Floquet stability analysis of two-phase flows

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1. Motivation and objectives

The growth of ocean waves due to the action of wind, where large waves are excited by the interaction between the wind and a corrugated liquid-air interface, is reminiscent of the growth of secondary instabilities in response to a primary disturbance in a locally parallel flow [e. g., see, for example, Buckley & Veron (2016) for a systematic experimental study of this configuration with laminar starting conditions]. Over the past few decades, several theories (Miles 1957; Phillips 1957; Belcher & Hunt 1993) have been proposed to explain the physical mechanism for the excitation of these waves in various physical regimes, including those where the boundary layer above the interface is assumed to be turbulent.

Motivated by these studies, and by the experimental observation of the growth of these waves even when the starting conditions are laminar, a secondary instability analysis of two-phase (liquid-gas) flows was sought. Secondary stability analysis of periodic water waves was previously performed by Longuet-Higgins in a pair of companion papers (Longuet-Higgins 1978*a*,*b*). The underlying concept of expanding both the periodic base state and the superimposed disturbance field in truncated Fourier series bears resemblance to the Floquet analysis later introduced by Herbert (1983, 1984, 1988) in the study of the secondary instability of Tollmien-Schlichting waves in laminar boundary layers.

The wave analyses by Longuet-Higgins (1978*a*,*b*) were based on the surmise that the flow in the water could be treated inviscidly with potential flow theory and that the flow in the air is dynamically insignificant. These assumptions are not valid at large times when the initially thin vortex sheet localized at the interface has diffused into the two phases, and for large differences in the free-stream velocities of the air and the water where deviations of the air velocity from the free-stream velocity at locations near the interface are significant. Thus, in this work, the Floquet ansatz is extended to accommodate flows in two phases coupled at their interface without making any physical simplifications while explicitly considering the effects of surface tension. Direct application of the instability analysis to wind-driven periodic water waves is ongoing and will be deferred to future work.

In Section 2, the relevant equations required for the secondary instability analysis are developed. Some preliminary test cases are presented in Section 3 and the findings are concluded in Section 4.

2. Formulation of Floquet stability equations

First, the necessary equations for a single phase with the appropriate normalizations are formulated. The coupling conditions necessary for the solution of the problem, as well as a coordinate transform to simplify the numerical simulation of the derived equations, are then introduced.

2.1. *Linearized Navier-Stokes equations for a single phase*

Suppose the local fluid velocity at any point in a predefined domain $\mathbf{u_f} = (u_f, v_f, w_f)^T$ can be decomposed as a superposition of a two-dimensional steady base flow $\mathbf{U_f} = (U_f, V_f, W_f)^T = [U_f(x,y), V_f(x,y), 0]^T$ and small disturbances

$$\mathbf{u'_f} = [u'_f(x,y,z,t), v'_f(x,y,z,t), w'_f(x,y,z,t)]^T]. \tag{2.1}$$

Here, the subscript f refers to the phase being considered $f \in \{l, g\}$ where l refers to the liquid and g refers to the gas. Assuming the disturbances are small, and considering a reference frame moving with the phase speed of the primary disturbance C^I aligning the x-coordinate with the motion of the primary disturbance, the nondimensional Navier-Stokes equations in Cartesian coordinates can be linearized as follows

$$\rho_f^* \left(\frac{\partial u'_f}{\partial t} + U_f \frac{\partial u'_f}{\partial x} + u'_f \frac{\partial U_f}{\partial x} + V_f \frac{\partial u'_f}{\partial y} + v'_f \frac{\partial U_f}{\partial y} \right) = -\frac{\partial p'_f}{\partial x} + \frac{\mu_f^*}{\mathrm{Re}} \nabla^2 u'_f, \tag{2.2}$$

$$\rho_f^* \left(\frac{\partial v'_f}{\partial t} + U_f \frac{\partial v'_f}{\partial x} + u'_f \frac{\partial V_f}{\partial x} + V_f \frac{\partial v'_f}{\partial y} + v'_f \frac{\partial V_f}{\partial y} \right) = -\frac{\partial p'_f}{\partial y} + \frac{\mu_f^*}{\mathrm{Re}} \nabla^2 v'_f, \tag{2.3}$$

