A dynamic localization model with stochastic backscatter

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1. Motivation and objectives

1.1 The dynamic localization procedure

The modeling of subgrid scales in large-eddy simulation (LES) has been rationalized by the introduction of the dynamic localization procedure (Ghosal *et al.* 1993, 1994). This method allows one to compute rather than prescribe the unknown coefficients in the subgrid-scale model. Formally, the LES equations are supposed to be obtained by applying to the Navier-Stokes equations a "grid filter" operation defined as:

$$\overline{\psi}(\mathbf{x}) = \int_V d\mathbf{y}\; G\left(\frac{\mathbf{x}-\mathbf{y}}{\Delta}\right)\; \psi(\mathbf{y}), \tag{1}$$

where G is a kernel damping the fluctuations with a characteristic length shorter than Δ. The resulting equations (here we only consider incompressible flows)

$$\partial_t \overline{u}_i + \partial_j(\overline{u}_j \overline{u}_i) = \nu_0 \nabla^2 \overline{u}_i - \partial_i \overline{p} - \partial_j \tau_{ij}, \tag{2}$$

contain an unknown "subgrid stress" tensor τ_{ij} that needs to be modeled:

$$\tau_{ij} = \overline{u_i\, u_j} - \overline{u}_i\, \overline{u}_j. \tag{3}$$

Though the subgrid stress itself is unknown, an identity between subgrid stresses generated by different filters has been derived (Germano *et al.* 1991) and is the basic ingredient of the dynamic procedure:

$$L_{ij} = T_{ij} - \widehat{\tau_{ij}}, \tag{4}$$

where $L_{ij} = \widehat{\overline{u}_i \overline{u}_j} - \widehat{\overline{u}}_i\, \widehat{\overline{u}}_j$ is the Leonard tensor and $T_{ij} = \widehat{\overline{u_i\, u_j}} - \widehat{\overline{u}}_i\, \widehat{\overline{u}}_j$ is the subgrid stress tensor generated by a second filter defined by:

$$\widehat{\psi}(\mathbf{x}) = \int_V d\mathbf{y}\; \widehat{G}(\frac{\mathbf{x}-\mathbf{y}}{\widehat{\Delta}})\; \psi(\mathbf{y}). \tag{5}$$

Here $\widehat{G}$ is a kernel damping the fluctuations with a characteristic length shorter than $\widehat{\Delta}$. It will be referred to as the "test filter". If models τ_{ij}^M, T_{ij}^M are used for these quantities, the difference between the right- and the left-hand sides of relation (4),

$$E_{ij} \equiv L_{ij} + \widehat{\tau}_{ij}^M - T_{ij}^M \neq 0, \tag{6}$$

may be used as "quality indicators" for the subgrid-scale models. In practice, if the models contain a small number of unknown parameters (like in the Smagorinsky (1963) model $\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2C\Delta^2|\overline{S}|\overline{S}_{ij}$), the dynamic procedure proposes to determine these parameters by minimizing the quantity:

$$\mathcal{F}[C] = \int_V d\mathbf{y}\ E_{ij}[\mathbf{y}; C]\ E_{ij}[\mathbf{y}; C]. \tag{7}$$

This procedure partially removes the arbitrariness inherent to modeling in LES. However, the success of the dynamic procedure still strongly depends on the quality of the model for which it is implemented.

1.2 Dynamic localization model with k-equation

This model is motivated by the following considerations. When no constraint is imposed on the Smagorinsky coefficient C, the minimum of the functional $\mathcal{F}[C]$ is achieved for a field C, which can be locally negative. In this case, the model exhibits local reverse energy transfer. This is one of the simplest adaptations of the standard Smagorinsky expression to allow for a variable C and backscatter (Piomelli *et al.* 1991). However, it leads to some difficulties. A negative eddy viscosity generates an exponential amplification of local disturbances instead of the traditional exponential damping. The resulting backscatter is an "auto-catalytic" phenomenon which does not correspond to the real physics of reverse energy transfer in turbulent flows. As a consequence, unphysical instabilities in the LES equations have been observed when the coefficient C in the Smagorinsky model is determined by an unconstrained (no positivity required) variational procedure. The backscatter appears to be unsaturated and the model is unusable. However, Ghosal *et al.* (1993, 1994) have stressed that the reverse flow of energy must be quenched at the latest when all the subgrid-scale energy has been removed, and they have proposed to use the alternate representation

