5.5) with nonzero α_j and F_j.

The physical interpretation of this decomposition may be further clarified by going to a probability (rather than moment-generating functional) description. We may write the stationary Hopf equation in "Vlasov-equation" format (dropping the magnetic and scalar variables for notational simplicity) as

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \int \prod_{\mathbf{x}} d\mathbf{u}(\mathbf{x})\; d\mathbf{x}\; i\mathbf{f}\cdot\dot{\mathbf{u}}\; P[\mathbf{u}]e^{i\int \mathbf{f}\cdot\mathbf{u}} \\ &= \int \prod_{\mathbf{x}} d\mathbf{u}(\mathbf{x})\; d\mathbf{x}\; \dot{\mathbf{u}}\; P[\mathbf{u}]\cdot\frac{\delta}{\delta \mathbf{u}} e^{i\int \mathbf{f}\cdot\mathbf{u}} \\ &= -\int \prod_{\mathbf{x}} d\mathbf{u}(\mathbf{x})\; d\mathbf{x}\; \frac{\delta}{\delta \mathbf{u}}\cdot(\dot{\mathbf{u}}\; P[\mathbf{u}])e^{i\int \mathbf{f}\cdot\mathbf{u}} = 0\end{aligned} \tag{5.9}$$

Inverse functional Fourier transforming with respect to $\mathbf{f}(\mathbf{x})$ yields

$$\frac{\partial P}{\partial t} = -\int d\mathbf{x}\; \frac{\delta}{\delta \mathbf{u}}\cdot\left(\dot{\mathbf{u}}P[\mathbf{u}]\right) = 0 \tag{5.10}$$

Separating the dissipative (D), transfer (T) and magnetic (M) contributions yields

$$0 = \sum_{j=D,T,M}\left(\frac{dP}{dt}\right)_j \qquad \dot{\mathbf{u}} = \sum_{j=D,T,M}(\dot{\mathbf{u}})_j \tag{5.11}$$

The $(\dot{\mathbf{u}})_j$ are just the functional Fourier transforms of $\frac{\partial}{\partial \xi_j}$. We now rewrite this as

$$0 = \frac{d}{dt} \sum_{j=D,T,M} P_j \tag{5.12}$$

where the "partial probabilities"

$$P_j(t) \equiv \int_{-\infty}^{t} dt' \left(\frac{dP}{dt'}\right)_j + P_j(-\infty) = -\int_{-\infty}^{t} dt' \int d\mathbf{x} \, \frac{\delta}{\delta \mathbf{u}} \cdot (\dot{\mathbf{u}})_j \, P[\mathbf{u}] + P_j(-\infty) \tag{5.13}$$

The constants of integration $P_j(-\infty)$ are arbitrary; if we choose them to satisfy

$$\sum_{j=D,T,M} P_j(-\infty) = 1 \tag{5.14}$$

then

$$\sum_{j=D,T,M} P_j(t) = 1 \tag{5.15}$$

for all t. Similarly, $P_j(t)$ will remain bounded in the interval [0,1] for all t if initially so bounded, just as

$$P[\mathbf{u}(\mathbf{x},t)] \prod_{\mathbf{x}} d\mathbf{u}(\mathbf{x},t) = P[\mathbf{u}(\mathbf{x},0)] \prod_{\mathbf{x}} d\mathbf{u}(\mathbf{x},0) \tag{5.16}$$

guarantees the boundedness of P[u].

Hence we see that the *time rate of change of the probability* contributed by dissipation, transfer, and magnetic processes (with total probability change = 0 in steady state) is formally equivalent to the time rate of change of the *probability contributed by dissipative, transfer, and magnetic "states"* (with total probability change = 0 in a closed system.) In other words, we have shifted our perspective slightly, from solving for stationary probabilities with independent variable **u** and parameter j, to solving for nonstationary probabilities with independent variable j and parameter **u**. The F_j's represent the net probability flux or transition rate between pairs of states. Note that when $\frac{\partial \phi}{\partial \xi}$ vanishes, we obtain that $\nu \nabla^2 \omega$ weakly vanishes (its ensemble average with any n-point function of **u**, **B**, c, and their derivatives vanishes) implying that P_D vanishes. In other words, P_D may be interpreted as a measure of the intensity of vortex reconnection, P_T as a measure of vortex stretching, and P_M as a measure of magnetic stretching.

