The helical decomposition and the instability assumption

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1. Motivations and objectives

Direct numerical simulations (Domaradzki & Rogallo, 1990, Yeung & Brasseur, 1991, Okhitani & Kida, 1992) show that the triadic transfer function $T(k,p,q)$ peaks sharply when q (or p) is much smaller than k. The triadic transfer function $T(k,p,q)$ gives the rate of energy input into wavenumber k from all interactions with modes of wavenumber p and q, where k,p,q form a triangle. This observation was thought to suggest that energy is cascaded downscale through non-local interactions with local transfer and that there was a strong connection between large and small scales. Both suggestions were in contradiction with the classical Kolmogorov picture of the energy cascade.

In fact, the large peaks in $T(k,p,q)$ have no direct physical significance. Their origin lies in the Fourier representation of the differential advection (i.e. distortion) of small scales by large scales. It is only the *difference* between the large peaks which has a physical meaning. That difference represents an advection in Fourier space, i.e. a $\partial/\partial k$ term (Waleffe, 1992). With regard to the energy cascade, the large local transfers in non-local triads are not the primary downscale cascading interactions, their net effect is actually a *reverse* cascade, in the inertial range. A worthy note on this point is that the non-local interactions with large *local downscale* transfer are also present in 2D turbulence, and it is well-known that there the energy cascade is reversed in a $-5/3$ range.

The helical decomposition has been found useful in distinguishing between kinematically independent interactions. That analysis has gone beyond the question of non-local interaction with local transfer. In particular, an assumption about the statistical direction of triadic energy transfer in any kinematically independent interaction was introduced (the *instability assumption*). That assumption is not necessary for the conclusions about non-local interactions with local transfer recalled above. In the case of turbulence under rapid rotation, the instability assumption leads to the prediction that energy is transferred in spectral space from the poles of the rotation axis toward the equator. The instability assumption is thought to be of general validity for any type of triad interactions (e.g. internal waves). The helical decomposition and the instability assumption offer detailed information about the homogeneous statistical dynamics of the Navier-Stokes equations.

The objective of this work was to explore the validity of the instability assumption and to study the contributions of the various types of helical interactions to the energy cascade and the subgrid-scale eddy-viscosity. This was done in the context of spectral closures of the Direct Interaction or Quasi-Normal type.

2. Accomplishments

2.1 Helical decomposition and non-local interactions

The helical decomposition of the velocity field shows that there are two distinct classes of triad interactions. Only one of these classes is such that the energy transfer is mostly between the two longest legs of the triad when the third leg is much smaller than the other two (non-local triad). That class of helical interactions is then wholly responsible for the large local transfers in non-local triads observed in DNS (Domaradzki and Rogallo, 1990, Yeung and Brasseur, 1991, Okhitani and Kida, 1992). Somewhat surprisingly, however, the sum of all such helical interactions gives a *reverse cascade* of energy, from large to small wavenumbers, even though the large local transfers are from the medium to the largest leg of the triad. This is an exact result for an extended inertial range. The meaning of that result is that the peaks in the transfer function $T(k,p,q)$, observed when q, say, is much smaller than k and p, can not be interpreted as representing the downscale transfer of energy.

Interactions between wavevectors in Fourier space are triadic as a result of the quadratic non-linearity of the Navier-Stokes equations. The incompressibility constraint requires that the velocity vector be perpendicular to the wavevector. That leaves only two degrees of freedom per wavevector and eight kinematically independent triad interactions. The helical decomposition is particularly appropriate because the non-linear term and each triad interaction independently conserves both energy and helicity. There are two helical modes per wavevector, a "+" mode of maximum helicity and a "−" mode of minimum helicity. The triadic transfer function is the sum of the eight kinematically independent helical triadic transfers,

$$T(k,p,q) = \sum_{i=1}^{8} T^{(i)}(k,p,q).$$

Normalizing all wavenumbers in the triad by the middle one, let v, 1, w represent the smallest, middle, and largest wavenumbers, respectively, ($v \le 1 \le w \le 1+v$) and s_v, s_1, s_w the signs of the corresponding helical modes. A direct consequence of energy and helicity conservation is the following important relation,

$$\frac{T^{(i)}(v,1,w)}{s_w w - s_1} = \frac{T^{(i)}(1,w,v)}{s_v v - s_w w} = \frac{T^{(i)}(w,v,1)}{s_1 - s_v v}. \tag{1}$$

One nice advantage of this decomposition is that it distinguishes clearly between *non-local interactions with non-local transfers* and *non-local interactions with local transfers*. Non-local interactions (i.e. $v \ll 1 \sim w$) of the $s_1 = s_w$ type have local energy transfer, while non-local $s_1 = -s_w$ interactions have non-local transfer. This can be deduced from (1). Hence, the *non-local interactions with large local transfers* observed in the DNS are the result of $s_1 = s_w$ interactions.