$$\rho_f^* \left(\frac{\partial w'_f}{\partial t} + U_f \frac{\partial w'_f}{\partial x} + V_f \frac{\partial w'_f}{\partial y} \right) = -\frac{\partial p'_f}{\partial z} + \frac{\mu_f^*}{\mathrm{Re}} \nabla^2 w'_f, \tag{2.4}$$

$$\frac{\partial u'_f}{\partial x} + \frac{\partial v'_f}{\partial y} + \frac{\partial w'_f}{\partial z} = 0, \tag{2.5}$$

where the Reynolds number $\mathrm{Re} = \rho_l UL/\mu_l$ is defined without loss of generality using the density ρ_l and viscosity μ_l of the liquid, as well as some characteristic base speed U and length scale L. The equations also contain the density ratio $\rho_f^* = \rho_f/\rho_l$ and the viscosity ratio $\mu_f^* = \mu_f/\mu_l$ for the particular phase in consideration.

For quasi-parallel flows, the base flow velocities can be decomposed into

$$U_f(x,y) = U_f^O(y) - C^I + \sum_{m=-\infty}^{\infty} U_{f,m}^I(y) e^{\mathrm{i}m\alpha x}, \tag{2.6}$$

$$V_f(x,y) = V_f^O + \sum_{m=-\infty}^{\infty} V_{f,m}^I(y) e^{\mathrm{i}m\alpha x}, \tag{2.7}$$

where U_f^O and V_f^O are, respectively, the x- and y-components of the base flow associated with the free-stream boundary condition(s), and $U_{f,m}^I$ and $V_{f,m}^I$ are, respectively, the magnitudes of the x- and y-components of the harmonic variation of the primary disturbance with wavenumber $m\alpha$. Here, α is the fundamental wavenumber of the primary disturbance field being considered. Note that $U_{f,0}^I = V_{f,0}^I = 0$. Also, by continuity, V_f^O is constant. In the application of this analysis to the secondary instability of viscous boundary layers, for example, $U_{f,m}^I$ and $V_{f,m}^I$ are proportional to the magnitude of the eigensolutions to the primary instability problem if the normal-mode ansatz for the primary instability analysis is proportional to $e^{\mathrm{i}m\alpha x}$. In the application of this analysis to periodic water waves, $U_{f,m}^I$ and $V_{f,m}^I$ are the m-th Fourier coefficients of the underlying flow in the presence of the wavy interface subtracting the contribution from the free-stream velocity. $U_{f,-m}^I$ and $V_{f,-m}^I$ are then the complex conjugates of $U_{f,m}^I$ and $V_{f,m}^I$, respectively. The Floquet ansatz for the fluctuations is now introduced

$$\mathbf{q_f}(x,y,z,t) = \sum_{n=-\infty}^{\infty} \mathbf{q_f}^{(n)}(y) e^{\mathrm{i}(n\alpha+\gamma)x+\mathrm{i}\beta z-\mathrm{i}\sigma t}, \tag{2.8}$$

$$\mathbf{q_f} = (u'_f, v'_f, w'_f, p'_f)^T, \tag{2.9}$$

where $\gamma \in [0, 0.5\alpha]$ denotes the wavenumber detuning factor. For subharmonic modes, for example, $\gamma = 0.5\alpha$. Also, σ denotes the shifted frequency such that it should be purely imaginary in the moving frame for a temporally unstable mode moving at the same speed as the primary disturbance under consideration. Substitution of this ansatz into the linearized Navier-Stokes equations yields the following relations for each Floquet mode n

