On the dynamics of approximating schemes for dissipative nonlinear equations

By Don A. Jones

1. Motivation and objectives

Since one can rarely write down the analytical solutions to nonlinear dissipative partial differential equations (PDEs), it is important to understand whether, and in what sense, the behavior of approximating schemes to these equations reflects the true dynamics of the original equations. Further, because standard error estimates between approximations of the true solutions coming from spectral methods—finite difference or finite element schemes, for example—and the exact solutions grow exponentially in time, this analysis provides little value in understanding the infinite time behavior of a given approximating scheme.

The notion of the *global attractor* has been useful in quantifying the infinite time behavior of dissipative PDEs, such as the Navier-Stokes equations. Loosely speaking, the global attractor is all that remains of a sufficiently large bounded set in phase space mapped infinitely forward in time under the evolution of the PDE. Though the attractor has been shown to have some nice properties—it is compact, connected, and finite dimensional, for example—it is in general quite complicated.

Nevertheless, the global attractor gives a way to understand how the (infinite time) behavior of approximating schemes such as the ones coming from a finite difference, finite element, or spectral method relates to that of the original PDE. Indeed, one can often show that such approximations also have a global attractor. We therefore only need to understand how the structure of the attractor for the PDE behaves under approximation. This is by no means a trivial task. Several interesting results have been obtained in this direction. However, we will not go into the details. We mention here that approximations generally lose information about the system no matter how accurate they are. There are examples that show certain parts of the attractor may be lost by arbitrary small perturbations of the original equations (see Humphries, Jones, Stuart, 1993, and the references therein for a description of some of the results).

Under certain hypothesis on the approximation, one can be guaranteed some structures of the attractor survive the approximation. For example, consider the ordinary differential equation (ODE)

$$\frac{dx}{dt} = X(x), \tag{1}$$

where we suppose that $x \in \mathbb{R}^n$ and that X is a C^1 function. Further, we suppose that the system (1) is dissipative and hence has a global attractor. Now suppose that (1) is approximated by

$$\frac{dy}{dt} = X(y) + Y(y), \tag{2}$$

where $y \in \mathbb{R}^n$ and Y is a C^1 function. Suppose further that

$$\|Y\|_{C^1} \leq \epsilon$$

for some suitable $\epsilon > 0$. That is, (1) and (2) may be viewed as small C^1 perturbations of one another. This seems to be a natural condition to require of a perturbation in order to say something about how the global attractor of (2) relates to that of (1). Indeed, such systems have been studied by several authors and increasingly stronger results have been obtained (see *e.g.* Pliss & Sell, 1992, and the references therein). It is known, for example, that normally hyperbolic, invariant manifolds persist under such perturbations.

In order to apply these results to PDEs, one must first construct finite systems of ODEs that have the same global attractor as the infinite-dimensional PDE. This has been done for several dissipative PDEs including, for example, the Kuramoto-Sivashinsky equation, Cahn-Hilliard equation, Ginzburg-Landau, certain reaction-diffusion equations, and the Navier-Stokes equations, Kwak, 1991. Such systems are called *inertial forms.*

To be more specific, each of these PDEs can be viewed as an ordinary differential equation on a suitably chosen Hilbert space, H. We denote by $(\cdot, \cdot)$ the inner product and $|\cdot|$ the norm on H. Then these equations take the form

$$\frac{du}{dt} + Au + R(u) = f \tag{3}$$

$$u(0) = u_0.$$

Typically, the operator Au is $-\nabla^2$ with Dirichlet or periodic boundary conditions. For the Navier-Stokes equations, for example, the term $R(u)$ is the divergence free part of $(u \cdot \nabla)u$ (see Temam 1988).

In all cases but the Navier-Stokes equations, the existence of inertial forms (IF) has been proven by showing the existence of an Inertial Manifold. To date, inertial manifolds have been constructed as a graph in phase space of a Lipschitz function Φ (see Foias, Sell, Temam, 1988). An inertial manifold (IM) for a dissipative evolution partial differential equation is a smooth finite-dimensional manifold in phase space, which is positively invariant under the solution operator and which uniformly attracts every bounded subset of phase space at an exponential rate. It is clear that if the IM exists, then it must contain the global attractor. Moreover, the reduction of the partial differential equation to the IM yields the inertial form.