2.2 The instability assumption

The analysis sketched above highlights some of the characteristics of the triadic transfers but leaves open the question of statistical direction of the energy exchanges.

These directions are determined by the *Instability assumption: statistically, the triadic energy transfer is from the mode whose coefficient in the triadic equations (1) is opposite in sign to the other two coefficients.* In other words, $T^{(i)}(v,1,w)$ is negative if $s_1 = -s_w$ (and thus $T^{(i)}(1,w,v)$ and $T^{(i)}(w,v,1)$ are positive), and $T^{(i)}(1,w,v)$ is negative if $s_1 = s_w$ (with $T^{(i)}(v,1,w)$ and $T^{(i)}(w,v,1)$ positive). This assumption was motivated by the stability characteristics of the triadic interactions in (Waleffe, 1992). It amounts to an assumption about the triple correlations.

The instability assumption is in agreement with the numerical observations that energy is transferred statistically *from the medium to the largest wavenumber* in non-local triads. As mentioned above, the large local transfers in non-local triads can only come from $s_1 = s_w$ interactions, and thus the numerical observations imply that the *sum* of all $s_1 = s_w$ interactions must extract energy from the medium wavenumber and feed energy into the longest wavenumber. The instability assumption leads to the same result through the stronger statement that this is so for *each* interaction $s_1 = s_w$.

Analyses of numerical simulation data have focused so far only on the total triadic energy transfer $T(k,p,q) = \sum_i T^{(i)}(k,p,q)$. The total triadic energy transfer is the net result of 8 kinematically independent interactions. In a recent paper, Okhitani and Kida (1992) have classified interactions according to the sign of the total triadic transfer into each leg of the triad. They associate to each interaction a triplet of signs defined as (sign$[T(w,v,1)]$, sign$[T(1,w,v)]$, sign$[T(v,1,w)]$). These triplets can *a priori* take any of 6 different values ($(+,+,+)$ and $(-,-,-)$ are not allowed from energy conservation). Their classification should not be confused with the helical classification used in this and previous papers. They observe that statistically only $(+,+,-)$, $(+,-,+)$ and $(+,-,-)$ appear. The instability assumption is not only consistent with this observation but also predicts it. The total energy transfer is a combination of the two helical classes of transfers determined by the instability assumption. These two helical classes are $(+,+,-)$ when $s_1 = -s_w$ and $(+,-,+)$ when $s_1 = s_w$. The only possible *net sums* of these two classes of helical interactions are $(+,+,-)$, $(+,-,+)$ *and* $(+,-,-)$, exactly as observed by Okhitani and Kida. In other words, it is impossible to observe a net interaction in which the largest wavenumber in a triad loses energy on average.

2.3 Comparison with the closures

The signs of the helical triadic transfers predicted by the instability assumption have been compared to those obtained from the spectral closures (Quasi-Normal or Direct Interaction type). All such closures prescribe the same following form for the shell-averaged triadic transfer function $T^{(i)}(k,p,q)$ due to the i-th interaction (for isotropic flow):

$$T^{(i)}(k,p,q) = \pi^2 kpq\, gg^*\, \theta_{kpq}\, (s_q q - s_p p) \times \\ [(s_q q - s_p p)U(p)U(q) + (s_k k - s_q q)U(k)U(q) + (s_p p - s_k k)U(k)U(p)]$$

