Turbulence modeling for non-equilibrium flows

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1. Motivation

Two projects are reported herein. The first is the development and testing of an eddy viscosity transport model. This project also is a starting point for our work on developing computational tools for solving turbulence models in complex geometries—the computational aspect is collaborative with Nagi Mansour. The second project is a stochastic analysis of the realizability of Reynolds stress transport models. This work was motivated by private discussions with Charles Speziale (Boston University) and related collaborative work with him, and also by Steve Pope's presentation at the (Bill) Reynolds Turbulence Symposium in Monterey this year (Pope 1993).

1.1 Eddy viscosity transport

Momentum mixing-length, eddy viscosity models have been popular since the idea was introduced by Prandtl. They invoke a quasi-equilibrium assumption, essentially requiring that the mixing of lumps of fluid takes place in a time short compared to that of the mean flow evolution. In many situations of fluid dynamical interest, mixing cannot be assumed instantaneous and a dynamical equation for the turbulence is required. A rather important example is that of boundary layers subjected to strong pressure gradients: on a practical level, it has been found that mixing length models are unable to predict this type of strongly non-equilibrium flow. A reasonable step is to formulate an analytical eddy viscosity transport model that extends the mixing-length idea by admitting non-equilibrium effects; this is what I have done.

This research has a strong practical incentive. Engineering fluid dynamicists are recognizing a need for more elaborate turbulence models as the complexity of the flows they calculate increases. This has led to a willingness, even a desire, to introduce turbulent transport models into prediction codes. Currently most aerodynamics codes (that solve the Reynolds averaged Navier-Stokes equations) use algebraic eddy viscosity models. It has been found that algebraic models are unable to predict the region of reversed flow near the trailing edge of an airfoil at angle of attack. The failure in this particular case indicates a general inadequacy for calculating complicated mean flows. This shortcoming of simple algebraic models motivated recent research into dynamical equations for eddy viscosity, including that reported herein (Durbin *et al.* 1994). Baldwin and Barth (1990) first proposed that an eddy-viscosity transport equation might be effective in complex aerodynamic flows. The Baldwin-Barth study led to further development of their formulation by Spalart and Almaras (1992) and was a primary impetus for the present work.

Notable features of the present model are that it uses an elliptic relaxation equation to avoid damping functions, that it is formulated solely in terms of local variables, and that it is tensorially and Galilean invariant. In these respects, the model was formulated with complex flows in mind.

1.2 Realizability

The exact, unclosed Reynolds-stress transport equations are usually the starting point for formulations of second-moment closure models. Modeling consists of replacing unclosed terms by semi-empirical formulae that express these terms as functions of the dependent variables. After introducing models, quantities with names like '$\overline{u^2}$' no longer represent non-negative functions obtained by squaring and averaging a random variable; rather, they are simply the dependent variables of the model; they are obtained as the solution to a differential equation. However, it is desirable to formulate the equations of the model so that variables like $\overline{u^2}$ do maintain their non-negativity. In essence, this is the issue of realizability in second-order turbulence closure modeling.

The present report demonstrates how realizability can be addressed by a constructive method. This involves formulating a stochastic process for which the Reynolds-stress model is the exact evolution equation of second moments. The model then is guaranteed to be realizable because it is exact for a well defined stochastic process. In the present analysis, second-moment closure models of the type currently in use are shown to be exact for the statistics of a particular form of Langevin equation. When that Langevin equation is well-defined, the Reynolds-stress model is guaranteed to be realizable.

It is assumed that the appropriate physics are accommodated by the moment closure; the stochastic analysis is purely a mathematical method for analyzing such models. The realizability criteria that are derived are *sufficient*, but not necessary, conditions; also, the analysis is only of homogeneous turbulence.

2. Accomplishments

2.1 The eddy viscosity transport model

The model is described at length in Durbin *et al.* (1994). It consists of a parabolic transport equation for the eddy viscosity:

$$\frac{D}{Dt}\nu_T = \nabla(\nu + \nu_T)\nabla\nu_T + P_\nu - c_2|S|\nu_T - c_4\frac{\nu_T^2}{L_\nu^2}\,; \tag{1}$$

and an elliptic relaxation equation for P_ν:

$$L_p^2\nabla^2 P_\nu - P_\nu = c_3|\nabla\nu_T|^2 - |S|\nu_T\,. \tag{2}$$

The dependent variable P_ν contains the turbulence production. The elliptic relaxation in (2) introduces a wall effect that suppresses the production of eddy viscosity near a surface. In (1) and (2) the $c's$ are model constants, the $L's$ are length scales, and $|S|^2 = {}^1\!/_2\,(\partial_i U_j + \partial_j U_i)(\partial_j U_i + \partial_i U_j)$ is a measure of the mean rate of strain.

