A method for obtaining a statistically stationary turbulent free shear flow

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1. Motivation and objectives

The long-term goal of the current research is the study of Large-Eddy Simulation (LES) as a tool for aeroacoustics. New algorithms and developments in computer hardware are making possible a new generation of tools for aeroacoustic predictions, which rely on the physics of the flow rather than empirical knowledge. LES, in conjunction with an acoustic analogy (Lighthill 1952), holds the promise of predicting the statistics of noise radiated to the far-field of a turbulent flow.

While there have been preliminary studies where LES was used in aeroacoustic calculations, a thorough examination of LES's predictive capabilities has not been undertaken. Recent advances in subgrid-scale models (Ghosal *et al.* 1992) have shown promising results, but accurate acoustic predictions present a far more stringent test of LES's fidelity. LES does not resolve all scales in the simulated flow, and results of a given simulation depend on the subgrid-scale model employed. Previous verification of LES results has focused on one point statistics. The application of an acoustic analogy requires a two point space-time correlation, and it is unclear how accurately LES will reproduce this quantity.

It has been shown that the dominant features of far-field noise are associated with the energy containing range of scales in the near-field, offering hope that LES can supply accurate predictions. However, in practical applications, the subgrid-scale energy can represent a significant fraction of the total. Further, as Crighton (1988) points out, care must be taken when applying acoustic analogies. Computationally, spurious but efficient low order acoustic sources may result from discretization errors, boundary conditions, etc. It is important to assess whether or not the subgrid-scale model acts as such a low order acoustic source.

LES's predictive ability will be tested through extensive comparison of acoustic predictions based on a Direct Numerical Simulation (DNS) and LES of the same flow, as well as *a priori* testing of DNS results. The method presented here is aimed at allowing simulation of a turbulent flow field that is both simple and amenable to acoustic predictions. A free shear flow that is homogeneous in both the streamwise and spanwise directions and which is statistically stationary will be simulated using equations based on the Navier Stokes equations with a small number of added terms. Studying a free shear flow eliminates the need to consider flow-surface interactions as an acoustic source. The homogeneous directions and the flow's statistically stationary nature greatly simplify the application of an acoustic analogy.

2. Accomplishments

A method allowing simulation of a statistically stationary free shear flow has been developed. The method is an extension of that presented by Spalart (1988) to the case of a wake or coflowing jet in the small deficit limit. The results are similar to those for the sink-flow boundary-layer presented in Spalart (1986). The derivation is more rigorous than either of the above analyses due to the simplification introduced by explicitly considering the small deficit limit. Some limited testing of the method has been carried out, and a more detailed validation is in progress.

2.1 Mathematical formulation

The flow to be simulated is that of a plane wake or coflowing jet. Self similar behavior is assumed and forms the basis for the rest of the analysis. The classical conditions necessary for self-similarity in a plane wake (in the small deficit limit) are

$$\delta \propto \sqrt{x} \tag{1}$$

$$U_o \propto \frac{1}{\delta} \propto \frac{1}{\sqrt{x}} \tag{2}$$

where δ is the wake thickness, $U_o = \Delta u_{max}$ is the maximum velocity deficit, and x denotes the streamwise direction.

A coordinate transformation is defined such that

$$(x, y, z, t) \rightarrow (x, \eta, z, t) \tag{3}$$

where $\eta = y/\delta(x)$. The spatial coordinates have been normalized by some initial length scale L_o, and time is normalized by some initial time scale t_o. Thus all quantities are non-dimensional. In the new set of coordinates, the profile thickness is independent of x. Lines of constant η have slope S where for the above coordinate system

$$S = \eta \frac{d\delta}{dx}. \tag{4}$$

The Jacobian of the coordinate transformation in space may then be written

$$J = \begin{pmatrix} 1 & 0 & 0 \\ S & T & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

where $T = \delta$ for the coordinate change given in (3). The actual Jacobian is a 4x4 matrix, but as no transformation is carried out in time, the extra elements are deleted for simplicity.