We denote by P the orthogonal projection of the space H onto the span of the first M eigenfunctions of A, and $Q = I - P$. We set $p = Pu, q = Qu$. Then the evolution equation (3) is equivalent to the system

$$\frac{dp}{dt} + Ap + PR(p + q) = Pf,$$

$$\frac{dq}{dt} + Aq + QR(p + q) = Qf.$$

If the IM is given as a graph of a Lipschitz function $\Phi : PH \mapsto QD(A)$ for M sufficiently large, then on this manifold the solutions of (3) are of the form $u(t) = p(t) + \Phi(p(t))$. Moreover, in this case, the inertial form is given by

$$\frac{dp}{dt} + Ap + PR(p + \Phi(p)) = Pf \quad p \in PH. \tag{4}$$

Equation (4) does not have the same solutions as Equation (3) (only on M). Rather, it has the same infinite time behavior as original PDE. Most importantly, it is an ODE. In view of the work of Pliss & Sell (1992) mentioned above, there is an advantage in approximating (4) in the C^1 sense. A candidate for such a system is

$$\frac{dp}{dt} + Ap + PR(p + \Phi_{app}(p)) = Pf \quad p \in PH, \tag{5}$$

with

$$\sup_{p \in PH} \left(|A(\Phi_{app}(p) - \Phi(p))| + \|A(D\Phi_{app}(p) - D\Phi(p))\|_{\mathcal{L}(PH,QH)} \right) \leq \epsilon,$$

and where $D\Psi$ denotes the Fréchet derivative of the function Ψ. In this case, under reasonable assumptions on R, the vector field in the approximate inertial form (5) may be viewed as a small C^1 perturbation of the vector field in the inertial form (4).

2. Accomplishments

The main goal of this method of reduction is to implement the reduced ordinary differential system (5) in long-time simulations of solutions to the PDE, (3). Even in the case that the IM or a smooth function Φ does not exist, the theory suggests looking for a global function Φ_{app} whose graph in phase space approximates the attractor. Indeed, many Φ_{app} have been constructed. These approximations have been implemented in numerical schemes for a variety of equations and settings (see Jones, Margolin, Titi, 1993, and the references therein). We will discuss the effectiveness of these schemes below.

Perhaps the most important role the IF, Equation (4), can play, as mentioned above, is to understand how the dynamics of approximating schemes relates to that of the original PDE. The first attempt at approximating Φ in the C^1 sense was in Jones, Titi, 1993. There, Φ was viewed as the asymptotically stable stationary solution of a certain PDE. One can then approximate Φ by integrating this PDE forward for a short time.

However, the situation may be much simpler than this. Consider the spectral approximation of (3) based on the eigenfunction of the linear operator A. One obtains the approximation

$$\frac{du_N}{dt} + Au_N + P_N R(u_N) = P_N f \tag{6}$$

with initial data $u_N(0) = u_{0,N}$. As shown in Foias, Sell, Temam, 1988, and Foias, Sell, Titi, 1988, if N is chosen sufficiently large, there exists a global function Φ_N such that M_N =Graph(Φ_N) is an inertial manifold for (6). On this manifold, solutions are of the form $u_N(t) = p_N(t)+\Phi_N(p_N(t))$ with $p_N(t) = Pu_N(t)$. Further, on this manifold, (6) reduces to

$$\frac{dp_N}{dt} + Ap_N + PR(p_N + \Phi_N(p_N(t))) = Pf, \tag{7}$$

where P is defined above. Notice also that Equation (7) remains of dimension M, the same dimension as (4), as $N \to \infty$. Again, it is not that (7) has the same solutions as (6) (only on M_N), but rather it has the same global attractor as (6) (since the IM contains the attractor).