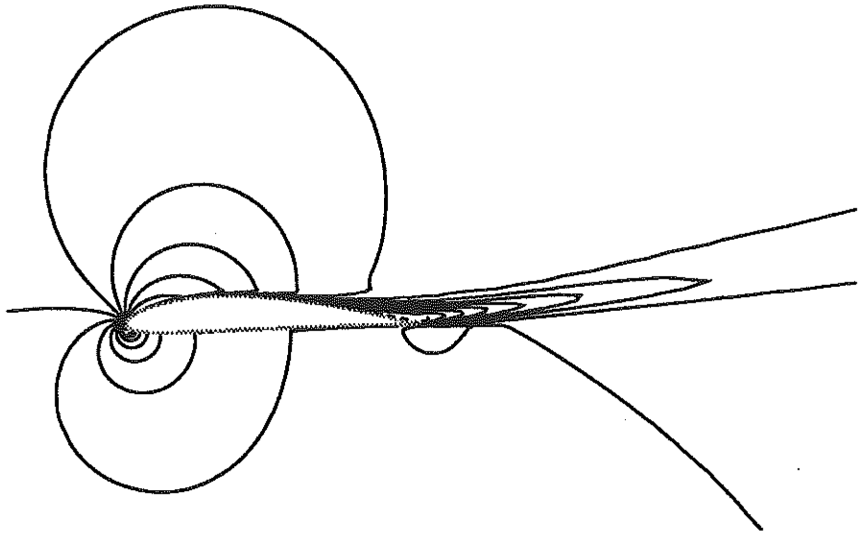

FIGURE 1. Contours of constant U, showing trailing edge separation and flow geometry.

The ellipticity in (2) is important near surfaces; away from surfaces, or regions of strong inhomogeneity, the model relaxes to a parabolic transport equation. The first term on the right side of (1) describes turbulent and molecular transport; the third and fourth terms model dissipation of ν_T, as does the first term on the right side of (2); the second term on the right side of (2) allows for production of turbulence from mean flow gradients.

To complete the model, the length scales must be prescribed and then the constant coefficients chosen. A difficulty of a model with only one primary dependent variable—or, more precisely, only one non-negative dependent variable—is that local turbulent length and time scales cannot be formed from turbulence quantities alone. For this reason L_p and L_ν have been prescribed as functions of the local mean rate of strain, as well as of ν_T:

$$L_\nu^2 = \frac{|S|^2}{|\nabla S|^2} + c_m \frac{|\nabla \nu_T|^2}{|S|^2} \tag{3}$$

$$L_p^2 = c_p^2 \min\big(L_\nu^2, \max(\nu_T, c_l^2 \nu)/|S|\big)\,. \tag{4}$$

The boundary conditions to the model are

$$\nu_T = \hat{n} \cdot \nabla \nu_T = 0 \tag{5}$$

at a no-slip surface with unit normal $\hat{n}$,

$$\hat{n} \cdot \nabla \nu_T = P_\nu = 0 \tag{6}$$

at the edge of a boundary-layer, or prescribed free-stream values far from an airfoil. No surface boundary condition needs to be imposed on P_ν to satisfy (5).