$$\nu_t = 2\ C'\ \overline{\Delta}\ k^{1/2} \tag{8}$$

instead of the Smagorinsky scaling. Here, k represents the subgrid-scale energy for which a separate transport equation is needed. The basic DLM(k) equations are given in (Ghosal *et al.* 1993, 1994). It can be shown that this model is stable. This approach involving the subgrid-scale kinetic energy is the first self-consistent model in which backscatter is accounted for in the framework of the dynamic procedure.

The combination of a locally negative transport coefficient and a saturation process is reminiscent of some instabilities in complex fluids. For example, phase separation in multicomponent mixtures can be described by instabilities created by a negative diffusion coefficient and saturated by surface tension effects (which are usually modeled by "hyperdiffusivity" terms). Roughly speaking, the DLM(k) picture for backscatter is similar. At some locations in the fluid, the eddy viscosity becomes negative. In a first stage, this generates an instability characterized by an exponential amplification of the local disturbances. In a second stage, a saturation process arrests the further growth of the instability. Later, the rapid changes in the turbulent velocity are likely to modify the local conditions, and the viscosity is expected to go back to positive values. This process does not explicitly take into account the possible stochastic nature of backscatter, but is not incompatible with a "molecular representation" of the small eddies. Indeed, the large diversity of small-scale eddies suggests that the turbulent fluid should behave more like a very complex (in a rheological sense) medium, and one should not expect the eddy viscosity to remain positive at every space time point.

Although preliminary tests of this model have been satisfactory, the use of a negative eddy viscosity to describe backscatter is probably a crude representation of the physics of reverse transfer of energy. Indeed, the model is fully deterministic. Knowing the filtered velocity field and the subgrid-scale energy, the subgrid stress is automatically determined. Obviously, this is only an approximation. It is very unlikely that the small scales influence the large scale evolution only through k. This is nevertheless an improvement when compared to the traditional Smagorinsky model in which no information from the small scales is included. However, we know that the LES equations cannot be fully deterministic since the small scales are not resolved. This stems from an important distinction between equilibrium hydrodynamics and turbulence. In equilibrium hydrodynamics, the molecular motions are also not resolved. However, there is a clear separation of scale between these unresolved motions and the relevant hydrodynamic scales. The result of molecular motions can then be separated into an average effect (the molecular viscosity) and some fluctuations. Due to the large number of molecules present in a box with size of the order of the hydrodynamic scale, the ratio between fluctuations and the average effect should be very small (as a result of the "law of large numbers"). For that reason, the hydrodynamic balance equations are usually purely deterministic. In turbulence however, there is no clear separation of scale between small and large eddies. In that case, the fluctuations around a deterministic eddy viscosity term could be significant. An eddy noise would then appear through a stochastic term in the subgrid-scale model and could be the source of backscatter. Some existing models have already represented reverse energy transfers by random terms. For example, a random eddy force derived from the eddy damped quasi-normal Markovian approximation has been used with some success in LES of isotropic turbulence by Chasnov (1991). This idea has been extended to boundary layers by Mason & Thomson (1992) and a similar approach has also been used by Leith (1990) to study LES of mixing layers. However, all these stochastic models contain an arbitrary parameter that must be tuned to obtain satisfactory results. Here we present an alternative subgrid-scale model in which the dynamic procedure is combined with a stochastic representation of backscatter. Following the dynamic procedure, no arbitrary parameter will be introduced in the model. Such a model represents a more traditional picture (Kraichnan, 1976; Leslie & Quarini, 1979) of backscatter than a negative eddy viscosity based model. However, it must be stressed that the true energy transfers between small and large scales are probably much more complex than that described by either an eddy viscosity or an eddy noise formalism. Both these models probably remain rather crude approximations to the real physical process.

