Notes on rotating turbulence

By O. Zeman

1. Motivations and objectives

The purpose of this work was to investigate the turbulent constitutive relation when turbulence is subjected to solid body rotation. Rotating turbulent flows exist in many industrial and geo- and astrophysical applications.

2. Accomplishments

2.1 Note on spectra and decay of rotating homogeneous turbulence

Recently, Squires, Chasnov, Mansour, & Cambon (1993) (hereon SCMC) addressed the problem of asymptotic behavior of homogeneous turbulence. Briefly, to summarize their results and findings, they applied a spectral LES method to achieve an asymptotic, equilibrium evolution of initially isotropic turbulence subjected to solid body rotation of angular speed Ω. The computations, which were run for times of order $O(10^3)$ of initial turbulence time scales, confirmed the prediction of the asymptotic decay laws. The authors used two different (initial) spectral forms of the energy spectrum $E(k)$ at low wavenumber k:

$$E(k) = 2\pi k^2 A_o + ... \quad \text{and} \quad E(k) = 2\pi k^4 A_2 + ...$$

In nonrotating turbulence the two spectral forms are known to produce different time decay exponents (n) of the turbulent kinetic energy $\frac{1}{2}q^2 = \int_0^\infty E(k)dk \propto t^{-n}$. For the k^2 spectrum, $n = 6/5$, and for the k^4 spectrum, $n = 10/7$. In the presence of rotation the following asymptotic decay laws were proposed in SCMC:

$$q^2 \propto A_o^{2/5} t^{-3/5} \Omega^{3/5} \quad (k^2 \text{ spectrum}) \tag{1}$$

$$q^2 \propto A_2^{2/7} t^{-5/7} \Omega^{5/7} \quad (k^4 \text{ spectrum}) \tag{2}$$

The above laws were confirmed by the LES computations within a few percent. Computations also indicated that the rotating turbulence has a tendency toward a two-dimensional state in the sense that the spectral energy tends to concentrate at wavenumbers normal to the rotation axis, i.e. the gradients with respect to the wavenumber parallel to $\mathbf{\Omega}$ become relatively small. At the same time the turbulence remained remarkably close to isotropy if measured by departure from the isotropy tensor $b_{ij} =< u_i u_j > /q^2 - 1/3\delta_{ij}$. This suggests a turbulence structure consisting of vortices aligned with the rotation axis and of jet-like (fluctuating) flow parallel to the rotation axis.

The purpose of this note is to explain the behavior of the rotating turbulence on the basis of a model for the spectral energy transfer, and to propose modification

of the turbulence spectrum when the rotation is much more rapid than the eddy turnover time scale. We start with a simplified Lagrangian description of the relation between stress and mean strain in rotating turbulence. The resulting relationship is then used to describe the energy transfer from larger to smaller scales in the spirit of the analysis described in Tennekes & Lumley (1972) for nonrotating turbulence.

Denoting Lagrangian fluctuating velocity components by v_i and taking the rotation vector to be $\mathbf{\Omega} = (0, 0, \Omega)$, one can write equations for the motion of fluid particles originating at some point in space and time $(\mathbf{a_o}, t_o)$ as follows:

$$\frac{dv_i}{dt} = -v_j U_{i,j} + 2\epsilon_{ij3}\Omega v_j + \Pi_i \tag{3}$$

Here, U_i represents a background mean velocity field which is considered as slow varying with respect to the characteristic Lagrangian (turbulence) time scale τ_L and length scale ($\propto q\tau_L$). $\Pi_i's$ are random forcing terms comprising the effect of the fluctuating pressure and higher order correlations. The viscous terms are taken to be negligible on account of the high turbulence Reynolds number assumption ($Re_T \propto q^2\tau_L/\nu >> 1$). The velocities v_i are functions of position and time $v_i(\mathbf{X}, t)$ of the fluid particle, with the initial position $\mathbf{X}(t_o) = \mathbf{a_o}$. Because turbulence is statistically homogeneous, we shall suppress the space dependence and utilize the ensemble-average identity $< v_i v_j > (t) =< u_i u_j > (t)$, i.e., the one-point Lagrangian averages (over all initial locations) are equal to the Eulerian averages (over the flow volume). A useful reference for Lagrangian description of turbulence is Monin & Yaglom (1971).