Now Equation (7) will play the role of the approximate inertial form, Equation (4). That is, we take $\Phi_{app} = \Phi_N$. Moreover, it was shown in these two papers that

$$\sup_{p\in PH} |A(\Phi(p) - \Phi_N(p)| \leq \frac{C}{\lambda_N^{1-\beta}}$$

for some $0 \leq \beta \leq 1/2$ which depends on the nonlinear term R. However, more is true.

Theorem *Suppose that M is so large (determined by the spectral properties of A) that Φ, Φ_N as described above exist. Then for all $\epsilon > 0$ there exists a $N(\epsilon)$ such that*

$$\sup_{p\in PH} [|A(\Phi(p) - \Phi_N(p))| + \|A(D\Phi(p) - D\Phi_N(p))\|_{\mathcal{L}(\mathcal{PH},\mathcal{QH})}] \leq \epsilon$$

for all $N \geq N(\epsilon)$.

Proof. See Jones, Titi (1993).

The IF, Equation (4), has the same dynamics as the original PDE. Moreover, the above shows that the spectral method based on the eigenfunctions of A is a small C^1 perturbation of the IF for N sufficiently large, since Equation (6) and (7) have the same attractor. Thus, this spectral method preserves certain structures of the attractor of the PDE, for example, the ones studied in Pliss, Sell, 1991, for N sufficiently large.

A similar type of analysis may be possible for finite element methods. To do this properly, we should turn to a specific PDE. However, we will attempt to keep the exposition as general as possible. We denote by $\{V^h\}_{h>0}$ a finite dimensional subspace of differentiable functions (most typically piecewise linear functions), where one can think of h as being the maximum partition size. Then one attempts to approximate solutions $u(t)$ of (3) by functions $u^h(t)$ in V^h. The functions u^h solve

$$(u_t^h, \chi) + (A^{1/2}u^h, A^{1/2}\chi) + (R(u^h), \chi) = (f, \chi) \tag{8}$$

$$u^h(0) = u_0^h \in V^h,$$

where one can think of $A^{1/2}$ as d/dx in the 1D case.

The operator A^h can be defined from the equation $(A^h\psi, \chi) = (A^{1/2}\psi, A^{1/2}\chi)$. Further, projecting R and f onto the space V^h, Equation (8) takes the form

$$\frac{du^h}{dt} + A^h u^h + R^h(u^h) = f^h. \tag{9}$$

Since A is assumed to be self-adjoint, A^h is also. Moreover, the spectrum of A^h can be shown to approximate that of A. Thus, the space V^h may be decomposed $V^h = P^h V^h \oplus Q^h V^h$. In the same manner that the Φ_N was constructed in Foias, Sell, Titi, 1988, a global function Φ^h may be constructed for Equation (9) (see Jones, Stuart, 1993, for the details) such that M^h =Graph(Φ_h) is an inertial manifold for (9). On this manifold, solutions are of the form $u^h(t) = p^h(t) + \Phi^h(p^h(t))$, where $p^h = P^h u^h$. On the IM, (9) reduces to

$$\frac{dp^h}{dt} + A^h p^h + P^h R^h(p^h + \Phi^h(p^h)) = P^h f \tag{10}$$

for h sufficiently small. As in the case of the spectral method, the dynamics of (10) are the same as that of (9). Moreover, the dimension of (10) remains fixed (roughly on the order of M) as $h \to 0$. One would like to show C^1 closeness of (10) and (4). However, at this point all we have is the following

Theorem *For h sufficiently small there exists a function Φ^h such that (10) holds. Moreover,*

(i) for any $p \in PH$ there exists $C(p) > 0$ such that

$$\|(p + \Phi(p)) - (P^h p + \Phi^h(P^h p))\| \leq C(p)h;$$

(ii) for any $p^h \in P^h H$ there exists $C(p^h) > 0$ such that

$$\|(Pp^h + \Phi(Pp^h)) - (p^h + \Phi^h(p^h))\| \leq C(p^h)h.$$

Proof. See Jones, Stuart (1993).