2. Accomplishments

2.1 Stochastic dynamic localization model

The DLM(k) proposed by Ghosal *et al.* (1993, 1994) accounts for backscatter through a purely deterministic eddy viscosity. Let us now adopt a different point of view and assume that backscatter may be represented by a stochastic forcing term. At grid level, the proposed model is:

$$\partial_j \tau_{ij} = \partial_j (C\ \beta_{ij}) + f_i, \tag{9}$$

where $\beta_{ij} = -2\ \Delta^2\ |\overline{S}|\ \overline{S}_{ij}$ corresponds to the standard Smagorinsky model and f_i is an eddy force. For the sake of simplicity we will choose the simplest temporal behavior for $\mathbf{f}$ by assuming that the eddy force is a white noise process. The general form of its two-point, two-time correlation is then given by:

$$\langle f_i(\mathbf{r},t) f_j(\mathbf{r}',t')\rangle = A^2(\mathbf{r},t) H_{ij}(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{10}$$

where $(\mathbf{r},t)$ and $(\mathbf{r}',t')$ are two space-time points. The operator $\langle \ldots \rangle$ will denote the averaging over all possible realizations of the random force conditioned on a given velocity field $\mathbf{u}(\mathbf{r},t)$. The functions H_{ij} characterizing the forcing correlation will be discussed later. We only assume that the prefactor A^2 is chosen so that $H_{ii}(0) = 1$. In what follows, f_i is supposed to be divergence free (this can always be ensured by suitably modifying the pressure term). The choice of a solenoidal force is not essential but it simplifies the following discussion because the pressure then does not involve the random fields used to model the stochastic force. In some sense, the white noise process can be seen as the "most stochastic" choice. Thus, comparison between the stochastic dynamic localization model, DLM(S) defined by (9), and the DLM(k) should show what are the respective advantages (if any) of stochastic and deterministic models for backscatter.

It should be noted that the DLM(S) only models the divergence of the subgrid-scale stress. This is justified because the divergence is the only quantity needed in the LES equations. Also, the introduction of a stochastic force is much easier in the formulation (9). Thus, the quantity $\partial_j E_{ij}$ should be used in the dynamic procedure instead of E_{ij}. However, this would lead to major difficulties: The unknown quantities A and C would be determined by stochastic, integro-differential equations. The resolution of such equations would dramatically reduce the performances of the model. To avoid these problems, we propose to base the minimization procedure on the quantity $\langle E_{ij}\rangle$ instead of $\partial_j E_{ij}$ where the average is performed over all the possible realizations of the random noise f_i *conditioned on a fixed velocity field* $\mathbf{u}(\mathbf{r},t)$. This is a convenient approximation which results in the following simplifications. First, the error tensor $\langle E_{ij}\rangle$ is deterministic and totally independent of the random forces:

$$\langle E_{ij}\rangle = L_{ij} + \widehat{C\ \beta_{ij}} - C\ \alpha_{ij}, \tag{11}$$

where $\alpha_{ij} = -2\,\widehat{\Delta}^2|\widehat{\overline{S}}|\,\widehat{\overline{S}}_{ij}$. Also, C is now determined by minimizing $\int_V d\mathbf{y}\langle E_{ij}\rangle\langle E_{ij}\rangle$ which is exactly the same variational problem as in the deterministic models. Finally, since the DLM(S) is supposed to model backscatter by the eddy force, it is natural to assume that the Smagorinsky term is purely dissipative. The parameter C has then to be determined by the constrained dynamic localization model, DLM(+) (see Ghosal *et al.* 1993, 1994).