Now, neglecting the effect of Π_i and assuming that the gradients $U_{i,3}$ of the slow-varying velocity field (in the rotation direction) are negligible, it is possible to formulate a stress-strain relation $< v_1 v_2 > \propto -S_{12}$ ($S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$ is the slow strain tensor). This is achieved first by integrating (3) to obtain expressions for v_1, v_2:

$$v_1(t) = v_{1o}(t_o) - U_{1,j}\int_{t_o}^{t} v_j(t')dt' + 2\Omega\int_{t_o}^{t} v_2(t')dt'$$

$$v_2(t) = v_{2o}(t_o) - U_{2,j}\int_{t_o}^{t} v_j(t')dt' - 2\Omega\int_{t_o}^{t} v_1(t')dt'$$

Further manipulations yield an expression

$$< v_1 v_2 > =< u_1 u_2 > = -[c_1 \frac{\frac{1}{3}q^2\tau_L}{1 + 4c_2\tau_L^2\Omega(2\Omega - R_{12})}]2S_{12} = -2\nu_T S_{12}. \tag{4}$$

Here, all the required Lagrangian time scales such as $T_{L\alpha} =< v_\alpha^2 >^{-1} \int_0^\infty < v_\alpha(t')v_\alpha(t_o) > dt'$ were written for simplicity as a single time scale τ_L, which is in turn proportional to the turbulence time scale ℓ/u' (ℓ and u' being characteristic length and velocity scales); R_{ij} is the asymmetric complement to S_{ij}. The presumed differences in timescale magnitudes are absorbed in the free constants c_i. Clearly, (4) expresses a turbulent constitutive relation in the presence of rotation; the effective

eddy viscosity ν_T is evidently diminished by a factor depending on $(\tau_L\Omega)^2$. Although in the following analysis the flow field represented by S_{ij} and R_{ij} is taken as random, it is of interest to interpret (4) considering a homogeneous rotating shear flow with $S_{12} = R_{12} = \frac{1}{2}U_{1,2} = \frac{1}{2}S$. Then we note that the nature of the constitutive relation (4) is such that ν_T is maximized for $(\Omega/S)_{max} = 0.125$. On the other hand, the LES results of Bardina *et al.* (1985) and linear stability analysis (see e.g. Speziale, 1991) indicate the maximum turbulence amplification at $(\Omega/S)_{max} = 0.25$. Since in the following analysis R_{12} is neglected, the exact value of $(\Omega/S)_{max}$ is irrelevant to our problem. A stress-strain relation analogous to (4) has also been derived in cylindrical coordinates for a turbulent line vortex by Zeman (1994a) (comparable when the vortex flow is in solid body rotation i.e. when $U_{azimuth} = r\Omega$).

Now we shall relax the relation in (4) so that $S_{12} = S(k')$ represents the strain of eddies (of size $\propto 1/k'$) larger than the wavenumber k of the stress $< u_1u_2 > (k)$. Following the line of reasoning in Tennekes & Lumley (Section 8.4) concerning the spectral transfer in nearly isotropic turbulence, the spectral energy flux $T(k)$ across the wavenumber k in the inertial subrange is effected mainly by local interaction so that $T(k) \propto -S_{ij}(k') < u_iu_j(k'') > \propto \nu_T(k'')S^2(k')$ where, approximately, $k/3 < k' \leq k$ and $k'' = 3k'$. As shown in Tennekes & Lumley, the quantities at k' or k'' are directly related to the same quantities at k, thus e.g., $S(k') \propto (E(k)k^3)^{1/2} \propto \tau_L^{-1}(k'')$. Utilizing (4) to express $T(k)$ in terms of quantities (depending now on $E(k)$, k, Ω) and neglecting the contribution ΩR_{12} in (4) ($R_{12}(k)$ is a random quantity with zero mean and $|R_{12}| << \Omega$), we obtain a relationship

$$T(k) \propto \nu_T(k'')S^2(k') \propto \frac{(Ek)^{3/2}k}{1 + c_3\frac{\Omega^2}{Ek^3}}.$$