One could try to solve the Vlasov-type equation (5.10) directly. For example, for the one-point velocity pdf, one could transform to the local principal axes frame where

$$\frac{\partial \dot{u}_i}{\partial u_j} \sim \delta_{ij} \tag{5.17}$$

Then one may verify that

$$P(\mathbf{u}(\mathbf{x})) = \prod_{i=1}^{3} \frac{1}{\dot{u}_i} + \int^{\mathbf{u}} d\mathbf{v} \cdot \mathbf{n} \times \dot{\mathbf{u}} \tag{5.18}$$

where

$$n_j(\mathbf{u}) \equiv \prod_{i\neq j} \frac{1}{\dot{u}_i} + \text{jth component of null eigenvector of}$$

$$\begin{pmatrix} \dot{u}_3 k_1 & \dot{u}_3 k_2 & -\dot{u}_1 k_1 - \dot{u}_2 k_2 \\ \dot{u}_2 k_1 & -\dot{u}_1 k_1 - \dot{u}_3 k_3 & \dot{u}_2 k_3 \\ -\dot{u}_2 k_2 - \dot{u}_3 k_3 & \dot{u}_1 k_2 & \dot{u}_1 k_3 \end{pmatrix} e^{\mathbf{k}\cdot\mathbf{u}} \tag{5.19}$$

However, this becomes intractable (and not particularly illuminating) if one is interested in more than just the velocity at one point. Moreover, this solution satisfies equation (5.10) pointwise; one may have to consider not the local condition (integrand of equation (5.10)) but the global one (equation (5.10)) in order to obtain stationary solutions of interest.

2.6. Driven steady state for particular generating functional

Consider the case in which the $F_j[\phi]$ are arbitrary (subject to equation (5.6)) but $\phi = \phi(B_1(\xi)B_2(\eta)B_3(\zeta))$. Then equations (5.3)-(5.5) become

$$\begin{pmatrix} \alpha_1 & 1-\alpha_2 & 0 \\ 0 & \alpha_2 & 1-\alpha_3 \\ 1-\alpha_1 & 0 & \alpha_3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \frac{1}{\phi' B} \tag{6.1}$$

$$k_j \equiv \frac{B_j'(\xi_j)}{B_j(\xi_j)} = \text{constant by eqn.(5.1)}$$

$$B \equiv B_1(\xi)B_2(\eta)B_3(\zeta)$$

If the determinant of the α matrix vanishes, detailed balance can occur. More generally, the stationary ensemble exhibits a net probability flux between pairs of states or a nonzero cyclic flow of probability through the three states. Equivalently, one may write the matrix equation as

$$\begin{pmatrix} k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \\ -k_1 & 0 & k_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} F_1/\phi' B - k_2 \\ F_2/\phi' B - k_3 \\ F_3/\phi' B - k_1 \end{pmatrix} \tag{6.2}$$

Because the determinant of the k matrix vanishes, one may solve for the net transition rates

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} k_2 \\ k_3 \\ k_1 \end{pmatrix} \phi' B \tag{6.3}$$

$$\text{when} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \text{null eigenvector of k matrix} \tag{6.4}$$

In general, one cannot set the net transition rates equal to zero because $\mathbf{k} \neq$ null eigenvector of the α matrix (i.e., regardless of the choice of k_j, one cannot solve for

α_j given vanishing F_j because the k matrix is noninvertible.) Hence, these solutions differ fundamentally (Graham 1973) from the previous ones which exhibited detailed balance because the entropy production

$$\dot{S} = \sum_{j=D,T,M} \dot{P}_j \log P_j \neq 0 \quad \text{for } F_j \neq 0 \tag{6.5}$$

Translating back into hydromagnetic language, we have

$$\frac{\partial}{\partial \xi} \to \nabla \times \nabla \times \nu \frac{\delta}{\delta \mathbf{g}(\mathbf{x})} \tag{6.6}$$

$$\frac{\partial}{\partial \eta} \to \nabla \times \frac{\delta}{\delta \mathbf{f}(\mathbf{x})} \times \frac{\delta}{\delta \mathbf{g}(\mathbf{x})} \tag{6.7}$$

$$\frac{\partial}{\partial \zeta} \to \nabla \times \frac{\delta}{\delta \mathbf{l}(\mathbf{x})} \times \frac{1}{\mu\rho} \frac{\delta}{\delta \mathbf{h}(\mathbf{x})} \tag{6.8}$$