Since the force disappears from this first part of the dynamic procedure due to the conditional averaging over all realizations of the random force, we need an extra-relation for determining its amplitude. It can be obtained by noting that, even though the average effect of the force vanishes in the equations for the mean momentum, it will lead to a finite effect in the energy balance equation. Two equivalent energy equations may be obtained for the quantity $\widehat{\overline{E}} = \widehat{\overline{u}}_i\widehat{\overline{u}}_i/2$:

$$\partial_t\widehat{\overline{E}} = \ldots - \widehat{\overline{u}}_i\partial_j\left(C\ \alpha_{ij} + P\ \delta_{ij}\right) + \mathcal{E}_F, \tag{12a}$$

$$\partial_t\widehat{\overline{E}} = \ldots - \widehat{\overline{u}}_i\partial_j\left(\widehat{C\ \beta_{ij}} + L_{ij} + \widehat{p}\ \delta_{ij}\right) + \mathcal{E}_{\hat{f}}, \tag{12b}$$

where $\ldots$ stands for the viscous and inertial terms that are identical in both these equations. The pressure terms P and p are determined to keep the velocity divergence free at grid and test level. The quantities $\mathcal{E}_F$ and $\mathcal{E}_{\hat{f}}$ represent the energy input in the system respectively by F_i (the test level eddy force) and $\hat{f}_i$ (the filtered grid level eddy force). The difference between the right-hand sides of Eqs. (12a) and (12b)

$$Z \equiv \mathcal{E}_F - \mathcal{E}_{\hat{f}} - g \neq 0 \tag{13}$$

plays exactly the same role for the energy transfer as the quantities E_{ij} for the subgrid-scale stress. Here g is a known quantity (C has been determined by the DLM(+)) given by

$$g = \widehat{\overline{u}}_i\partial_j\left(C\ \alpha_{ij} + P\ \delta_{ij} - \widehat{C\ \beta_{ij}} - L_{ij} - \widehat{p}\ \delta_{ij}\right). \tag{14}$$

The minimization of the quantity $\mathcal{Z} = \int_V\ d\mathbf{r}\ \langle Z\rangle^2$ can now be used as a variational determination of the parameters that enter the model for the eddy force. At this point, little has been said about the statistical characteristics of the stochastic force itself. The variational problem presented here could be used together with a wide variety of choices for the eddy force.

Let us now discuss some additional assumptions that will greatly simplify the DLM(S) equations. From a computational point of view, it will be very convenient to consider the limiting case for which f_i at different grid points may be assumed to be completely uncorrelated. This avoids the non-trivial problem of generating random numbers with complex spatial correlations. This is also physically plausible since the eddy force is assumed to model random phenomena due to structures smaller than the mesh grid. Thus, the function H_{ij} will be assumed to be negligible for distances $|\mathbf{r}-\mathbf{r}'|$ larger than the mesh grid. Here we also assume that the probability distribution function of f_i corresponds to a Gaussian process. In that case, a force characterized by the correlations (10) leads to an average energy input

$$\mathcal{E}_f = \frac{1}{2}A^2(\mathbf{r},t)H_{ii}(0) = \frac{1}{2}A^2(\mathbf{r},t). \tag{15}$$

Let us now show how the dynamic procedure can be used to determine this energy input. The variational formulation presented above does not directly involve $\mathcal{E}_f$. However, we can easily relate it to $\mathcal{E}_f$ using Kolmogorov type ideas about energy transfer in the inertial range. Indeed, all the models used in LES - and model (9) is not an exception - are based on such arguments. Thus, the transfer rate is supposed to be independent of the filter width (in the inertial range), and we may assume that the backscatter rate has the same property $\langle\mathcal{E}_f\rangle = \langle\mathcal{E}_F\rangle$. Moreover, the test-filter acts like a local averaging of the random numbers f's which are almost uncorrelated (the two-point correlation is assumed to decay rapidly) and which vanish on the average. Thus, $\widehat{f}$ is much smaller than f, and $\langle\mathcal{E}_{\hat{f}}\rangle$ can be neglected when compared to $\langle\mathcal{E}_f\rangle$. It can then be assumed that $\langle\mathcal{E}_{\hat{f}}\rangle \ll \langle\mathcal{E}_f\rangle = \langle\mathcal{E}_F\rangle$. Thus, relation (13) gives