In the inertial subrange the spectral energy flux $T(k)$ across each wavenumber is constant and equal to the dissipation ϵ, and the above equation can be written as

$$\epsilon = \alpha^{-3/2}\frac{(Ek)^{3/2}k}{1 + c_3\frac{\Omega^2}{Ek^3}}, \tag{5}$$

where α is the Kolmogorov constant and c_3 is another free coefficient. Evidently, (5) represents an implicit relation for the energy spectrum $E(k, \epsilon, \Omega) = 0$ in the presence of rotation. For more insight into the meaning of (5), it is useful to define a rotation (cut-off) wavenumber k_Ω

$$k_\Omega = (\frac{\Omega^3}{\epsilon})^{1/2}, \tag{6}$$

which delimits the region of the spectrum where the rotation effects are important, i.e., $k < k_\Omega$; (note that k_Ω^{-1} is analogous to the Ozmidov length in stratified turbulence). In the region where $k << k_\Omega$, (5) results in an explicit expression

$$E(k) \propto \epsilon^{2/5}\Omega^{4/5}k^{-11/5}, \tag{7}$$

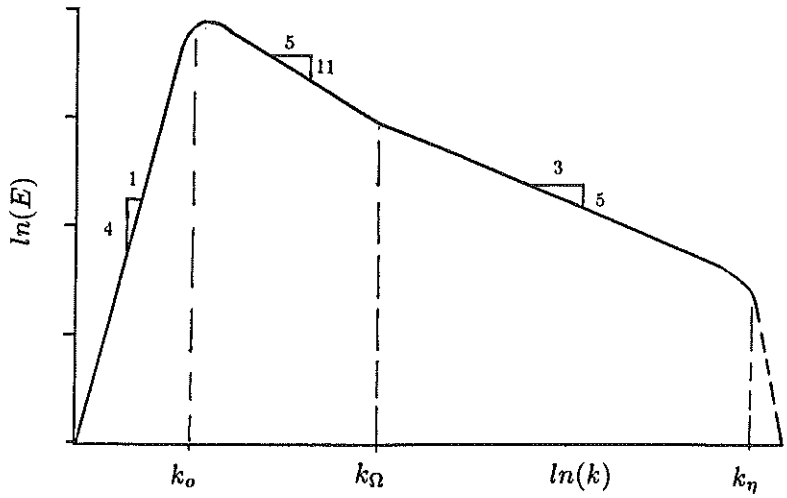

FIGURE 1. Sketch of the turbulence energy spectrum subjected to rotation; k_o and k_η are, respectively, the large-eddy and Kolmogorov scale wavenumbers, k_Ω is defined in (6).

while for $k >> k_\Omega$ the Kolmogorov inertial subrange $E(k) = \alpha\epsilon^{2/3}k^{-5/3}$ is recovered. The sketch of the spectrum (with exaggerated slope change) is depicted in Fig. 1. A general solution of (5) (with $c_3 = 1$) is obtained in the form

$$x = (\frac{y^5}{1 + 2y + y^2})^{1/4}, \tag{8}$$

where $x = k/k_\Omega$ and $y = Ek^3/\Omega^2$. Fig. 2 shows the solutions of the above equation emphasizing the rotation-affected range by plotting $Ek^{11/5}$ and $Ek^{5/3}$. It is seen that the spectrum of the form (7) is approximately valid for $k/k_\Omega \leq 10^{-1}$. It should be noted that expressing the eddy viscosity (in square brackets) in (4) in terms of spectral quantities at a given k, one obtains

$$\nu_T(k) = \frac{(E/k)^{1/2}}{1 + c_4/y(k)}, \tag{9}$$

hence the parameter Ω^2/Ek^3 is the measure of the damping effect of rotation on the local eddy viscosity. It is of interest that the same parameter appears in the expression for the subgrid-scale eddy viscosity in the LES of SCMC. Although the functional dependence of $\nu_T(k)$ on y is far more complicated, both CSMC and expression (9) give the same asymptotic dependence $\nu_T(k) \propto y$ if $y << 1$.

