Optimal active control for Burgers equation

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A method for active fluid flow control based on control theory is discussed. Dynamic programming and fixed point successive approximations are used to accommodate the nonlinear control problem. The long-term goal of this project is to establish an effective method applicable to complex flows such as turbulence and jets. However, in this report, the method is applied to stochastic Burgers equation as an intermediate step towards this goal. Numerical results are compared with those obtained by gradient search methods.

1. Motivation and objectives

There is current research at Stanford to develop active feedback control schemes based on optimal control theory to control turbulence. In particular, an optimal control method based on a gradient search algorithm is discussed in Choi, Teman and Moin (1993) and Bewley and Moin (1994). Such gradient schemes, however, are not guaranteed to converge to the global minimum of the cost functional and thus may suffer from degraded performance when compared with the "optimal" in a given situation.

The objective of the current research is to investigate an alternative method to the gradient search method without increasing computational complexity. The approach we take is to impose the convexity onto the cost functional and derive the analytical optimal controlled solutions for a set of linearized systems. This eliminates the minimization process of the cost functional. Then the optimal controlled solution for stochastic Burgers equation is found by Fixed Point Theorem. The resulting method is compared with the gradient method through numerical simulation, then assessments for applicability to more complex flow dynamics are made.

2. Scheme for optimal control

2.1 System model

We consider stochastic Burgers equation as a system model:

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\frac{u^2}{2} + F + \chi. \tag{1}$$

Initial condition and boundary conditions are given by $u(x, t_0) = u_0(x)$, $x \in (0,1)$ and $u(0,t) = u(1,t) = 0$, $t \in [t_0, T]$. Also, Re, $F(x,t)$, and $\chi(x,t)$ denote Reynolds number, a forcing term, and a normally distributed random forcing term with zero mean and unit variance, respectively.

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In order to control the velocity gradient $v = \partial u/\partial x$, we introduce the dynamics of the velocity gradient, which can be formally obtained by differentiating Burgers equation with respect to x such that

$$\frac{\partial v}{\partial t} = \frac{1}{Re}\frac{\partial^2 v}{\partial x^2} - u\frac{\partial v}{\partial x} - u_x v + f + \xi \tag{2}$$

where $f(x,t) = \partial F(x,t)/\partial x$ denotes the control forcing term for the differential form of Burgers equation and $\xi = \partial\chi(x,t)/\partial x$, a formally differentiated random forcing term.

2.2 Cost functional

We consider a control problem in which the cost functional to be minimized is given by

$$J = \frac{1}{2}E[\int_{t_0}^{T}\int_{0}^{1}(m_d(\frac{\partial u}{\partial x})^2 + l_d(f(x,t))^2)dxdt], \tag{3}$$

where $E[.]$ denotes a mathematical expectation. In a more general setting, the state, control, and the cost functional can be formed by

$$X = \begin{pmatrix} u \\ \frac{\partial u}{\partial x} \end{pmatrix}, \quad U = \begin{pmatrix} F \\ f \end{pmatrix},$$

and

$$J = \frac{1}{2}E[\int_{t_0}^{T}\int_{0}^{1}(m_d(X^T QX) + l_d(U^T RU))dxdt], \tag{4}$$

where

$$Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \text{ and } R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

If we select q_1 small relative to q_2 and r_2 small relative to r_1, we can formulate a problem similar to that discussed in Choi *et al.* (1993). However, this introduces a higher dimension of the system dynamics. For computational simplicity, control of the decoupled (1-D) system rather than the augmented (2-D) system is considered by selecting $q_1 = r_1 = 0$, which results in (3).

2.3 Optimal control strategy

A brief summary of the optimal control strategy is now given:

* *Linearize system:* Consider the system given by the linear equation

$$\frac{\partial v}{\partial t} = \frac{1}{Re}\frac{\partial^2 v}{\partial x^2} - u_{opt}\frac{\partial v}{\partial x} - (u_{opt})_x v + f + \xi, \tag{5}$$

where u_{opt} denotes the solution of Burgers equation (1) when the optimal control f_{opt} is applied to the system (2).