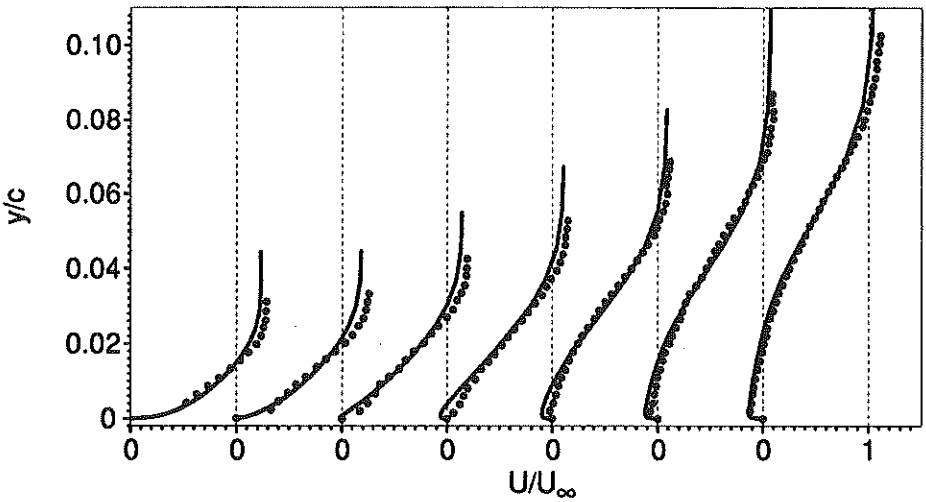

FIGURE 2. Mean velocity profiles at locations along the suction surface of an airfoil at angle of attack, showing trailing edge separation. The profiles are at $x/c = 0.62, 0.675, 0.731, 0.786, 0.842, 0.897$ and 0.953. The circles are experimental data.

The selection of model constants is described in Durbin *et al.* (1994). To summarize, the present values are $c_2 = 0.85$, $c_4 = 0.2$, $c_l = 3.3$, $c_p = 1.2$ $c_m = 2$ and $c_3 = (1 - c_2)/\kappa^2 + 1 - c_4/(1 + \kappa^4 c_m)$ where κ is the von Karman constant.

Figures 1 and 2 illustrate the model solutions. They are from a computation of flow around a NACA4412 airfoil at 14° angle of attack. Contours of constant U-component of mean velocity are plotted in figure 1. These illustrate that the flow separates just upstream of the trailing edge. The whole computational domain is not shown: the inflow is 14 chords upstream and is angled upward by 14°.

Figure 2 contains mean velocity profiles on the suction surface near the trailing edge. The last two profiles show that the model predicts the correct degree of separation. The Baldwin-Lomax, equilibrium eddy-viscosity model fails to predict separation in this flow; the rapid evolution of the flow as it undergoes separation violates the assumption of near equilibrium. The operation of the non-equilibrium model would seem to be as follows: the eddy viscosity transports mean momentum from the free-stream to the surface, counter balancing deceleration by the adverse pressure gradient. As the boundary layer thickens, eddy viscosity will be produced in the outer region of the boundary layer. However, this process is not instantaneous. If the pressure gradient decelerates the near-wall flow before the eddy viscosity can increase, the flow will separate. The transport equation describes the temporal evolution and spatial redistribution of the eddy viscosity as the flow passes through separation. Equilibrium, or algebraic, models assume that the eddy viscosity adjusts instantaneously to the level of shear and hence they overpredict the turbulent transport in this trailing edge flow.

2.2 Stochastic analysis of realizability

The stochastic method will be presented by describing how it is used to derive a sufficient condition for realizability of the most general, linear, second-order closure model for Reynolds stresses. However, this method is far more widely useful and can be applied to all existing Reynolds stress transport models, including those that are non-linear in the Reynolds stress tensor. The method is presented at greater length in Durbin and Speziale (1993). It involves showing the Reynolds-stress model to be the exact second-moment equation of a Langevin stochastic differential equation. The exact moment equation of a well defined stochastic process is realizable, by definition—for instance, quantities like $\overline{u^2}$ have non-negative values that could be obtained by squaring and averaging a random variable.

In homogeneous turbulence, the General Linear Model consists of the following ordinary differential equation for the evolution of the Reynolds stress tensor

$$\frac{d\overline{u_iu_j}}{dt} = -\frac{c_1}{T}(\overline{u_iu_j} - \frac{2}{3}k\delta_{ij}) - c_2(P_{ij} - \frac{2}{3}P\delta_{ij}) \\ -c_3(D_{ij} - \frac{2}{3}P\delta_{ij}) - c_s k S_{ij} + P_{ij} - \frac{2}{3}\varepsilon\delta_{ij}. \tag{7}$$

In this equation, $P_{ij} = -\overline{u_iu_k}\partial_k U_j - \overline{u_ju_k}\partial_k U_i$ is the production tensor, $D_{ij} = -\overline{u_iu_k}\partial_j U_k - \overline{u_ju_k}\partial_i U_k$ is a tensor introduced by Launder *et al.* (1975), $P = {}^1/_2\, P_{ii}$ is the rate of turbulent energy production, and ε is its rate of dissipation. All the c_i's are empirical, numerical constants and T is a time-scale.