If there exists a self-similar spectrum as sketched in Fig. 1 (with $k_o < k_\Omega << k_\eta$), then (7) also contains information concerning the turbulence energy decay.

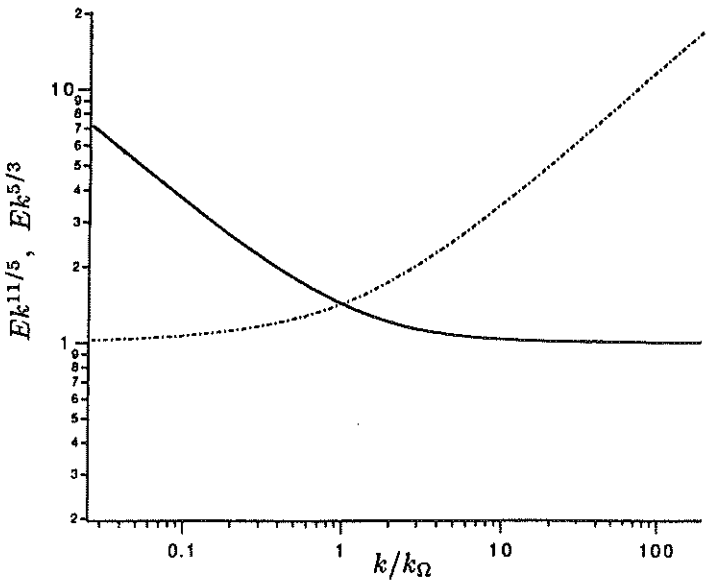

FIGURE 2. Spectral energy quantities $Ek^{11/5}$ (—·—) and $Ek^{5/3}$ (———) vs. k/k_Ω, based on the solution to (8). par

Assuming a k^4 spectrum for $k \leq k_o$, then $\frac{1}{2}q_{,t}^2 \propto A_2(k_o^5)_{,t} \propto -\epsilon$, and using (7) one obtains

$$q^2 \propto t^{-10/21}\Omega^{20/21}A_2^{2/7}. \tag{10}$$

The exponent $n = 10/21$ is lower than $n = 5/7$ in (2) derived from the dimensional analysis in SCMC. If, however, the rotation damping factor in (5) can be generalized to $1 + c_4 y^{-1} \propto y^{-m}$, one obtains the relation $n = 10/(7 + 14m)$. Hence, to satisfy the decay exponent $n = 5/7$ proposed in (2), m must take on the value $m = 1/2$. This leads to a spectral form

$$E(k) \propto (\epsilon\Omega)^{1/2}k^{-2},$$

different from (7). Analogous relations can be obtained for the k^2 spectrum to satisfy (1).

In summary, from Lagrangian analysis a relation between turbulent stress and strain in rotating homogeneous turbulence was inferred. This relation was utilized to derive the spectral energy flux and, ultimately, the energy spectrum form. If the rotation wavenumber k_Ω lies in the inertial subrange, then for wavenumbers less than k_Ω the turbulence motions are affected by rotation and the energy spectrum slope is modified. The present findings provide a new insight into the nature of the rotation effects on turbulence and, needless to say, their confirmation by (numerical) experiments would be desirable. It may, however, be difficult to experimentally distinguish the change in the spectral slope around the rotation wavenumber. The energy decay laws inferred in CSMC and the present results suggest a modification of the ϵ model equation and eddy viscosity in $k - \epsilon$ models. This is a subject of the following note.

2.2 A note on the eddy viscosity in rotating turbulence

A suggested generalization of the expression for the eddy viscosity (in the constitutive relation $\overline{u_i u_j} = -2\nu_T S_{ij} + \frac{1}{3}q^2\delta_{ij}$) in rotating turbulence derived by Zeman (1994b) (Eq. (4) in preceding Section 2.1) is

$$\nu_T = \frac{\nu_{To}}{1 + c_5\Omega_k(\Omega_k - \epsilon_{ijk}R_{ij})\tau^2}. \tag{11}$$

Here, ν_{To} is the appropriate eddy viscosity for nonrotating flow; otherwise the notation is as in Section 2.1, i.e., $q^2 = \overline{u_j u_j}$ is twice turbulent kinetic energy (TKE), $\tau = q^2/\epsilon$, and $R_{ij} = \frac{1}{2}(U_{i,j} - U_{j,i})$ is the rotation tensor. The optimal value of the numerical constant c_5 was found to be $c_5 \approx 0.1$.