$$\rho_f^* \left[-\mathrm{i}\sigma u'_f + \mathrm{i}(n\alpha+\gamma)U_f u'_f + u'_f \frac{\partial U_f}{\partial x} + V_f \mathcal{D} u'_f + v'_f \frac{\partial U_f}{\partial y} \right] = -\mathrm{i}(n\alpha+\gamma)p'_f + \frac{\mu_f^*}{\mathrm{Re}}[\mathcal{D}^2 - (n\alpha+\gamma)^2 - \beta^2]u'_f, \tag{2.10}$$

$$\rho_f^* \left[-\mathrm{i}\sigma v'_f + \mathrm{i}(n\alpha+\gamma)U_f v'_f + u'_f \frac{\partial V_f}{\partial x} + V_f \mathcal{D} v'_f + v'_f \frac{\partial V_f}{\partial y} \right] = -\mathcal{D}p'_f + \frac{\mu_f^*}{\mathrm{Re}}[\mathcal{D}^2 - (n\alpha+\gamma)^2 - \beta^2]v'_f, \tag{2.11}$$

$$\rho_f^* \left[-\mathrm{i}\sigma w'_f + \mathrm{i}(n\alpha+\gamma)U_f w'_f + V_f \mathcal{D} w'_f \right] = -\mathrm{i}\beta p'_f + \frac{\mu_f^*}{\mathrm{Re}}[\mathcal{D}^2 - (n\alpha+\gamma)^2 - \beta^2]w'_f, \tag{2.12}$$

$$\mathrm{i}(n\alpha+\gamma)u'_f + \mathcal{D}v'_f + \mathrm{i}\beta w'_f = 0, \tag{2.13}$$

where $\mathcal{D} = \partial/\partial y$. Substituting (2.6) and (2.7) into the above equations and truncating the infinite sums over m and n to $[-M, M]$ and $[-N, N]$, respectively, then yields the following temporal eigenproblem

$$\mathsf{A_f}\hat{\mathbf{q}}_\mathbf{f} = \mathrm{i}\sigma \mathsf{B_f}\hat{\mathbf{q}}_\mathbf{f}, \tag{2.14}$$

where

$$\hat{\mathbf{q}}_\mathbf{f} = \begin{bmatrix} \mathbf{q_f}^{(-N)T} & \cdots & \mathbf{q_f}^{(n)T} & \cdots & \mathbf{q_f}^{(N)T} \end{bmatrix}^T, \tag{2.15}$$

$$\mathsf{A_f} = [\mathsf{A_{f(i,j)}}], \tag{2.16}$$

$$\mathsf{B_f} = [\mathsf{B_{f(i,j)}}], \tag{2.17}$$

$$\mathsf{A_{f(n,n)}} = \begin{bmatrix} \Gamma_f^n & \rho_f^* \mathcal{D} U_f^O & 0 & \mathrm{i}(n\alpha+\gamma)\mathsf{I} \\ 0 & \Gamma_f^n & 0 & \mathcal{D} \\ 0 & 0 & \Gamma_f^n & \mathrm{i}\beta\mathsf{I} \\ \mathrm{i}(n\alpha+\gamma)\mathsf{I} & \mathcal{D} & \mathrm{i}\beta\mathsf{I} & 0 \end{bmatrix}, \tag{2.18}$$

$$\mathsf{A}_{\mathsf{f(n,n-m)}} = \begin{bmatrix} \Gamma^n_{f,m} - \rho^*_f \mathcal{D} V^I_{f,m} & \rho^*_f \mathcal{D} U^I_{f,m} & 0 & 0 \\ \mathrm{i}m\alpha\rho^*_f V^I_{f,m} & \Gamma^n_{f,m} + \rho^*_f \mathcal{D} V^I_{f,m} & 0 & 0 \\ 0 & 0 & \Gamma^n_{f,m} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.19}$$

$$\mathsf{B}_{\mathsf{f(i,j)}} = \delta_{ij}\mathsf{B}_{\mathsf{f(i)}} \quad \text{with} \quad \mathsf{B}_{\mathsf{f(i)}} = \begin{bmatrix} \mathsf{I} & 0 & 0 & 0 \\ 0 & \mathsf{I} & 0 & 0 \\ 0 & 0 & \mathsf{I} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.20}$$