Following Spalart (1988), a transformation is made to the dependent variables as well. Let the Cartesian velocity components be denoted by (u^*, v^*, w^*), then the contravariant velocity components will be denoted by $(\tilde{u}, \tilde{v}, \tilde{w})$. The two sets are related through J^{-1}, giving

$$\begin{pmatrix} \tilde{u} \\ T\tilde{v} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -S & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^* \\ v^* \\ w^* \end{pmatrix}. \tag{6}$$

Use of the contravariant velocity components preserves the form of the transport terms in the new coordinate system, i.e.

$$u^*\frac{\partial}{\partial x} + v^*\frac{\partial}{\partial y} + w^*\frac{\partial}{\partial z} = \tilde{u}\frac{\partial}{\partial x} + \tilde{v}\frac{\partial}{\partial \eta} + \tilde{w}\frac{\partial}{\partial z}. \tag{7}$$

As Spalart shows, the continuity equation becomes

$$\tilde{u}_x + \tilde{v}_\eta + \tilde{w}_z + \frac{S_\eta}{T}\tilde{u} + \frac{T_\eta}{T}\tilde{v} = 0. \tag{8}$$

The x-momentum equation becomes

$$\begin{aligned}\tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{v}\tilde{u}_y + \tilde{w}\tilde{u}_z &= -p_x + \frac{S}{T}p_\eta \\ &+\frac{1}{Re}\Bigg(\left(-\frac{S_x}{T} + \frac{2SS_\eta}{T^2} - \frac{(1+S^2)T_\eta}{T^3}\right)\tilde{u}_\eta + \tilde{u}_{xx} \\ &\qquad + \frac{1+S^2}{T^2}\tilde{u}_{\eta\eta} - \frac{2S}{T}\tilde{u}_{x\eta} + \tilde{u}_{zz}\Bigg).\end{aligned} \tag{9}$$

where p is the kinematic pressure.

Normalized velocity components are introduced, and the equations are expressed in terms of the velocity deficit. The variable change is

$$\begin{pmatrix}\tilde{u}(x,\eta,z,t)\\ \tilde{v}(x,\eta,z,t)\\ \tilde{w}(x,\eta,z,t)\\ p(x,\eta,z,t)\end{pmatrix} \rightarrow \begin{pmatrix}U_\infty + U_o(x)u(x,\eta,z,t)\\ U_o(x)v(x,\eta,z,t)\\ U_o(x)w(x,\eta,z,t)\\ U_o^2(x)p(x,\eta,z,t)\end{pmatrix} \tag{10}$$

The maximum deficit is used to scale the velocity components so that the mean and r.m.s of the velocity components are independent of downstream location in the new coordinate system.

The resulting equations are not presented as they are quite cumbersome. One should note that new terms involving x derivatives of the velocity scale are introduced. These terms are recast in terms of the undifferentiated velocity deficit using the assumed self-similar streamwise evolution. Using Eqs. (2) and (4) and recalling that $T = \delta$, it can be shown that

$$\frac{dU_o}{dx} = -\frac{S_\eta}{T}U_o, \tag{11}$$

and

$$\frac{d^2U_o}{dx^2} = \left(2\frac{S_\eta^2}{T^2} - \frac{S_{x\eta}}{T}\right)U_o. \tag{12}$$

The equations are further simplified by choosing the length scale L_o to be δ_o, the dimensional value of the layer thickness at the downstream station to be simulated.

This implies the dimensionless quantity δ is 1. Then $T = 1$, η derivatives of T are zero, and x derivatives are removed through the identity

$$T_x = S_\eta. \tag{13}$$

Further, if δ is unity, η and y become equivalent, and all instances of η in the equations may be replaced with y. Then defining the parameter

$$\epsilon = \frac{U_o}{U_\infty}, \tag{14}$$

the continuity equation becomes

$$\frac{S_y}{\epsilon} + u_x + v_y + w_z = 0. \tag{15}$$

The x-momentum equation becomes

$$u_t + uu_x + vu_y + wu_z + \left(\frac{u_x}{\epsilon} - \frac{S_y u}{\epsilon} - S_y u^2\right) = -p_x + Sp_y + \text{viscous terms}. \tag{16}$$

The viscous terms are quite complicated and are omitted for simplicity.

A Galilean transformation is now applied. A new coordinate x' is defined such that

$$x' = x - U_\infty t. \tag{17}$$

Applying this transformation, the terms in the momentum equations resulting from convection at U_∞ (e.g. the u_x/ϵ term in (16)) vanish.