By equation (6.1), $B_j = \exp(k_j \xi_j)$ where

$$k_1 \xi \to \int_{-\infty}^{\infty} d\mathbf{x}\, \mathbf{g}(\mathbf{x}) \cdot \nabla \times A_D \tag{6.9}$$

$$k_2 \eta \to \int_{-\infty}^{\infty} d\mathbf{x}\, [\mathbf{g}(\mathbf{x}) \cdot \nabla \times (A_T - A_D) + \mathbf{f}(\mathbf{x}) \cdot A_T] \tag{6.10}$$

$$k_3 \zeta \to \int_{-\infty}^{\infty} d\mathbf{x}\, [\mathbf{h}(\mathbf{x}) \cdot A_M + \mathbf{l}(\mathbf{x}) \cdot \nabla \times A_M] \tag{6.11}$$

These expressions are the minimum required to exhibit nonvanishing $\partial B_j(\xi_j)/\partial \xi_j$; they incorporate Ampere's Law and the definition of vorticity.

By imposing that the derivatives satisfy

$$\frac{\partial (k_m \xi_m)}{\partial \xi_n} = k_m \delta_{mn} \tag{6.12}$$

we obtain

$$\mathbf{k}_1 \equiv \nabla \times \nabla \times \nu \nabla \times A_D \tag{6.13}$$

$$\mathbf{k}_2 \equiv \nabla \times (A_T \times \nabla \times (A_T - A_D)) \tag{6.14}$$

$$\mathbf{k}_3 \equiv \nabla \times (\nabla \times \frac{A_M}{\mu\rho}) \times A_M \tag{6.15}$$

$$\nabla \times \nabla \times \nabla \times (A_T - A_D) = 0 \tag{6.16}$$

$$\nabla \times (A_T \times \nabla \times A_D) = 0 \tag{6.17}$$

"Uncurling" equations (6.13)-(6.17) yields "constants" of integration:

$$-\nabla^2 A_D = \nabla\phi_D + \mathbf{k}_1 \times \mathbf{x}/3\nu \tag{6.18}$$

$$-\nabla^2 A_T = \nabla\phi_T + \mathbf{k}_1 \times \mathbf{x}/3\nu \tag{6.19}$$

$$A_T = n_1(\mathbf{x})\nabla \times A_D + P_1, \quad P_1 \times \nabla \times A_D = \nabla\psi_1 \tag{6.20}$$

$$A_T = n_2(\mathbf{x})\nabla \times A_T + P_2, \quad P_2 \times \nabla \times A_T = \nabla\psi_2 + \mathbf{k}_2 \times \mathbf{x}/3 \tag{6.21}$$

$$A_M = n_3(\mathbf{x})\nabla \times A_M + P_3, \quad P_3 \times \nabla \times A_M = \nabla\psi_3 + \mu\rho\mathbf{k}_3 \times \mathbf{x}/3 \tag{6.22}$$

while the magnetic equation (1.8) with $\lambda = 0$ becomes

$$A_T \times A_M = \nabla \times A_M + \nabla\psi_4 \tag{6.23}$$

(Compare this with the MFE prediction:)

$$A_T \times A_M = \alpha A_M + \eta\nabla \times A_M \tag{6.23a}$$

This gives us enough unknown fields to satisfy the equations (6.18)-(6.23).

Solenoidality will be satisfied if

$$\nabla \cdot A_T = \nabla \cdot A_M = 0 \tag{6.24}$$

which suggests a vector potential or stream function representation for A_M and A_T. For the case in which ϕ is linear in B, the stationarity (vorticity) condition is simply

$$k_1 + k_2 + k_3 = 0 \tag{6.25}$$

For ϕ analytic in B or containing negative powers of B, cancellation of each power of B requires that additional conditions be imposed, which we will not discuss further here.