$$\Gamma^n_f = -\frac{\mu^*_f}{\mathrm{Re}}\left[\mathcal{D}^2 - (n\alpha+\gamma)^2\mathsf{I} - \beta^2\mathsf{I}\right] + \rho^*_f\left[\mathrm{i}(n\alpha+\gamma)(U^O_f - C^I\mathsf{I}) + V^O_f\mathcal{D}\right], \tag{2.21}$$

$$\Gamma^n_{f,m} = \mathrm{i}[(n-m)\alpha+\gamma]\rho^*_f U^I_{f,m} + \rho^*_f V^I_{f,m}\mathcal{D}, \tag{2.22}$$

where I is an appropriately sized identity matrix. Note that because of the truncated sum over m, the coefficients $U^I_{f,m}$ and $V^I_{f,m}$ disappear when $|m| > M$.

2.2. *Coupling the two phases*

Here, it is assumed that the gas phase is situated above the liquid phase along the y-coordinate, and gravity is oriented in the negative y-direction. Also, the undisturbed interface between the two phases lies initially at $y = 0$ so the gas phase occupies $y > 0$ and the liquid phase occupies $y < 0$. The subsequent analysis is generalizable to other configurations.

The two phases will need to be coupled in the neighborhood of $y = 0$ through a coupling equation. The coupling equation is derived from the kinematic boundary condition and involves the interface displacement h in the following manner

$$\left[\frac{\partial}{\partial t} + U(y=h)\frac{\partial}{\partial x}\right] h = v'(y=h). \tag{2.23}$$

Let the Floquet ansatz be assumed for h as well, with the addition of a time-invariant zeroth-order term h_0 that is assumed to vary only in the streamwise direction

$$h(x,y,z,t) = h_0(x) + \sum_{n=-\infty}^{\infty} h'^{(n)} e^{\mathrm{i}(n\alpha+\gamma)x + \mathrm{i}\beta z - \mathrm{i}\sigma t} \tag{2.24}$$

$$= \sum_{n=-\infty}^{\infty} \left[h_0^{(n)} e^{\mathrm{i}n\alpha x} + h'^{(n)} e^{\mathrm{i}(n\alpha+\gamma)x + \mathrm{i}\beta z - \mathrm{i}\sigma t}\right]. \tag{2.25}$$

Here, it is necessarily demanded that $h'^{(n)}$ are small. Technically, the magnitude of h_0 can be arbitrary, but it will soon be seen that direct substitution of (2.25) into (2.23) incurs a truncation error. Notice that some of the terms in (2.23) involve values at $y = h$. If h_0 is small, it can be neglected, and $U(y = 0)$ and $v'(y = 0)$ can be used in (2.23). Otherwise, a Taylor expansion is performed, the first-order terms are retained as a first approximation, and $U(y = h) = U(y = 0) + h_0\, \partial U/\partial y|_{y=0}$ and $v'(y = h) = v'(y = 0) + h_0\, \partial v'/\partial y|_{y=0}$ can be written. The error incurred in this procedure will then necessarily be $O(h_0^2)$. In the next subsection, a coordinate transformation that reduces the error associated with this approximation if $h_0 \gg h'^{(n)}$ will be introduced. Given these considerations, the Floquet ansatz can now be substituted, and the coupling equation for mode n (taking all relevant values of U and v' at $y=0$) can be written as

$$
\begin{aligned}
\left[\mathrm{i}(n\alpha+\gamma)\left(U^O-C^I\right)\right]h'^{(n)} + \sum_{m=-\infty}^{\infty}\left\{\left\{\mathrm{i}[(n-m)\alpha+\gamma]U_m^I\right\}h'^{(n-m)}\right\} \\
+ \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left\{\left\{\mathrm{i}[(n+k-m)\alpha+\gamma]h_0^{(k)}\mathcal{D}U_{m-k}^I\right\}h'^{(n-m)}\right\} \\
+ \sum_{m=\infty}^{\infty}\left\{\left[\mathrm{i}(n\alpha+\gamma)h_0^{(m)}\mathcal{D}U^O\right]h'^{(n-m)}\right\} \\
- \left[v'^{(n)} + \sum_{m=-\infty}^{\infty}h_0^{(m)}\mathcal{D}v'^{(n-m)}\right] = \mathrm{i}\sigma h'^{(n)}.
\end{aligned}
\tag{2.26}
$$