We will show (7) to be the second moment of the stochastic differential equation

$$du_i = -\frac{c_1}{2T}u_i dt + (c_2 - 1)u_k\partial_k U_i dt + c_3 u_k \partial_i U_k dt \\ + \sqrt{c_0\varepsilon}\, d\mathcal{W}_i(t) + \sqrt{c_s\varepsilon}\, M_{ik}\, d\mathcal{W}'_k(t) \tag{8}$$

with

$$c_0 = \frac{2}{3}\left[c_1\frac{k}{\varepsilon T} - 1 + (c_2 + c_3)\frac{P}{\varepsilon}\right] - \frac{c_s}{3}M^2\,. \tag{9}$$

In (8), $\mathcal{W}$ and $\mathcal{W}'$ are independent Weiner processes ($\overline{d\mathcal{W}'_i d\mathcal{W}_j} = 0$). $\mathbf{M}$ is a symmetric matrix that is required to satisfy

$$M_{ij}^2 - \frac{1}{3}M^2\delta_{ij} = -\frac{k}{\varepsilon}S_{ij} \tag{10}$$

where $M_{ij}^2 = M_{ik}M_{kj}$, $M^2 = M_{kk}^2$, and S_{ij} is the rate of strain tensor, defined as $[\partial_i U_j + \partial_j U_i]/2$. The matrix $\mathbf{M}$ can be constructed as follows: in incompressible flow, $\mathbf{S}$ is a sym metric matrix with eigenvalues that sum to zero; it can be diagonalized by the unitary matrix $\mathbf{U}$ of eigenvectors:

$$\mathbf{S} = \mathbf{U}\cdot \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]\cdot {}^{\mathbf{t}}\mathbf{U} \tag{11}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are the eigenvalues in decreasing order. They satisfy

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{12}$$

In terms of these eigenvalues

$$\mathbf{M} = \mathbf{U} \cdot \mathrm{diag}\left[0, \sqrt{\lambda_1 - \lambda_2}, \sqrt{\lambda_1 - \lambda_3}\right] \cdot {}^t\mathbf{U}\,. \tag{13}$$

By virtue of (12) and (13), $M^2 = 3\lambda_1$.

The second moment equation for (8) is derived by applying the rules

$$\begin{aligned} \overline{dW_i} &= 0 \\ \overline{dW_i dW_j} &= dt\delta_{ij} \\ \overline{u_j dW_i} &= 0\,. \end{aligned} \tag{14}$$

Using these to compute $d\overline{u_i u_j}/dt$ for (8) shows that (7) is formally the exact second moment equation of the stochastic process (8) (Durbin and Speziale 1993). The only issue is whether the Langevin equation (8) is well-defined.

If the coefficients are all bounded, then (8) will be well-defined if the square roots are real valued. This requires c_s and c_0 to be non-negative. c_s is an empirical constant of the model (7) that can be chosen to be positive. c_0 is a function given by (9). The condition that $c_0 \geq 0$ translates to

$$c_1 \geq 1 - (c_2 + c_3)\frac{P}{\varepsilon} + \frac{3}{2}c_s\lambda_1 \tag{15}$$

where $M^2 = 3\lambda_1$ has been used and the time-scale has been set to $T = k/\varepsilon$. The right side of this inequality depends on the flow. As long as (15) is met, solutions to the general linear model are certain to be realizable; however, (15) is not guaranteed to be met, so the general linear model need not always have realizable solutions. Our experience with models that are special cases of (7) is that (15) is usually satisfied.

In extreme cases, far from equilibrium, unrealizable solutions to the general linear model can be generated. However, the present analysis shows that one easily can modify the model to guarantee realizable solutions: just alter c_1 so that (15) is always satisfied. For instance, if $c_1 = 1.8$ is usually a satisfactory value, then replacing this constant by the function

$$c_1 = \max\left[1.8,\ 1 - (c_2 + c_3)\frac{P}{\varepsilon} + \frac{3}{2}c_s\lambda_1\right] \tag{16}$$

will enforce the inequality (15). This ensures realizability in extreme cases without altering the model in usual cases.