In the context of our "partial probability" picture, a solution with nonzero transition rates corresponds to a probability packet cycling consecutively between ω stretching, B stretching (and reconnection, by the induction equation), and ω reconnection. If instead one writes the Hopf equations in terms of $\mathbf{u}\cdot\nabla\mathbf{u}$ and $\mathbf{B}\cdot\nabla\mathbf{B}$, depletion of state "M" implies zero magnetic tension, which may have implications for coronal mass ejection (Low 1990).

2.7. Driven steady state for particular detailed imbalance

Consider the case in which there is no detailed balance and the (hence nonvanishing) $F_j = F_j(\phi)$. Then equations (5.3)-(5.5) become

$$\left(\alpha_1 \frac{\partial}{\partial\xi} + (1-\alpha_2)\frac{\partial}{\partial\eta}\right)\phi = F_1(\phi) \tag{7.1}$$

$$\left(\alpha_2 \frac{\partial}{\partial \eta} + (1-\alpha_3)\frac{\partial}{\partial \zeta}\right)\phi = F_2(\phi) \tag{7.2}$$

$$\left(\alpha_3 \frac{\partial}{\partial \zeta} + (1-\alpha_1)\frac{\partial}{\partial \xi}\right)\phi = F_3(\phi) \tag{7.3}$$

again with the stationarity condition

$$\sum_{j=1}^{3} F_j(\phi) = 0 \tag{7.4}$$

Rewrite this as

$$\alpha_1 \frac{\partial S_1}{\partial \xi} + (1-\alpha_2)\frac{\partial S_1}{\partial \eta} = 1 \tag{7.5}$$

$$\alpha_2 \frac{\partial S_2}{\partial \eta} + (1-\alpha_3)\frac{\partial S_2}{\partial \zeta} = 1 \tag{7.6}$$

$$\alpha_3 \frac{\partial S_3}{\partial \zeta} + (1-\alpha_1)\frac{\partial S_3}{\partial \xi} = 1 \tag{7.7}$$

where

$$S_j(\phi) \equiv \int \frac{d\phi}{F_j(\phi)} \tag{7.8}$$

Solving, we obtain

$$S_1 = S_1\left(\int^{\xi} d\xi'\,(1-\alpha_2) - \int_0^{\eta} d\eta'\,\alpha_1\right) + \int^{\xi} \frac{d\xi'}{\alpha_1} \tag{7.9}$$

$$S_2 = S_2\left(\int^{\eta} d\eta'\,(1-\alpha_3) - \int_0^{\zeta} d\zeta'\,\alpha_2\right) + \int^{\eta} \frac{d\eta'}{\alpha_2} \tag{7.10}$$

$$S_3 = S_3\left(\int^{\zeta} d\zeta'\,(1-\alpha_1) - \int_0^{\xi} d\xi'\,\alpha_3\right) + \int^{\zeta} \frac{d\zeta'}{\alpha_3} \tag{7.11}$$

$$\text{where } \frac{\partial \alpha_1}{\partial \xi} = \frac{\partial \alpha_2}{\partial \eta} = \frac{\partial \alpha_3}{\partial \zeta}, \qquad \alpha_j(0) = 1 \tag{7.12}$$

and α_1, α_2 and α_3 are independent of η, ζ and ξ respectively. We translate back into hydromagnetic language as in the previous section except that we choose

$$k_1\xi \to \int_{-\infty}^{\infty} d\mathbf{x}\,\mathbf{g}(\mathbf{x}) \cdot \nabla \times A_D \tag{7.13}$$

$$k_2\eta \to \frac{1}{2}\left(\int_{-\infty}^{\infty} d\mathbf{x}\,\left[\mathbf{g}(\mathbf{x}) \cdot \nabla \times (A_T - A_D) + \mathbf{f}(\mathbf{x}) \cdot A_T\right]\right)^2 \tag{7.14}$$

$$k_3\zeta \rightarrow \frac{1}{2}\left(\int_{-\infty}^{\infty} d\mathbf{x}\,[\mathbf{h}(\mathbf{x})\cdot A_M + \mathbf{l}(\mathbf{x})\cdot\nabla\times A_M]\right)^2 \tag{7.15}$$