$$\langle Z\rangle = \langle\mathcal{E}_f\rangle - g. \tag{16}$$

Let us now consider a simple model for the stochastic force:

$$f_i = P_{ij}(\mathcal{A}\ e_j) \tag{17}$$

where $\mathcal{A}$ is a (dimensional) coefficient which plays a role similar to the Smagorinsky coefficient C. The operator $P_{ij} = \delta_{ij} - \nabla^{-2}\nabla_i\nabla_j$ takes out the divergence of the vector $\mathcal{A}e_j$. Here e_i are random numbers for which the probability distribution function is supposed to be Gaussian and isotropic:

$$\langle e_i(\mathbf{r},t)\rangle = 0, \tag{18a}$$

$$\langle e_i(\mathbf{r},t)e_j(\mathbf{r}',t')\rangle = \frac{1}{3}\ \delta_{ij}\ \delta(t-t')\ \delta_{\mathbf{r},\mathbf{r}'}, \tag{18b}$$

Here, we focus on the discrete equations and consequently we have used the Kroneker symbol. In agreement with the arguments leading to relation (16), the two-point correlation vanishes for $\mathbf{r}\neq\mathbf{r}'$. Comparison of (17) and (18) with (10) shows that $H_{ij}(\mathbf{r},\mathbf{r}') = P_{ik}P_{jk}\delta_{\mathbf{r},\mathbf{r}'}/2$ and $A^2 = 2\mathcal{A}^2/3$. Thus, the energy input is:

$$\langle\mathcal{E}_f\rangle = \frac{1}{2}A^2 = \frac{1}{3}\mathcal{A}^2. \tag{19}$$

The variational problem can then be used to determine $\mathcal{A}$ by minimizing

$$\mathcal{Z}[\mathcal{A}] = \int_V d\mathbf{y}\ \left(\frac{\mathcal{A}^2}{3} - g\right)^2. \tag{20}$$

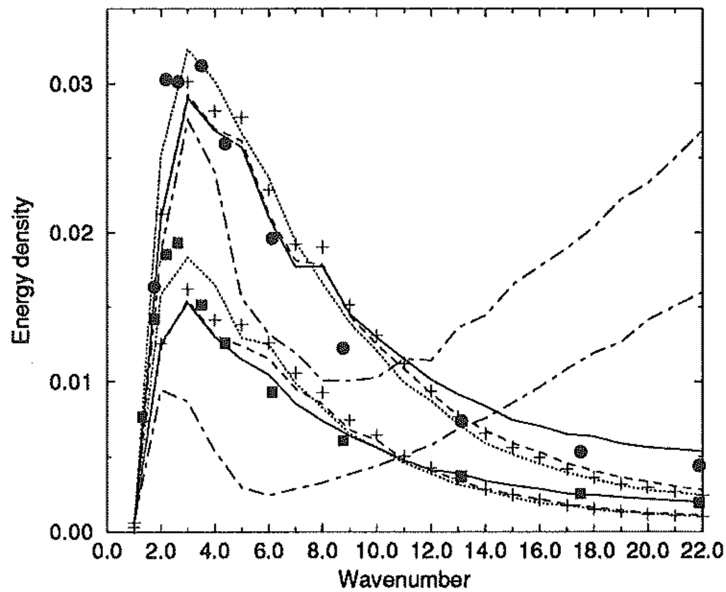

FIGURE 1. Time evolution of spectra in decaying isotropic turbulence —— : DLM(k); +: DLM(+); ---- : DM; ········ : DLM(S); —·— : no model; ●: experiment (t=1.55); ■ : experiment (t=2.70).

Following the method used by Ghosal *et al.* (1993, 1994) this leads to

$$\mathcal{A}^2 = 3\ [g]_+, \tag{21}$$

where $[x]_+ = (x + |x|)/2$. Thus, if we consider that the random numbers e_i are uncorrelated for different grid points, the DLM(S) is defined by (9) and (17) in which C has to be determined by the DLM(+) and the forcing amplitude $\mathcal{A}$ is given by the explicit relation (21).