The slope of the new coordinate lines S will now be expressed in the new coordinate system. Returning to the unmoving coordinate system, using (2) and (4) it can be shown that

$$S = \frac{U_o^2 \delta}{2} \frac{d}{dx}\left(\frac{1}{U_o^2}\right) y. \tag{18}$$

A quantity that was only a function of x in the stationary frame becomes a function of both x' and t in the moving frame, i.e.

$$F(x) = F(x' + U_\infty t). \tag{19}$$

Differentiating with respect to time in the new frame shows

$$\frac{1}{U_\infty} \frac{\partial F}{\partial t} = F' = \frac{dF}{dx}. \tag{20}$$

Applying (20) to (18) gives S in the new coordinate system as

$$S = \frac{U_o \delta}{2} \frac{\partial}{\partial t}\left(\frac{1}{U_o^2}\right) \epsilon y. \tag{21}$$

A parameter α is defined by

$$\alpha = \frac{U_o \delta}{2} \frac{\partial}{\partial t} \left(\frac{1}{U_o^2} \right), \tag{22}$$

and the scaling of U_o in (2) implies that α is constant. It follows that $S = \alpha \epsilon y$.

The small deficit limit is then enforced, letting $\epsilon \to 0$, giving for the continuity and momentum equations

$$\nabla \cdot \vec{u} = -\alpha, \tag{23}$$

and

$$\vec{u}_t + \vec{\omega} \times \vec{u} = -\nabla p + \alpha \begin{pmatrix} u \\ -v \\ w \end{pmatrix} + \frac{1}{Re} \nabla^2 \vec{u}. \tag{24}$$

These are the equations that will be integrated to yield the turbulent flow field. Note that (20) indicates that $V = -\alpha y$, and that the fluctuating velocity components are divergence free. Also note that all added viscous terms dropped out when the limit was taken.

2.2 Implementation

The modified equation set was implemented in the spectral code used by Rogers and Moser (1993) in their study of self-similar turbulent mixing layers. The numerical method (Spalart, Moser and Rogers 1991) employs Fourier analysis in conjunction with periodic boundary conditions in both the streamwise and spanwise directions. The application of periodic conditions in the streamwise direction is an approximation as the length and time scales for the fluctuating quantities are x dependent. The approximation should be a good one for the normalized components in this coordinate system as the variation of these scales is slow compared with the variation of δ. The non-homogeneous direction is handled through the use of rapidly decaying spectral basis functions in conjunction with two slowly decaying additional functions that exactly represent the irrotational component of the solution far from the vortical region.

The major question in implementing the modified equations is how to evaluate α. An initial value may be calculated for α by taking the second moment of the mean x-momentum equation, setting U_t to zero, and solving for α. Making use of the fact that the mean field is a function only of y, and integrating by parts, discarding the boundary terms, gives

$$\alpha = \frac{\int \frac{1}{2} y^2 \overline{(\vec{u} \times \vec{\omega})}_x + \frac{1}{Re} U dy}{\int y^2 U dy} \tag{25}$$

This value is then held, and the mean flow is allowed to evolve until a statistical steady state is reached. The flow in the interim has no physical meaning.

It may appear that a similar scheme may be used to constantly update α as the flow is evolving, in effect setting α to the value it would have if the flow was stationary at that moment, hopefully speeding evolution to the stationary state. In practice, however, this presents problems. It appears that a feedback loop involving the Reynolds stresses results. Further investigation in this area is clearly needed. For the present work this is not critical as the initial conditions to be used are fully developed wakes simulated by Moser and Rogers (1994). Calculating α based on the initial condition and fixing it should not result in a significant evolutionary period.

2.3 Testing

The first test case was a laminar wake. The initial condition was simply a Gaussian. The flow evolved into a shape that was nearly Gaussian (a Gaussian does not solve the modified equation set) and remained stationary. A second test consisted of an initial condition of y^2 times a Gaussian, such that the profile had the same mass flow and second moment. The flow evolved to the same steady solution as was reached starting with a Gaussian.

The first turbulent test case is that of a low Reynolds number turbulent wake. This test is currently underway. The initial condition was taken from a temporally evolving simulation done by Moser and Rogers (1994). The results will be compared qualitatively to the results obtained from the temporal simulation to ensure that the results produced by the modified equation set are reasonable.

3. Future plans

Upon completion of the testing with the low Reynolds number wake, a higher Reynolds number DNS will be carried out, building the database necessary for aeroacoustic predictions. *A priori* testing of the DNS results will be carried out, comparing the acoustic prediction from the DNS data and a filtered version of the same flow-field. It has been seen previously that LES simulations are capable of producing better results than indicated by *a priori* tests. Therefore, the code will also be modified, and an LES of the same flow will be carried out. Using these databases, the issues discussed in section 1 will be investigated.

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