The quadratic powers are the minimum required to give nonvanishing $\partial(k_2\eta)/\partial\eta$ and $\partial(k_3\zeta)/\partial\zeta$. Stationarity implies

$$\sum_{j=1}^{3}\frac{\partial\phi}{\partial S_j} = 0 \tag{7.16}$$

which is satisfied if

$$\phi(S_1, S_2, S_3) = \phi\big(b_1S_1 + b_2S_2 - (b_1+b_2)S_3\big) \tag{7.17}$$

or more generally

$$\phi(S_1, S_2, S_3) = \int db_1db_2\ \rho(b_1, b_2)\ \phi\big(b_1S_1 + b_2S_2 - (b_1+b_2)S_3\big) \tag{7.18}$$

Normalization of probability imposes the constraint

$$\phi(0,0,0) = 1 \tag{7.19}$$

For the case of constant α_j, condition (6.12) implies that S_j is linear in the arguments displayed in (7.9)-(7.12) (not to be confused with (7.8)). The resulting solutions are analogous to the secular solutions of the wave equation, just as the solutions of §2.2-2.4 are analogous to the propagating solutions of the wave equation.

2.8. Realizability

For the solution described in §2.6, in the case that ϕ is linear in B, the probability density for the velocity and magnetic fields is a sum of (functional) delta functions, each of the form $\prod_{\mathbf{x}}\delta(\mathbf{u}(\mathbf{x})-\mathbf{A}_T(\mathbf{x}))\delta(\mathbf{B}(\mathbf{x})-\mathbf{A}_M(\mathbf{x}))$ multiplied by similar factors for the vorticity, current, scalar, and scalar source (and pressure and density, if the flow is compressible). Realizability imposes the constraint that the coefficients multiplying the delta functions be positive. For the solution described in §2.7, for the case of constant α_j, realizability constrains the functional dependence of ϕ upon the argument displayed in equation 7.18. For example, if ϕ is chosen to be exponential, the probability density for the velocity, vorticity, magnetic field, and current would be joint Gaussian, hence positive everywhere as desired. Realizability remains to be verified for the solutions exhibiting detailed balance.

More generally, it has been suggested (Kraichnan 1991) that realizability could be imposed upon the probability $P[\mathbf{u}]$ by introducing a complex probability amplitude $\psi[\mathbf{u}]$ and its complex conjugate such that

$$P[\mathbf{u}] = \psi^*[\mathbf{u}]\psi[\mathbf{u}] \tag{8.1}$$

which is positive semidefinite as desired. Although the Hopf equation could be written in terms of $\psi[\mathbf{u}]$ or its functional Fourier transform $\tilde{\psi}[\mathbf{f}]$ where

$$\phi[\mathbf{f}] = \sum_{\{\mathbf{g}(\mathbf{x})\}} \tilde{\psi}^*[\mathbf{g}-\mathbf{f}]\tilde{\psi}[\mathbf{g}] \tag{8.2}$$

it is not immediately obvious how to solve this infinite system of coupled equations or, equivalently, under what circumstances a given solution of the Hopf equation can be represented in the form (8.2).

One may, however, consider the equation of motion for $\tilde{\psi}[\mathbf{f}]$ rather than for $\phi[\mathbf{f}]$. If one imposes the single- valuedness of evolution under equations (1.1)-(1.3) upon the probability *amplitude* (a condition analogous to but slightly-more stringent than equation (5.16)), then, letting T denote the time evolution operator, we have (symbolically)

$$T\tilde{\psi}[\mathbf{f}] = \int T\{d\mathbf{u}\,\psi[\mathbf{u}]\}e^{-i\int \mathbf{f}\cdot\mathbf{u}} = \int d\{T^{-1}\mathbf{u}\}\;\psi[T^{-1}\mathbf{u}]e^{-i\int \mathbf{f}\cdot\mathbf{u}}$$
$$= \int d\mathbf{v}\;\psi[\mathbf{v}]e^{-i\int \mathbf{f}\cdot T\mathbf{v}} \tag{8.3}$$

Hence, $\tilde{\psi}[\mathbf{f}]$ obeys the same Hopf equation as $\phi[\mathbf{f}]$. In other words, any solution of the Hopf equation (e.g., the solutions exhibiting detailed balance) can be inserted into equation (8.2) to obtain a realizable ϕ. Guaranteeing realizability in this manner does have a drawback, however; the functional integration in equation (8.2) must be performed (at least in the vicinity of $\mathbf{f} = 0$) in order to evaluate moments (unless the moments are of at least first order in all vector components of all physical fields in the exponent of equation (1.6)).