On the Practical Side

The above theory suggests that there may be an advantage in enslaving the high Fourier modes (in the case of the spectral method based on the eigenfunctions of A) in terms of the lower modes through the function Φ_{app}. Shortly after the discovery of the IF, Equation (4), many Φ_{app} were constructed and studied for various equations (see Jones, Margolin, Titi, 1993, and the references therein), including the Navier-Stokes equations (see for example Jolly, 1993). Schemes based on enslaving $q \approx \Phi_{app}(p)$ are generally referred to as *nonlinear Galerkin methods* since $\Phi_{app} = 0$ gives the standard Galerkin scheme.

In Jones, Margolin, Titi, 1993, we evaluate the effectiveness of the nonlinear Galerkin method in the context of spectral method (such schemes have now been constructed for finite element and finite difference schemes; see the references in Jones it et al. 1993). The goal of this work is to understand under what conditions the nonlinear Galerkin methods lead to a significant improvement in accuracy over the standard Galerkin method from a purely numerical analysis point of view.

Recall that in general, if one approximates a smooth function u with respect to some basis, the rate of convergence is limited by the smoothness of the basis functions. If the basis elements are C^∞ functions as in the case of the eigenfunctions of the linear operator A, the rate of convergence is only limited by the smoothness of u and compatibility of the function u with the basis elements of the expansion at the boundary (the presence of Gibb's phenomenon, for example).

Thus, if the solutions $u(t)$ of equation (3) are very regular and compatible with the eigenfunctions $\{\varphi_j\}$ of A at the boundary, then the Fourier coefficients of the solution may decay very rapidly in wave number. Indeed, Foias, Temam, 1989, showed that, under such circumstances, solutions of the Navier-Stokes equation may decay exponentially in Fourier space. Similar results hold for the other equations mentioned above. Thus, the business of trying to approximate the q part of solutions via the function Φ_{app} may not be effective when the $q = Qu$ part of the solutions is exponentially small. That is, the approximation $\Phi_{app} = 0$, which leads to the standard Galerkin scheme, may already be good enough. It turns out that what controls the regularity and compatibility of solutions coming from the NSE and related equations is the compatibility with the basis functions and regularity of the forcing term f.

We demonstrate this for the Kuramoto-Sivashinsky equation (KS). A similar analysis holds for the NSE. This equation is given by

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = f(x)$$

$$u(0,x) = u_0(x)$$

$$u(t,x) = u(t,x+L) \quad L > 0,\ t \geq 0.$$

The KS equations appears in physics literature with $f = 0$. Here we have added an additional feature to the KS equation, namely, a forcing term f. We will use this forcing term to control the level of regularity of the solutions to the KS equation. It is clear that whenever $f(x)$ is an odd function then the space of odd functions is invariant under the solution operator for the KS equation. For simplicity, we will restrict ourselves to the odd case. Hence, under these assumptions, one can easily show that the KS equation is equivalent to the evolution equation

$$\frac{du}{dt} + Au - A^{1/2}u + B(u,u) = f,$$

$$u(0) = u_0$$

on the Hilbert space $H = \{u \in L^2((0,L))|u(x) = u(x+L), u(x) = -u(L-x), x \in \mathbb{R}\}$. Here $A = \frac{\partial^4}{\partial x^4}$; the eigenvalues of A are $\lambda_m = (2\pi m/L)^4$ corresponding to the eigenfunctions $\varphi_m = \sin(2\pi mx/L)$, for $m = 1, 2, \ldots$.

In either case, one can define the operators A^α for $\alpha \geq 0$. One defines $D(A^\alpha) = \{u \in H | u = \sum_{j=1}^{\infty} u_j \varphi_j, \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |u_j|^2 < \infty\}$. Consequently, functions in $D(A^\alpha)$ are more regular, and more compatible (whenever it applies) with the eigenfunctions of A at the boundary, for larger α.

We may then approximate solutions of the KS equation with either the standard Galerkin scheme

$$\frac{dy_n}{dt} + Ay_n + A^{1/2}y_n + P_nB(y_n, y_n) = P_nf \tag{11}$$

or by the nonlinear Galerkin method

$$\frac{dz_n}{dt} + Az_n + A^{1/2}z_n + P_nB(z_n + \Phi_{app}(z_n), z_n + \Phi_{app}(z_n)) = P_nf \tag{12}$$

for some clever choice for Φ_{app}.