* *Design a linear optimal controller*: The dynamic programming technique is applied to the linear distributed parameter system (5) to find the optimal control for the linear system (for detail, see Tzafestas and Nightingale 1968).
* *Compute optimal controlled solution u_{opt}*: Once the optimal controller for (5) is found, integrate Eq. (5) with respect to x to yield:

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} - u_{opt}\frac{\partial u}{\partial x} + F + \chi. \tag{6}$$

To solve (6) for the optimal controlled solution, we need to know the optimal solution before it is solved. Hence, for the moment, we replace u_{opt} in (5) and (6) by a known function $w(x,t)$ and consider a mapping $G(w)$ defined on a function space which maps $w(x,t)$ to the solution $u(x,t)$ of (6). Notice that for each w the optimal controller is designed for the linearized system, and thus each image u of $G(w)$ forms an optimal solution for the corresponding system. Now, consider a family of optimal controlled solutions generated by $G(w)$. Then it is clear that the fixed point of $G(w)$ (if it exists) is the optimal controlled solution u_{opt} of Burgers equation (6). To find the fixed point of $G(w)$, a method of successive approximations is employed.

3. Numerical simulation

An early evaluation of this new optimal control formulation is important for determining the promise of the approach. To accomplish this, an evaluation through comparison is performed between the current method and the gradient method investigated by Choi *et al.* A numerical example was taken from Choi *et al.* (1993) for a comparison study (distributed control problem, case(ii), where the weights $l_d = 1$ and $m_d = 1/dx$ in the cost functional, $Re = 1500$ and $dx = 2047$). However, only qualitative comparison is meaningful in the current comparison study since the control problem is set up differently from the gradient method by Choi *et al.* in order to keep the computational complexity low. That is, the current method uses the cost functional (3) as one of the simplest cases of the more general form (4) (see discussion in 2.2), while the gradient method by Choi *et al.* uses the integrated control F instead of f in the cost functional (3). Another difference is that the cost functional in Choi *et al.* (1993) is formulated without the integral sign with respect to time; hence, the cost is minimized at each instance of time rather than over a duration of time.

Two different values for the ratio l_d/m_d were considered. Case-1: the weights l_d and m_d were set to be identical with those in the example. Case-2: the weight l_d was reduced by a factor of 1000 to allow more control power, keeping the weight m_d the same. In each case, the time histories of the cost functional, control power used, and gradient at the wall ($x = 0$) were computed. The results are shown in Figs. 1-3. The velocities at time 2 second are shown in Fig. 4. The corresponding figures from Choi *et al.* (1993) are also shown in Figs. 1, 2, and 4 for comparison.

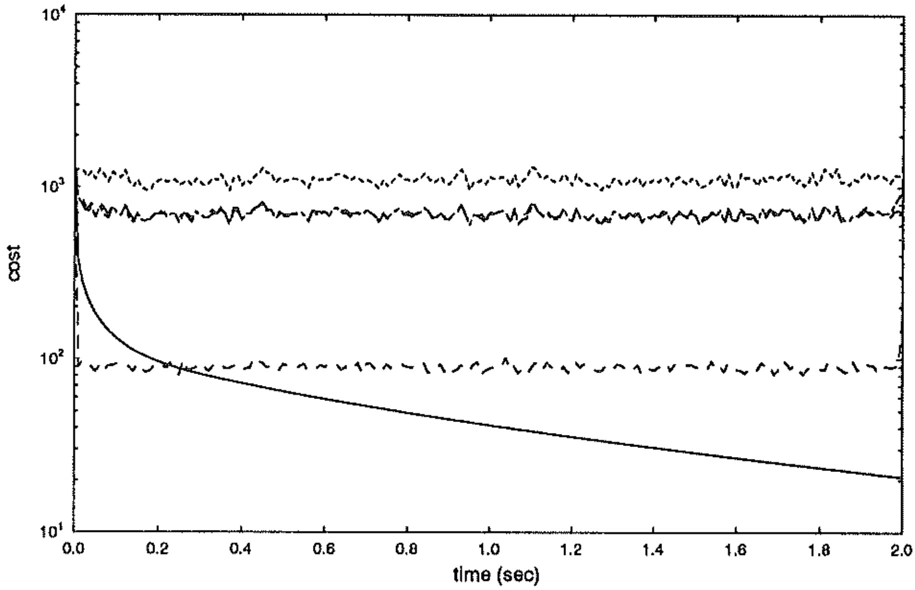

FIGURE 1. Time history of the cost functional. Legend: ——— , without forcing; - - - - , with random forcing and no control; — - — , with control and random forcing (case 1); — — — , with control and random forcing (case 2); —   — , with control and random forcing (Choi *et. al.*).