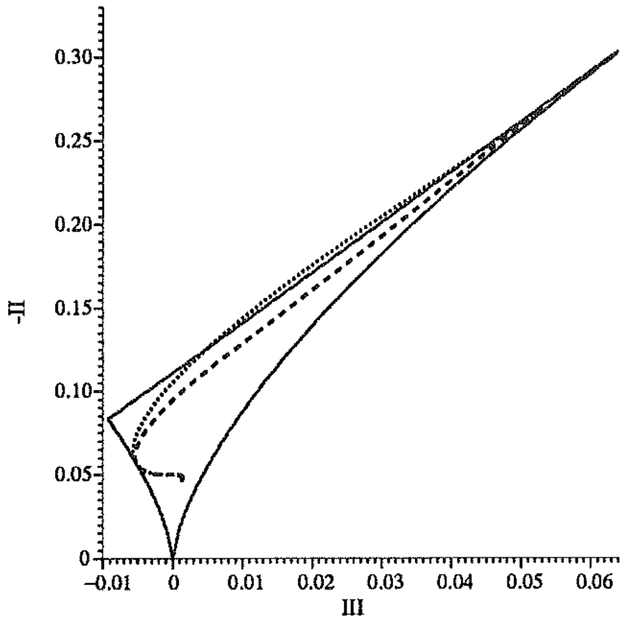

FIGURE 3. Solution trajectories of the LRR model projected onto the second-invariant–third-invariant plane. · · ·, original LRR model; - - -, modified, realizable LRR model.

Launder *et al.* (1975) imposed certain symmetry and normalization constraints that led them to specialize (7) by setting

$$c_2 = \frac{c+8}{11}; \; c_3 = \frac{8c-2}{11}; \; c_s = \frac{60c-4}{55} \tag{17}$$

where c is a constant they set to 0.4—this is the LRR model. With (17), the condition (16) becomes

$$c_1 = \max\left[1.8, \, 1 - 0.873\frac{P}{\varepsilon} + 0.545\lambda_1\right]. \tag{18}$$

Figure 3 shows trajectories of the LRR model for homogeneously sheared turbulence, with the initial condition

$$Sq^2/\varepsilon = 20; \; b_{11} = -0.27; b_{22} = -0.33; \; b_{33} = 0.6; \; b_{12} = b_{13} = b_{23} = 0.$$

These values give $P/\varepsilon = 0$, $-II = 0.27$ and $III = 0.053$, where II and III are the invariants of the Reynolds stress tensor. The trajectories in figure 3 are solutions to the LRR model projected into the second-third invariant plane. They start in the upper right corner of the triangle and ultimately are attracted to the equilibrium point in the lower center or the triangle. The initial value of $1 - 0.873P/\varepsilon + 0.545\lambda_1$ is 3.73 (note that $\lambda_1 = Sq^2/4\varepsilon$) so that (15) is violated if $c_1 = 1.8$. Realizable solutions must remain inside the curvilinear triangle of figure 3. The dotted curve shows that the LRR model exits the realizable region for this initial condition. Thus, although (15) is a sufficient but not necessary condition, it provides an insight into the existence of unrealizable trajectories. The solution trajectory remains inside the realizable region when the modification (18) is applied, as shown by the dashed curve in figure 3.

3. Future plans

Development of numerical tools for solving Reynolds-stress models in complex geometries is under way. These are needed so that we can test and further develop models. The computer program being developed is based on the incompressible Navier-Stokes solver (INS-2D) written by Stuart Rogers at NASA Ames. This is a general geometry code and the Reynolds stress solver is being written so that models can be implemented readily. The program under development is an extension of that used to calculate figure 1.

The greatest potential utility of Reynolds stress transport models is in strongly non-equilibrium flows. However, certain difficulties remain. For instance, all existing models predict incorrect rates of relaxation toward equilibrium in highly perturbed flows. Future work will include this area.

The analysis of realizability raises some intriguing questions. The realizability of models for non-homogeneous flows is largely unexplored. Although I have considered this issue for the type of near-wall models that I have developed, I have no mathematical understanding of why they seem to produce realizable solutions. The ellipticity of non-local wall effects suggests using maximum principles.

Stochastic differential equations are popular models for Lagrangian dispersion calculations. The present use of stochastic equations was solely to analyze statistical moment models. This work can be made the basis of self-consistent Reynolds-stress and Lagrangian dispersion models.

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