As mentioned, the rate of convergence of the two schemes is tied to the smoothness of the solutions and compatibility of the solutions with the basis functions. (Since we are considering the KS equation with periodic boundary conditions, compatibility of the solutions is not an issue here. However, such cases are studied in Jones, Margolin, Titi, 1993.) This is, in turn, tied to the smoothness and compatibility of the forcing term f. The rates of convergence of the two schemes is given by the following two theorems which is based on the work of Devulder, Marion, Titi, 1993, and whose proofs can be found in Jones, Margolin, Titi, 1993.

Theorem *Let $u(t) = p(t) + q(t)$ be a solution of the KS equation with u_0 on the attractor and $f \in D(A^\alpha)$. Suppose y_n solves (11) with $y_n(0) = P_nu(0)$. Then*

$$\|u(t) - y_n(t)\|_{L^2} \leq \frac{C_1(t)}{\lambda_{n+1}^{1+\alpha}}.$$

In general, requiring $f \in D(A^\alpha)$ for some $\alpha > 0$ requires not only that f be smooth, but also that f and its derivatives up to order 4α must satisfy the boundary conditions. Now we suppose that Φ_{app} satisfies certain conditions described in Jones, Margolin, Titi, 1993—such Φ_{app} abound. For the nonlinear Galerkin method, we have

Theorem *Let $u(t)$ be as in the previous theorem. Suppose z_n solves (12) with $z_n(0) = P_nu(0)$. Then*

$$\|u(t) - (z_n(t) + \Phi_{app}(z_n(t)))\|_{L^2} \leq \frac{C_1(t)}{\lambda_{n+1}^{\beta+\alpha}},$$

for some $\beta > 1$.

For most Φ_{app}, β is not larger than two. Now one can see the issue here. As f becomes more smooth, larger α, the difference in the theoretical rates of convergence of the two schemes decreases.

Let us push the smoothness of the solutions to an extreme. Consider the forced KS equation and suppose that for some $\sigma > 0$, $f \in D(e^{\sigma A^{1/4}})$; that is, f is in a Gevrey class (real analytic). Notice that

$$\|e^{\sigma A^{1/4}} f\|_{L^2}^2 = \sum_{j=1}^{\infty} e^{2\sigma\lambda_j^{1/4}} |f_j|^2 < \infty,$$

where $f = \sum_{j=1}^{\infty} f_j \varphi_j$. Under these assumptions for the forced KS equation, we have (Proposition 3.6 of Jones, Margolin, Titi, 1993) that the Fourier expansion of the solution converges exponentially fast. This means that the solutions are infinitely compatible with the basis functions at the boundary and analytic inside the domain. Hence, the high Fourier modes of the solutions have exponentially small norms, and there may be little advantage in approximating them. Indeed, in this case, we have

$$\|u(t) - y_m(t)\|_{L^2} \leq C_1(t) \frac{e^{-\sigma_1 \lambda_{m+1}^{1/4}}}{\lambda_{m+1}},$$

where y_m solves the Galerkin scheme (11)

As in the above theorem, we have

$$\|u(t) - (z_m(t) + \Phi_{app}(z_m(t)))\|_{L^2} \leq C_2(t) \frac{e^{-\sigma_1 \lambda_{m+1}^{1/4}}}{\lambda_{m+1}^{\beta}},$$

where z_m is the solution of the nonlinear Galerkin scheme (12).

Thus, in the case that the solutions have Gevrey class regularity (spatially real analytic), the nonlinear Galerkin method only leads to algebraic improvements in the upper bounds of the rates of convergence over the standard Galerkin scheme. This little improvement might not be significant in computations. Of course, the overall improvement depends on how small the constant σ_1 is.

We demonstrate this numerically. We choose Φ_{app} to be

$$\Phi_1(p) = A^{-1} Q_m (f - B(p, p)),$$

which we first studied for the NSE in Foias, Manley, Temam, 1988. For this choice, $\beta = 7/4$. For our first example, we force the KS equation with

$$f = \sum_{j=1}^{\infty} \frac{1}{j} \sin(jx).$$

Notice that for this choice $f \in D(A^\alpha)$ for $\alpha < 1/8$.