where $U(k) = \langle \mathbf{u}(\mathbf{k}) \cdot \mathbf{u}^*(\mathbf{k}) \rangle$, and $gg^* = (s_k k + s_p p + s_q q)^2 Q^2/(16k^2p^2q^2)$, with $Q^2 = 2k^2p^2 + 2p^2q^2 + 2q^2k^2 - k^4 - p^4 - q^4$ (g differs by a factor of 2 from its definition in (Waleffe, 1992)). The sign coefficients (s_k, s_p, s_q) denote the helical interaction under consideration, i.e. s_k mode for wavenumber k, etc. The parameter θ_{kpq} is a relaxation time scale for the triple moments. The relaxation is due to both viscous and non-linear effects. A simple prescription for θ_{kpq}, is $\theta_{kpq} = C[(k^3E(k))^{1/2} + (p^3E(p))^{1/2} + (q^3E(q))^{1/2}]^{-1}$ where the constant C is chosen to fit the Kolmogorov constant and $E(k)$ is the energy spectrum.

All that is needed for the comparison with the instability assumption is that θ_{kpq} be positive. It can be shown analytically (Waleffe, 1993) for a similarity range $E(k) = 2\pi k^2 U(k) \propto k^n$ that the instability assumption and the closures give the same direction for the triadic transfer if $n < 2$ when $s_l = -s_w$ and if $n < 1$ when $s_l = s_w$. These critical values of n correspond to equipartition of energy ($n = 2$) and of helicity ($n = 1$), respectively. Combining both results, the closures agree with the instability assumption whenever $n < 1$. The Kolmogorov spectrum has $n = -5/3$.

The knowledge of the direction for the triadic energy transfer can then be used to determine the cascade direction in a similarity range. The conclusions are that the $s_l = -s_w$ interactions always forward cascade energy to higher wavenumbers but the $s_l = s_w$ interactions, which are responsible for the non-local interaction with local transfer character of the total triadic transfer function, reverse cascade whenever $n > -7/3$ (see Waleffe, 1993 for details).

A quantitative analysis with the EDQNM model was made for a -5/3 range. The most interesting result is that 86% of the energy cascade is due to helical interactions of the form $s_v = -s_l = s_w$. Interactions with $s_l = s_w$ nearly cancel out, contributing only about -1% to the total cascade.

Decomposition of the subgrid-scale eddy-viscosity into the contributions from the two classes of helical interactions, and also from the forward and reverse cascading triads, shows that the cusp near the cut-off wavenumber arises from non-local reverse cascading $s_l = s_w$ interactions. The subgrid viscosity due to forward cascading interactions is approximately constant. One noteworthy observation is that the closures also give a mild cusp as $k \to 0$, and thus the eddy-viscosity does not tend to a constant as is usually believed.

3. Future plans

The instability assumption seems to be a solid and general assumption not limited to isotropic turbulence. It could likely be used in other areas, in stratified flows for instance. It would be nice to test the assumption more directly using DNS; only indirect verifications have been made so far (e.g. total triadic transfer but not each helical interaction independently). One would like to have a better theoretical understanding of why the instability assumption works. Why would the stability properties of a single triad determine the statistical dynamics of a large number of interacting triads?

The helical decomposition is a very appropriate way of looking at the fundamental kinematically independent triad interactions. The suggestion from the closures that one type of helical interaction is responsible for 86% of the energy cascade is puzzling and worthy of further investigation. The physical interpretation of these interactions and the link to some physical mechanism is desired. We still do not have a dynamical mechanism which leads to a $-5/3$ turbulent spectrum. If the cascade is due to a series of successive instabilities, which instability is it? The simplest one, the inflexional Kelvin-Helmholtz instability, reverse cascades. I have suggested before that the $s_1 = -s_w$ helical interactions, which forward cascade, have some similarities with the elliptical instability (Waleffe, 1990, 1992), but no work has been done to explore that any further.

Regarding the non-local interactions with local transfer, their importance is far less than a look at $T(k,p,q)$ suggests. This is because of all the subtle cancellations taking place, whether one looks at the total cascade or the effect of large scales on the small scales. The role of non-local interactions in turbulence still needs to be determined. They are responsible for the cusp in the subgrid viscosity. They could be linked to intermittency also. In fact, it is my opinion that the closures do not deal with them correctly. All the closures treat triads as essentially independent (through the Quasi-normal or Direct Interaction assumption) with all the other triads acting as a decorrelating background noise. This seems valid for local interactions. However non-local triads can not be treated as independent because of the near-cancellations between *several* triads needed to represent distortion of small scales by large ones.

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