This formulation implicitly assumes that U is continuous at $y=0$. By writing

$$
\hat{\mathbf{h}} = \left[h'^{(-N)T} \quad \cdots \quad h'^{(n)T} \quad \cdots \quad h'^{(N)T}\right]^T,
\tag{2.27}
$$

and

$$
\hat{\mathbf{q}} = \begin{pmatrix}\hat{\mathbf{q}}_{\mathbf{l}} \\ \hat{\mathbf{h}} \\ \hat{\mathbf{q}}_{\mathbf{g}}\end{pmatrix},
\tag{2.28}
$$

the system of equations for the two-phase system can finally be written as

$$
\mathsf{A}\hat{\mathbf{q}} = \mathrm{i}\sigma\mathsf{B}\hat{\mathbf{q}},
\tag{2.29}
$$

where

$$
\mathsf{A} = \begin{bmatrix}\mathsf{A_l} & 0 & 0 \\ \mathsf{A_{c,l}} & \mathsf{A_{c,c}} & \mathsf{A_{c,g}} \\ 0 & 0 & \mathsf{A_g}\end{bmatrix},
\tag{2.30}
$$

$$
\mathsf{B} = \begin{bmatrix}\mathsf{B_l} & 0 & 0 \\ 0 & \mathsf{I} & 0 \\ 0 & 0 & \mathsf{B_g}\end{bmatrix},
\tag{2.31}
$$

and the middle-row blocks of A and B are derived from (2.26). This temporal eigenproblem will require enforcement of the following boundary conditions

$$
u_l'(y\to-\infty)=0, \tag{2.32}
$$

$$
v_l'(y\to-\infty)=0, \tag{2.33}
$$

$$
w_l'(y\to-\infty)=0, \tag{2.34}
$$

$$
u_g'(y\to\infty)=0, \tag{2.35}
$$

$$
v_g'(y\to\infty)=0, \tag{2.36}
$$

$$
w_g'(y\to\infty)=0. \tag{2.37}
$$

These equations can be enforced numerically by replacing appropriate rows in A and B.

The eigenproblem will be well-posed after enforcement of the coupling conditions as well. By enforcing continuity of velocities at $y=h$, the velocity coupling conditions can be written as

$$u'_l + \frac{\partial(U_l + u'_l)}{\partial y} h = u'_g + \frac{\partial(U_g + u'_g)}{\partial y} h, \tag{2.38}$$

$$v'_l + \frac{\partial(V_l + v'_l)}{\partial y} h = v'_g + \frac{\partial(V_g + v'_g)}{\partial y} h, \tag{2.39}$$

$$w'_l + \frac{\partial w'_l}{\partial y} h = w'_g + \frac{\partial w'_g}{\partial y} h. \tag{2.40}$$

Similarly, the stress coupling conditions can be written as

$$\frac{1}{\mathrm{Re}} \left\{ \frac{\partial u'_l}{\partial y} + \frac{\partial v'_l}{\partial x} + \frac{\partial}{\partial y} \left[\frac{\partial(U_l + u'_l)}{\partial y} + \frac{\partial(V_l + v'_l)}{\partial x} \right] h \right\} = \frac{\mu_g^*}{\mathrm{Re}} \left\{ \frac{\partial u'_g}{\partial y} + \frac{\partial v'_g}{\partial x} + \frac{\partial}{\partial y} \left[\frac{\partial(U_g + u'_g)}{\partial y} + \frac{\partial(V_g + v'_g)}{\partial x} \right] h \right\}, \tag{2.41}$$