2.2 Results

The model described in the previous section has been extensively tested and compared to other models (the original dynamic model: DM; the constrained dynamic localization model: DLM(+), and the DLM(k)) for forced and decaying isotropic flows. We will not discuss in detail the conditions of these simulations which are the same as those presented in previous reports (Ghosal *et al.* 1993 and Ghosal 1993).

Fig. 1 shows the result of a simulation performed using 48 grid points in each direction. This seems to be the smallest simulation that would be consistent with the condition implicit in the idea of LES, viz., that the subgrid scales should carry significantly less energy than the resolved scales.

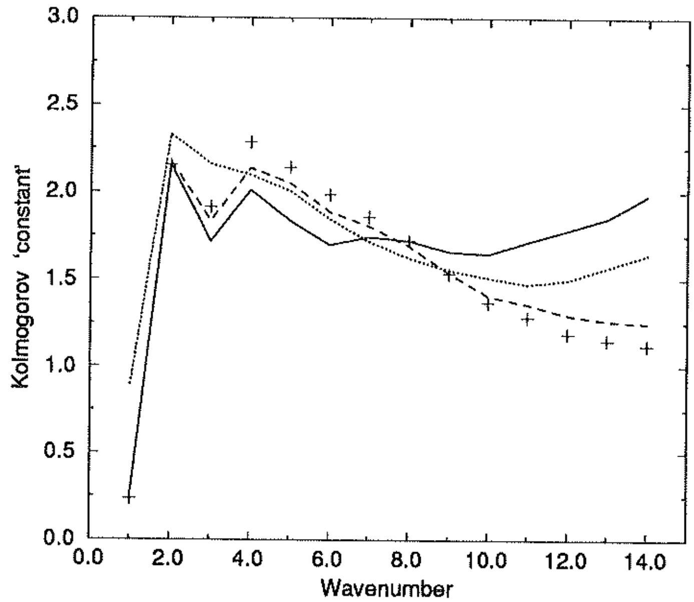

FIGURE 2. Prediction of Kolmogorov's 5/3 law and the Kolmogorov constant in forced isotropic turbulence at steady state ——— : DLM(k); +: DLM(+); ---- : DM; ········ : DLM(S).

All four models predict decays in the resolved energy that are in good agreement with the experiment (Comte-Bellot and Corrsin 1971).

In the asymptotic self-similar regime the energy decays as a power law $E \sim t^{-\alpha}$. It is not clear that such a self similar regime is reached in the present experiment since only three experimental points are available. However, the three experimental points almost lie on a straight line on a log-log plot. The decay exponent is thus estimated to be $\alpha \approx 1.20$. A least-square fit to the LES data yields the values $\alpha = 1.27$, DM; $\alpha = 1.21$, DLM(+); $\alpha = 1.28$, DLM(k); $\alpha = 1.17$, DLM(S). The predictions of LES are in good agreement with the value estimated from the experiment. These values are slightly lower than those obtained in the higher resolution LES with spectral eddy viscosity by Métais & Lesieur (1992). Results of running the simulation with no subgrid-scale model are also presented.

The average energy spectrum $E(\kappa)$ is obtained in forced turbulence. Fig. 2 shows $C_\kappa = \epsilon^{-2/3}\kappa^{5/3}E(\kappa)$ plotted against the wavenumber κ. According to Kolmogorov's 5/3 law (Kolmogorov, 1941), C_κ should be a constant in the inertial range. It is seen that the dynamic models with backscatter agree better with the 5/3 law than the purely dissipative models. Our best estimates for the Kolmogorov constant based on DLM(k) and DLM(S) are $C_\kappa \approx 1.8$ and $C_\kappa \approx 1.6$ respectively. The experimentally measured values of C_κ are in the range $1.3 - 2.1$ (Chasnov, 1991), though 1.5 is the