A certain class of moments can be evaluated without performing functional integration. Consider moments obtained by taking functional derivatives of

$$\tilde{P}[\mathbf{f}] \equiv \tilde{\psi}^*[\mathbf{f}]\tilde{\psi}[\mathbf{f}] \tag{8.4}$$

rather than of $\phi[\mathbf{f}]$. $\tilde{P}[\mathbf{f}]$ may be interpreted as the probability density for the conjugate field $\mathbf{f}$; its density in velocity space is analogous to the Wigner distribution function. One may verify that

$$i\frac{\delta \tilde{P}[\mathbf{f}]}{\delta \mathbf{f}(\mathbf{x})}|_{\mathbf{f}=0} = \sum_{\{\mathbf{u}(\mathbf{x})\}}\sum_{\{\mathbf{v}(\mathbf{x})\}} \mathbf{u}(\mathbf{x})\psi^*[\mathbf{u}+\mathbf{v}]\psi[\mathbf{v}] \tag{8.5}$$

$$= \mathrm{Tr}\big[\tilde{\rho}\Delta\mathbf{u}(\mathbf{x})\big] = \langle\Delta\mathbf{u}(\mathbf{x})\rangle \tag{8.6}$$

$$(\tilde{\rho})_{u,v} \equiv \psi[\mathbf{u}]\psi^*[\mathbf{v}], \qquad (\Delta\mathbf{u})_{u,v} \equiv \mathbf{u}-\mathbf{v} \tag{8.7}$$

If the (probability) density matrix $\tilde{\rho}$ is diagonal in the velocity-realization basis, then there is no phase correlation between states with different $\mathbf{u}(\mathbf{x})$ and

$$\langle\Delta\mathbf{u}(\mathbf{x})\rangle = 0 \tag{8.8}$$

In other words, $\langle\Delta\mathbf{u}(\mathbf{x})\rangle$ is a measure of the *coherent* velocity spread at a given point or of the phase coherence or interference between different realizations. The probability amplitudes satisfy the Hopf equation and hence can exhibit "propagative" behavior in the variables $\mathbf{f}$ and $\mathbf{x}$, analogous to the propagation of conventional wavefunctions in $\mathbf{x}$ and t. However, because the conjugate fields are dummy variables, it is not clear that interference between different realizations (analogous to the Aharonov-Bohm effect) would be physically observable.

In order to compute moments without having to perform a functional integration, we require that $\phi[\mathbf{f}]$ be local in $\mathbf{f}$ yet that $P[\mathbf{u}]$ be positive semidefinite. Consider (in one dimension for notational simplicity) the following piecewise prescription:

$$\begin{aligned} P[u] &\equiv \int_0^\infty df\ \phi[f]\ e^{-fu} \text{ for } u > 0 \\ &\equiv \int_{-\infty}^0 df\ \phi[f]\ e^{-fu} \text{ for } u < 0 \end{aligned} \tag{8.9}$$

where $\phi[f]$ takes the form given in equation (8.4). This expression for $P[u]$ is positive semidefinite for all real u. At $u = 0$, $P[u]$ is undefined; however, this does not affect moments since $P[0]$ is weighted by $u = 0$ in the integral over u. One may verify (using the Cauchy theorem) that taking functional derivatives of $\phi[f]$ and then setting $f = 0$ yields moments of the *absolute value* of u. Given the evolution equation (5.10) for $P[u]$, ϕ will satisfy the Hopf equation if surface terms $\phi[f = 0]$ and $\delta\phi[f = 0]/\delta f$ vanish (these arise from the functional integration by parts which is implicit in $\dot{u}P$.) The vanishing of the surface terms can be achieved by subtracting the constant $\psi[f = 0]$ from $\psi[f]$, which does not alter the validity of the Hopf equation for ψ. ϕ will satisfy the Hopf equation if ψ does and if cross terms involving functional derivatives of ψ and ψ^* vanish, which in turn can be achieved if equation (2.19) is satisfied and if one requires that the real part

$$Re\{\nabla \times (\mathbf{G}^{(-1)} \times \mathbf{G})\} = 0 \tag{8.10}$$

Normalization of probability (7.19) is replaced by the condition

$$\int_{-\infty}^{\infty} df\ \frac{\phi[f]}{|f|} = 1 \tag{8.11}$$

which, by linearity of the Hopf equation, can be satisfied by multiplying ψ by the appropriate constant factor. Whether this approach can be modified to easily generate moments of non-absolute- valued quantities remains to be seen.