In homogenous rotating turbulence with shear $S = \partial U_1/\partial x_2 = 2R_{12}$ and with the reference frame rotation $\Omega_j = \Omega_3$, (11) reduces to

$$\nu_T = \frac{\nu_{To}}{1 + c_5\Omega(\Omega - S/2)\tau^2}. \tag{12}$$

For a given value of the rotation-free viscosity ν_{To} and assumed constant value of the normalized shear $S\tau = Sq^2/\epsilon$, the eddy viscosity ν_T is solely a function of the ratio Ω/S and reaches maximum when $\Omega/S = 0.25$ in agreement with linear stability analysis. The function $\nu_T(\Omega/S)$ is symmetric about $\Omega/S = 0.25$ and falls off rapidly with increasing departure from 0.25. With the (tested) value of $S\tau = 12$, ν_T decreases by a factor of 14 as Ω/S changes from 0.25 to 0.25 ± 0.75.

Apart from the eddy viscosity, the rotation also affects the Kolmogorov energy cascade and therefore the rate of dissipation. The author proposed a correction to the ϵ equation to represent this effect (reported also in Hadid, Mansour & Zeman 1994). In the case of purely decaying turbulence subjected to rotation, the modified ϵ equation is

$$\frac{\partial\epsilon}{\partial t} = -\beta\frac{\epsilon^2}{q^2}, \tag{13}$$

where β is now a function of the rotation parameter $\omega = |\mathbf{\Omega}|\tau$, in the following way

$$\beta = 3.7 + \frac{5}{3}\frac{\omega^2}{1+\omega^2}. \tag{14}$$

In this formulation, (13) satisfies the decay law $\partial q^2/\partial t \propto t^{-n}$ so that the energy decay exponent n is 1.2 when $\omega = 0$ and $n = 0.6$ when $\omega >> 1$. The latter value is based on the asymptotic decay of rotating turbulence inferred from the scaling analysis and LES results of Squires *et al.* (1993) (when the energy spectrum $E(k)$ at the large scale end behaves as $E \propto k^4$). The form of the function (14) has been based on the analysis of Zeman (1994b). The model-experiment comparison for rotating decaying turbulence using (13) and (14) is shown in Fig. 3. The data are from the experiment of Wigeland & Nagib (1978).