We have found a stable periodic orbit with this forcing. Such trajectories are on the attractor. To obtain the "exact" periodic solution, we run a Galerkin scheme using 100 modes; when we reach .5 time units, we start recording the data every .002 time units up to a time of .7 time units (Figure 1). If the initial data is taken near the periodic orbit, it will take some .4 time units to converge to the periodic solution. Since the periodic solution is on the attractor, we expect that the rate of convergence in this case to be of the form

$$E_{gal} = \max_{.5 \le t \le .7} \|u(t) - y_m(t)\|_{L^2} \le \frac{c_1}{\lambda_{m+1}^{\alpha_1}},$$

of course, here we are ignoring errors due to the discretization of time. Also

$$E_1 = \max_{.5 \le t \le .7} \|u(t) - (z_m(t) + R_n \Phi_1(z_m(t)))\|_{L^2} \le \frac{c_2}{\lambda_{m+1}^{\alpha_2}}.$$

Since in this case $\lambda_m = m^4$, we have

$$\log E_{gal} = c_1 - 4\alpha_1 \log(m+1),$$

and

$$\log E_1 = c_2 - 4\alpha_2 \log(m+1).$$

Thus, a log-log plot of the error in terms of the wave number will easily determine the rate of convergence. In Figure 2, we have plotted the rate of change of the graph of the log-log plot of the error in terms of wave number. The theory suggests that the rate of convergence for the Galerkin method α_1 is less than or almost equal to $9/8$ and for the nonlinear Galerkin method α_2 is less than or almost equal to $15/8$. The results plotted in Figure 2 show that the Galerkin calculation asymptotes at a value of 1.1, whereas the nonlinear Galerkin asymptotes at a value of 1.92, which is in a good agreement with the theory.

Now we turn to the case when the force is in the Gevrey class (real analytic). We consider the KS equation with zero forcing. With the help of the software package AUTO, we start the calculation with initial data on the unstable manifold of a periodic orbit, which again is on the attractor. The solution converges to a steady state as time goes to infinity. Thus, this trajectory is contained in the global attractor, and the theory presented in Section 3 holds for this trajectory. We first compute this trajectory using 100 Fourier modes, which we will consider as our "exact" solution. The L^2 norm of this solution vs. time is shown in Figure 3. We integrate this trajectory out to 1.3 time units and record the solution every .01 time units.

We expect from the theory outlined in Section 3 that

$$E_{gal} := \max_{0 \le t \le 1.3} \|u(t) - y_m(t)\|_{L^2} \le \frac{c_1 e^{-\sigma_1 \lambda_{m+1}^{1/4}}}{\lambda_{m+1}},$$

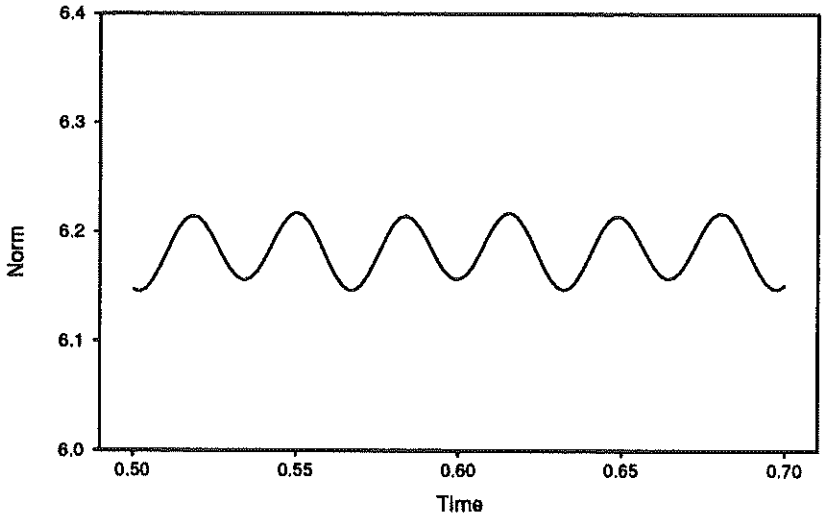

FIGURE 1. L^2 norm of the solution vs. time for the KS eq. of a Galerkin scheme with 100 modes forced with $f = \sum_{j=1}^{\infty} 1/j \sin(jx)$.