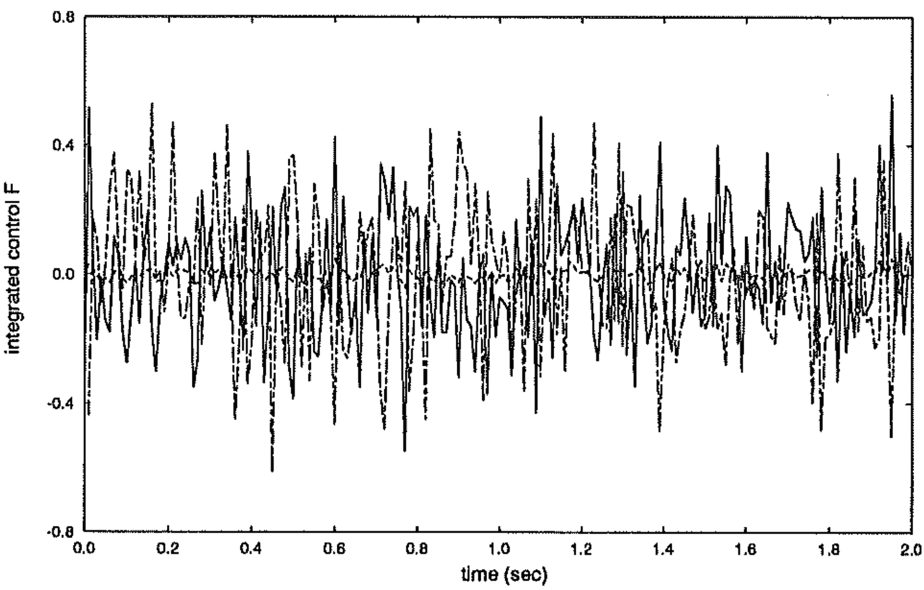

FIGURE 2. Time history of momentum forcing. Legend: ——— , control forcing (case 2); - - - - , control forcing (case 1); — - — , control forcing (Choi *et. al.*).

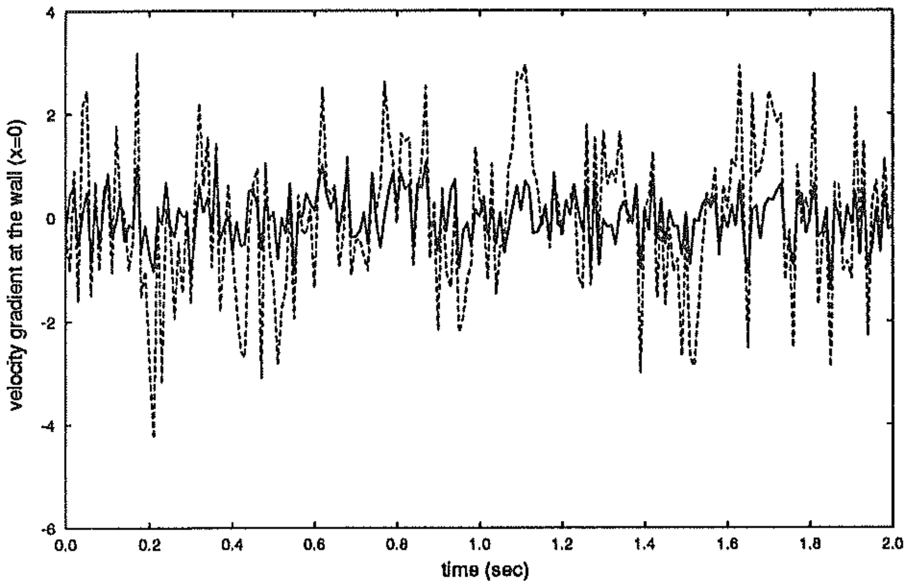

FIGURE 3. Wall velocity gradient at $x = 0$. Legend: —— , with control and random forcing (case 2); ---- , without control.