2.9. Time dependence

Time dependence may be incorporated into the framework of §2.5-7 by adding a term proportional to t to the arguments of ϕ in equations 5.2 and 7.18 and by relaxing the stationarity constraints (5.6, 6.25, 7.16). The coefficient of t in the argument of ϕ is chosen to be minus the sum of the coefficients of ξ, η and ζ.

Alternatively, one may take a simpler approach for the class of flow ensembles exhibiting "limited statistical linearity" (defined below). Consider the incompressible MHD equations without the buoyancy term or scalar equation. If we add a multiple β of the induction equation to the velocity equation, we obtain

$$-\frac{\partial}{\partial t}[\mathbf{u}(\mathbf{x},t)+\beta\mathbf{B}(\mathbf{x},t)] = \mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{B}\cdot\nabla\mathbf{B}+\beta\mathbf{u}\cdot\nabla\mathbf{B}-\beta\mathbf{B}\cdot\nabla\mathbf{u}+\frac{\nabla p}{\rho}-\frac{1}{R}\nabla^2\mathbf{u}-\frac{\beta}{R_M}\nabla^2\mathbf{B} \tag{9.1}$$

Setting $\beta \equiv \pm 1$ and defining

$$\mathbf{v} \equiv \mathbf{u}+\mathbf{B}, \quad \mathbf{w} \equiv \mathbf{u}-\mathbf{B} \tag{9.2}$$

we obtain

$$-\frac{\partial}{\partial t}\mathbf{v}(\mathbf{x},t) = \mathbf{w}\cdot\nabla\mathbf{v} + \frac{\nabla p}{\rho} - a\nabla^2\mathbf{v} - b\nabla^2\mathbf{w} \tag{9.3}$$

$$-\frac{\partial}{\partial t}\mathbf{w}(\mathbf{x},t) = \mathbf{v}\cdot\nabla\mathbf{w} + \frac{\nabla p}{\rho} - a\nabla^2\mathbf{w} - b\nabla^2\mathbf{v} \tag{9.4}$$

where

$$a \equiv \frac{1}{2}\left(\frac{1}{R}+\frac{1}{R_M}\right), \quad b \equiv \frac{1}{2}\left(\frac{1}{R}-\frac{1}{R_M}\right) \tag{9.5}$$

Pressure may be eliminated by using incompressibility and the solenoidality of the magnetic field and inserting the projection operator $\{1-(\nabla\nabla/\nabla^2)\}$ in front of the nonlinear terms.

A solution of the Hopf equations corresponding to equations (9.3-4) may be obtained by choosing a moment-generating functional

$$\phi[\mathbf{f},\mathbf{g},t] \equiv \left\langle e^{i\int_{-\infty}^{\infty} dx\,[\mathbf{f}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})+\mathbf{g}(\mathbf{x})\cdot\mathbf{w}(\mathbf{x})]}\right\rangle \tag{9.6}$$

which manifestly closes the equations, i.e., a ϕ whose off-diagonal second functional derivatives are linear combinations of its first functional derivatives. A moment-generating functional for $\mathbf{v},\mathbf{w}$ which satisfies this requirement is

$$\phi[\mathbf{f},\mathbf{g},t] \equiv \phi_1[\mathbf{f},t]\,e^{\int_{-\infty}^{\infty} dx\,\mathbf{g}\cdot\mathbf{c}_1} + \phi_2[\mathbf{g},t]\,e^{\int_{-\infty}^{\infty} dx\,\mathbf{f}\cdot\mathbf{c}_2} \tag{9.7}$$

where ϕ_1 and ϕ_2 are arbitrary. One may readily verify for this ϕ that

$$\frac{\delta^2\phi}{\delta f_i(\mathbf{x})\delta g_j(\mathbf{x})} = c_{1j}\frac{\delta\phi}{\delta f_i(\mathbf{x})} + c_{2i}\frac{\delta\phi}{\delta g_j(\mathbf{x})} - c_{1j}c_{2i}\phi \tag{9.8}$$