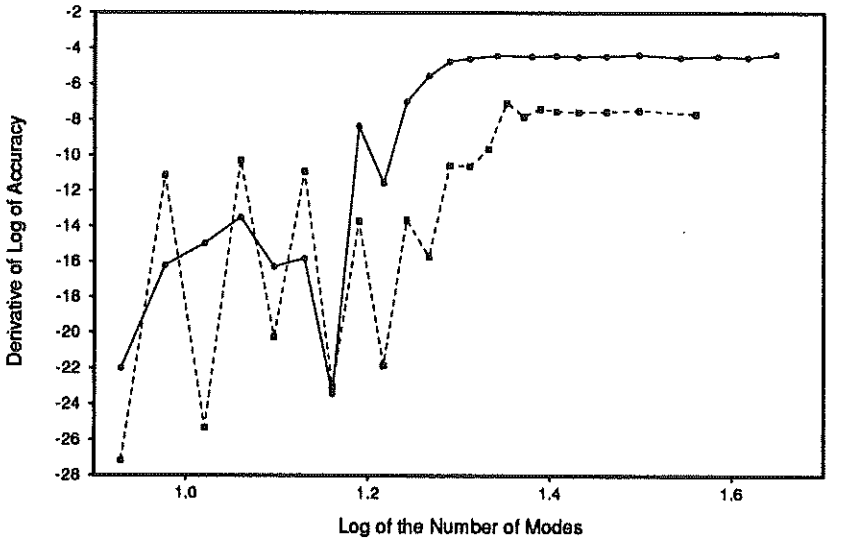

FIGURE 2. Rate of change of the log of the accuracy$\times 10^{11}$ vs. log of the number of modes. The Galerkin asymptotes at 4.4 and the nonlinear Galerkin at 7.7. The theory suggests that the Galerkin should asymptote near 4.5 and the nonlinear Galerkin near 7.5. ● , Gelerkin; ■ , FMT.

$$E_1 := \max_{0\le t\le 1.3} \|u(t) - (z_m(t) + R_n\Phi_1(z_m(t)))\|_{L^2} \le \frac{c_2 e^{-\sigma_1 \lambda_{m+1}^{1/4}}}{\lambda_{m+1}^{7/4}},$$

for some $\sigma_1 > 0$. Since $\lambda_{m+1} = (m+1)^4$, we have

$$\log E_{gal} = -\sigma_1(m+1) + \log(c_1\lambda_{m+1}^{-1})$$

and

$$\log E_1 = -\sigma_1(m+1) + \log(c_2\lambda_{m+1}^{-7/4}).$$

Thus, a log-linear plot of the error versus the number of modes should be nearly linear. This is confirmed in Figure 4. In fact, the two lines are parallel. That is, they have the same exponential rate of convergence (same σ_1). In addition, notice that the nonlinear Galerkin method still exhibits an algebraic improvement over the standard Galerkin. This is manifested by the fact that the graph for the nonlinear Galerkin is below the graph for the Galerkin.

Similar considerations apply where the compatibility of the solutions with the basis functions is an issue. Let us recall how this can come about. Consider Burgers' equation forced on the boundary

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0, \tag{13}$$

$$u(0,t) = 1 \quad u(1,t) = 0.$$

To formulate this problem in the same setting as Equation (3), one can set $v = u - (1-x)$ to obtain

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + v\frac{\partial v}{\partial x} + (1-x)\frac{\partial v}{\partial x} - v = 1 - x,$$

$$v(0,t) = 0 \quad v(1,t) = 0.$$

To apply the nonlinear Galerkin method, we must first expand the forcing term, here $f = 1 - x$, in terms of the eigenfunctions of the linear dissipative operator $\varphi_j(x) = \sin(j\pi x)$. We find

$$f = 1 - x = \sum_{j=1}^{\infty} \frac{4}{\pi j} \sin(j\pi x).$$

It is easily seen that, in terms of the spectral method based on the eigenfunctions of the Laplacian, the forcing term is only $L^2((0,1)), (\alpha = 0)$. We therefore expect the nonlinear Galerkin method to be significantly more accurate in this case.