Fig. 1 shows that both methods reduced the cost functional significantly. The integrated control power F in each case of the current method and the gradient method are shown in Fig. 2. It shows that the amount of control F used in Case-1 is much less than that in Case-2 and the gradient method. Therefore, Case-2 seems more comparable to the example with the gradient method with respect to the momentum forcing added to Burgers equation. This seems natural since the velocity gradient becomes large in magnitude, particularly when the random noise is present. Hence, it requires more control power f when the velocity gradient is fed back than when the velocity is fed back. Fig. 3 shows that the current method controls the velocity gradient effectively if enough control power is allowed. Both Case-1 and the gradient method needed more control power to reduce the gradient at the wall significantly. From Fig. 4 it can be seen that the current method reduced the velocity magnitude as well as the velocity gradient while the gradient method did not reduce the velocity magnitude as much. This may be explained as follows: the current method seems to control the velocity gradient by regulating the gradient magnitude uniformly. Then, since the velocity at the boundaries are fixed to be zero, the velocity magnitude becomes small. On the other hand, the gradient method seems to control the velocity gradient by linearly scaling down. Hence, it reduces the absolute magnitude of the higher velocity gradient more significantly. One final observation is that the control formulated by the current method seems to respond more than a one order of magnitude faster than that by the gradient method (see Case-2 in Fig. 1, and Fig. 7(b) in Choi *et al.* 1993). This is a very important advantage for non-stationary applications.

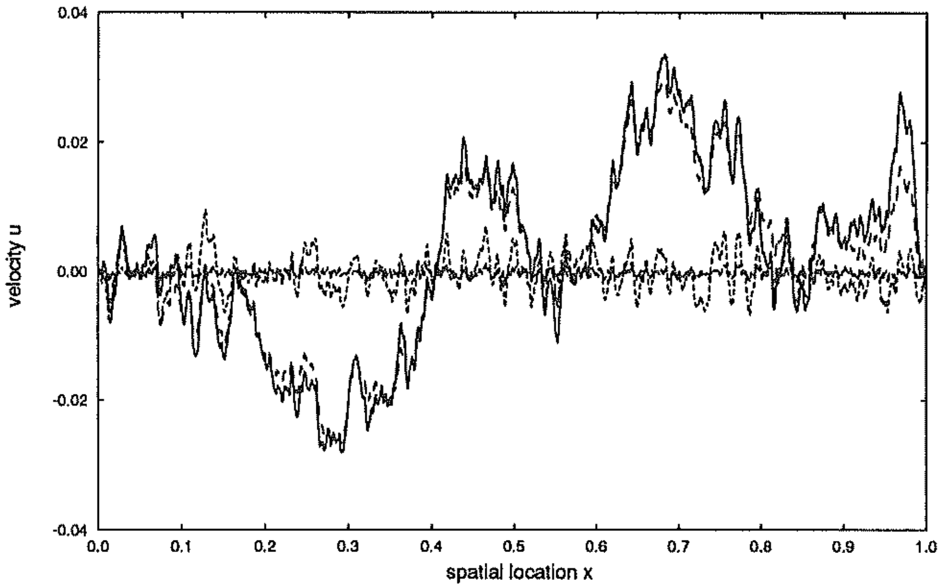

FIGURE 4. Velocity at $t = 2$. Legend: ——— , with random forcing and no control; ---- , with random forcing and control (case 1); —-— , with random forcing and control (case 2); — — — , with random forcing and control (Choi *et. al.*).

4. Conclusions

A method for active control of fluid flow dynamics was discussed. The simulation results show that the current control method works effectively and seems to be extendable to Navier-Stokes equations without major problems. Applications to turbulence and/or jet control will be attempted in the near future.

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