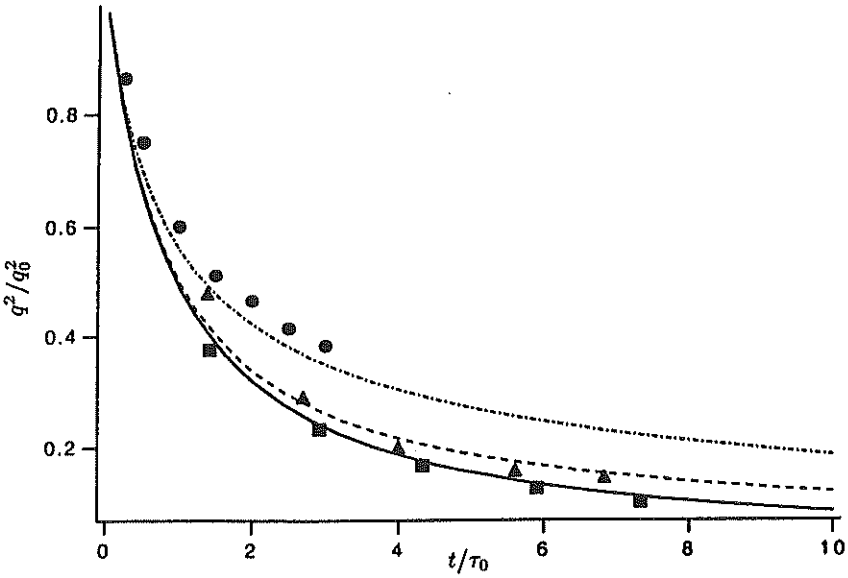

FIGURE 3. Decay of rotating turbulence; model-experiment comparison. Data points are from Wigeland & Nagib (1978). $\Omega_{\tau_0} = 0.12$ (——),0.47 (- - - -), 70.0 (—·—); $\Omega_{\tau_0} = 0.12$ (■), 0.47 (▲), and 70.0 (●).

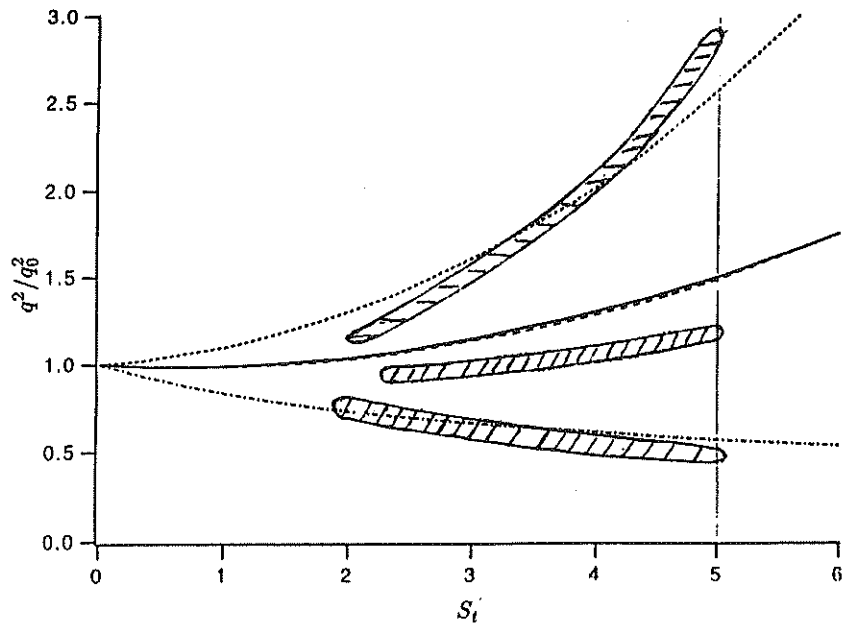

FIGURE 4. Shear-driven turbulence with rotation: k-ϵ model prediction (with modified viscosity in (12)); cross-hatched areas represent roughly the DNS results of Bardina *et al.* (1985). $\Omega/S = 0.0$ (——), 0.25 (- - - -), 0.5 (— — —), and -0.5 (—·—).

In the presence of shear $S = \partial U_1/\partial x_2$, the relevant $k-\epsilon$ model equations are

$$\frac{1}{2}\frac{\partial q^2}{\partial t} = P - \epsilon, \tag{15}$$

and (13) changes to

$$\frac{\partial \epsilon}{\partial t} = -\beta(\epsilon - 0.75P)\frac{\epsilon}{q^2}. \tag{13'}$$

The TKE production rate $P = 4\nu_T S_{ij} S_{ij} = \nu_T S^2$ is determined with the aid of the modified eddy viscosity in (12); the rotation parameter ω in (14) now has to include the contribution due to the presence of shear, i.e. $\omega^2 = (\Omega_k - \epsilon_{ijk} R_{ij})^2 \tau^2 = (\Omega - S/2)^2 \tau^2$. The results of comparison between the model represented by (13') and (15) and the DNS results of Bardina *et al.* (1985) are presented in Fig. 4. The trend in the turbulence evolution with varying Ω/S is apparently predicted although the $k-\epsilon$ model is incapable of predicting the rapid distortion regime during the initial development when $St < 1$. It is noted, however, that the majority of the Reynolds stress closure models are incapable of reproducing the Ω effect on shear turbulence, particularly for the case of maximum amplification when $\Omega/S = 0.25$.

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