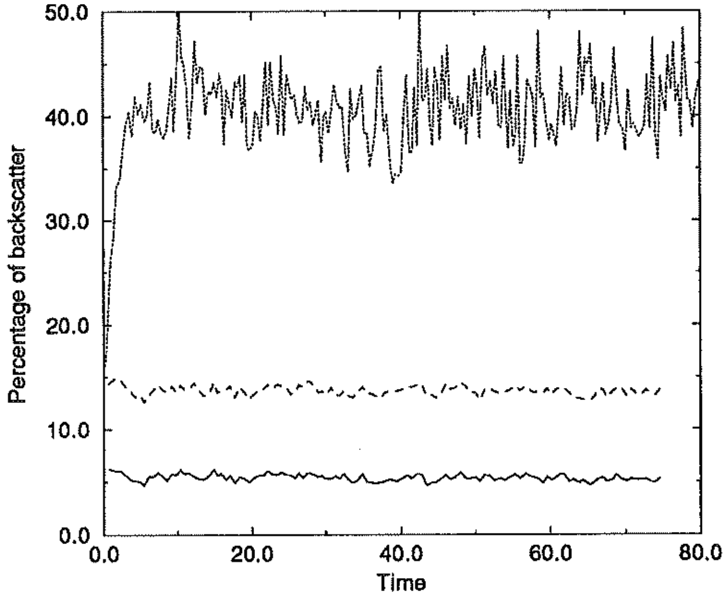

FIGURE 3. The level of backscatter as measured by energy transfer in forced isotropic turbulence at steady state ——— : DLM(k); ········ : DLM(S) and - - - - : as measured by fraction of points that have negative eddy-viscosity in DLM(k).

commonly accepted value (Saddoughi & Veeravalli, 1994). The spectra predicted by the DM and DLM(+) are almost identical to each other, and they seem to decay somewhat faster than the 5/3 law. As expected these models without backscatter seem to be too dissipative.

Fig. 3 shows the level of backscatter measured in two different ways. The solid line is the amount of energy being transferred from the subgrid to the large scales as a fraction of the net transfer as measured by

$$\frac{\int \left[\tau_{ij}\overline{S}_{ij}\right]_+ d^3\mathbf{x}}{\left|\int \tau_{ij}\overline{S}_{ij} d^3\mathbf{x}\right|} \qquad \text{for the DLM(k)}$$

$$\frac{\mathcal{E}_f}{\mathcal{E}_{sgs} - \mathcal{E}_f} \qquad \text{for the DLM(S)}$$

The dotted line is simply the fraction of grid points at which the Smagorinsky coefficient is positive (only for the DLM(k); in the DLM(S) C is constrained to be positive). Here, we notice a substantial difference between the two models accounting for backscatter. The deterministic DLM(k) predicts a much smaller amounts of backscatter than the stochastic DLM(S).

3. Future plans

The first test of the DLM(S) has shown that this model is able to reproduce most of the feature of isotropic turbulence. However, the way backscatter is accounted for does not seem to play a major role in the present simulations. The main difference in the predictions obtained by these models concerns the rate of backscatter which is significantly higher in the DLM(S) than in the DLM(k). However, it may be unsafe to conclude that one of the models performs better from this difference. Indeed, there is not a lot of measurement of backscatter rate, and in addition, this rate has been shown to be filter-dependent in DNS (Piomelli *et al.* 1991). Other performances of these two model are similar.

DLM(S) is cheaper to implement, but the DLM(k) provides more information since it predicts the subgrid-scale energy as well as the pressure. Complex flows could differentiate further between these models, and the next step in the validation of the stochastic model will be to implement it for more complex geometries. Preliminary work in the channel flow (Cabot, 1994) has shown that a slightly modified version of the DLM(S) from which the pressure totally disappears could be implemented more easily than the formulation presented here. The advantage of developing a model without explicit need of the pressure is obvious in complex geometries in which the pressure has to be obtained by a "Poisson solver". We thus plan to further develop this modified version of the DLM(S). Further tests in channel flows and mixing layer should follow.

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