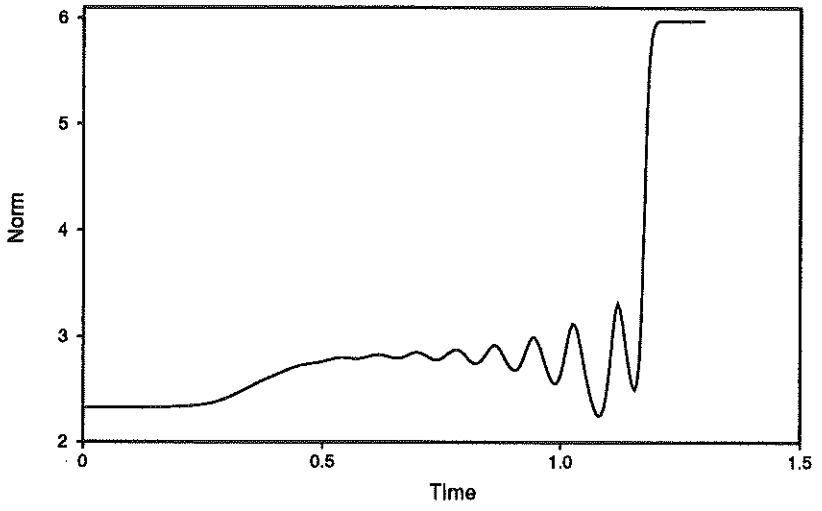

FIGURE 3. L^2 norm of "exact" solution vs. time for the KS equation with zero forcing with initial data on an unstable manifold.

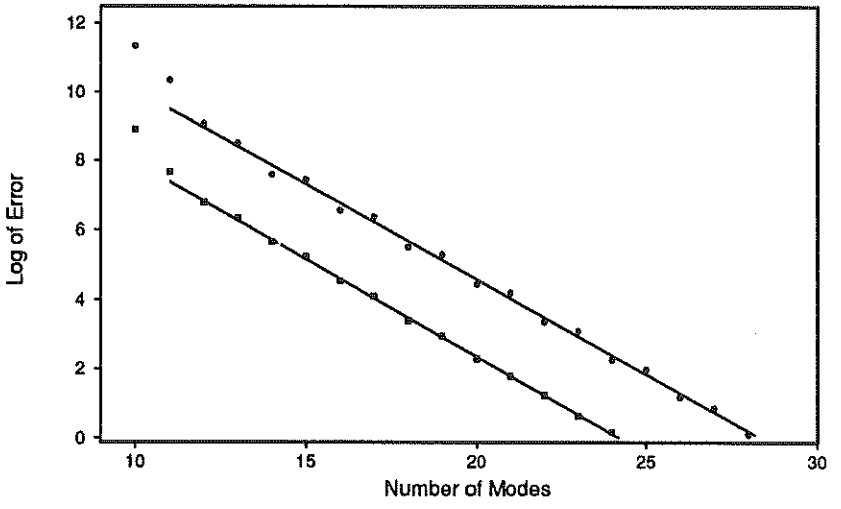

FIGURE 4. Log of the error$\times 10^{13}$ vs. the number of modes for the KS equation with zero forcing. Notice that the rate of convergence of both schemes is exponential. • , Gelerkin; ▪ , FMT

3. Future plans

The goal remains to understand what aspects of the long time behavior of infinite-dimensional PDEs are retained by their finite-dimensional approximations. Perhaps the ultimate test of an approximation will remain how well it predicts nature. However, there are a growing number of applications where one does not know *a priori* nor is there a way to test experimentally the behavior of some systems. Therefore, assuming the dynamics of the PDEs accurately reflects the physical phenomenon it is meant to depict, we hope understanding the behavior of approximating schemes of these PDEs will prove valuable in the future.

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