This implies that

$$\langle\mathbf{u}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t)\rangle = (\mathbf{c}_2-\mathbf{c}_1)\times\langle\mathbf{u}(\mathbf{x},t)\rangle - (\mathbf{c}_2+\mathbf{c}_1)\times\langle\mathbf{B}(\mathbf{x},t)\rangle + \mathbf{c}_1\times\mathbf{c}_2 \tag{9.9}$$

which offers another rigorous alternative to the conventional MFE model. Note that diagonal second functional derivatives such as $\delta^2\phi/\delta f_i\delta f_j$ do not reduce to linear functional derivatives; correlation functions such as $\langle u_i(\mathbf{x})u_j(\mathbf{x})\rangle$ or $\langle B_i(\mathbf{x})B_j(\mathbf{x})\rangle$ cannot in general be written in terms of $\langle u_i(\mathbf{x})\rangle$ and $\langle B_i(\mathbf{x})\rangle$.

The resulting closed system of 2 linear equations for the first functional derivatives of ϕ is readily diagonalized. For constant $\mathbf{c}_1$ and $\mathbf{c}_2$, the term in the Hopf equation corresponding to $\mathbf{w}\cdot\nabla\mathbf{v}+\frac{\nabla p}{\rho}$ in equation (9.3) simplifies:

$$\{1-(\nabla\nabla/\nabla^2)\}\nabla\cdot\frac{\delta}{\delta\mathbf{g}}\frac{\delta\phi}{\delta\mathbf{f}}\rightarrow\mathbf{c}_1\cdot\nabla\frac{\delta\phi}{\delta\mathbf{f}} \tag{9.10}$$

(similarly for equation (9.4)) Hence, one obtains propagating diffusive modes (an admixture of velocity and magnetic field) governed by a dispersion relation $\omega(\mathbf{k})$ which satisfies

$$(\omega-\mathbf{c}_1\cdot\mathbf{k}-iak^2)(\omega-\mathbf{c}_2\cdot\mathbf{k}-iak^2)+b^2k^4=0 \tag{9.11}$$

For $\mathbf{c}_1=\mathbf{c}_2$, the eigenmodes reduce to excitations which are purely kinetic or magnetic, diffusing on purely viscous or resistive time scales, respectively. For $R=R_M$, $\mathbf{v}$ and $\mathbf{w}$ decouple, corresponding to modes whose velocity and magnetic field oscillate in and out of phase, respectively. The case of nonconstant $\mathbf{c}_1$ and $\mathbf{c}_2$ and higher-order correlation functions may be computed by solving analogous but larger, inhomogeneous closed systems of equations, involving a diffusive kernel. Whether these modes are purely statistical (appearing only in the ensemble-averaged flow) or play a role in individual realizations remains to be clarified.

3. Future plans

The Hopf functional approach offers a new exact method for obtaining stationary MHD states, both with and without detailed balance, which generalize the usual ideal static or force-free, equilibrium, nonturbulent states. The treatment of time dependence beyond the usual, initial linear stability regime also appears possible. Closed-form analytic expressions for the mean emf and other correlation functions emerge without making perturbative, phenomenological, or statistical assumptions. Recognizing the wavelike character of the functional differential equations enables one to reduce them to a system of ordinary differential equations, a reduction of the number of degrees of freedom in the problem from N^{3L^3} to L^3 where N and L are large numbers on the order of the number of permitted values for velocity at a given point and the spatial extent of the system, respectively. The solutions obtained are not necessarily unique or complete but merely illustrative. The possibility of superposition indicates that matching to the mean flow may be necessary to determine relative strengths of different solutions unless it is possible to close the equations by substituting, for example, the mean emf and Lamb vector back into the stationary vorticity equation. Initial conditions or a variational criterion may play a role in selecting the correct ensemble or discarding spurious ones. Computation of moments and of the probability of arbitrary realizations of the velocity, magnetic field, scalar, and scalar source is currently under investigation.

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