$$\frac{1}{\mathrm{Re}} \left[\frac{\partial w'_l}{\partial y} + \frac{\partial v'_l}{\partial z} + \frac{\partial}{\partial y} \left(\frac{\partial w'_l}{\partial y} + \frac{\partial v'_l}{\partial z} \right) h \right] = \frac{\mu_g^*}{\mathrm{Re}} \left[\frac{\partial w'_g}{\partial y} + \frac{\partial v'_g}{\partial z} + \frac{\partial}{\partial y} \left(\frac{\partial w'_g}{\partial y} + \frac{\partial v'_g}{\partial z} \right) h \right], \tag{2.42}$$

$$\left\{ -\left(p'_l + \frac{1}{\mathrm{Fr}} h \right) + \frac{2}{\mathrm{Re}} \frac{\partial v'_l}{\partial y} + \frac{2}{\mathrm{Re}} \frac{\partial}{\partial y} \left[\frac{\partial(V_l + v'_l)}{\partial y} \right] h - \frac{\partial p'_l}{\partial y} h \right\} - \left\{ -\left(p'_g + \frac{\rho_g^*}{\mathrm{Fr}} h \right) + \frac{2\mu_g^*}{\mathrm{Re}} \frac{\partial v'_g}{\partial y} + \frac{2\mu_g^*}{\mathrm{Re}} \frac{\partial}{\partial y} \left[\frac{\partial(V_g + v'_g)}{\partial y} \right] h - \frac{\partial p'_g}{\partial y} h \right\} = -\frac{1}{\mathrm{We}} \left[(n\alpha + \gamma)^2 + \beta^2 \right] h, \tag{2.43}$$

where $\mathrm{We} = \rho_l U^2 L / \sigma_T$, $\mathrm{Fr} = U^2/gL$, σ_T is the liquid-gas surface tension, and g is the magnitude of the local gravitational acceleration. Note that these coupling conditions involve the products of terms of various orders for simplicity of presentation. Only first-order terms, such as those on the order of $O[(\partial U_l/\partial y) h'^{(n)}]$ and $O[(\partial u'_l/\partial y) h_0]$, should be retained in the final evaluation of the coupling conditions. When h_0 is small, all the terms involving the normal gradients of the fluctuation velocities disappear. For brevity, these conditions will not be expanded here, but the procedure is the same as that used to derive (2.26).

2.3. *Coordinate transformation tangential to the interface*

If the coefficients $U^I_{f,m}$ and $V^I_{f,m}$ are known *a priori*, then the system of equations discussed above can be solved directly for the growth rate with the aforementioned Taylor approximation. These coefficients are not necessarily available if, for example, details of the primary disturbance are obtained from a direct numerical simulation (DNS) and the time-invariant interface perturbation h_0 is large. A convenient coordinate transformation in this case would then be to map the y-coordinate to a coordinate that follows the unperturbed contour of the phase interface h_0 (Longuet-Higgins 1978*a*,*b*). Here, "unperturbed" refers to the absence of a secondary disturbance. In other words, a set of new coordinates is defined

$$\tau = t, \qquad \xi = x, \qquad \eta = y - h_0(x), \qquad \zeta = z. \tag{2.44}$$

The derivatives in the various coordinates can then be transformed as follows

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \sum_{m=-\infty}^{\infty} \left[\mathrm{i} m\alpha h_0^{(m)} e^{\mathrm{i} m\alpha \xi} \right] \frac{\partial}{\partial \eta}, \tag{2.45}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial \zeta}. \tag{2.46}$$

This transformation leads to the following modified forms for A_f and Γ_f, noting that V_f^O remains constant under this transformation

$$\mathsf{A}_{\mathsf{f(n,n)}} = \begin{bmatrix} \Gamma_f^n & \rho_f^* \mathcal{D}^\eta U_f^O & 0 & \mathrm{i}(n\alpha+\gamma)\mathsf{I} \\ 0 & \Gamma_f^n & 0 & \mathcal{D}^\eta \\ 0 & 0 & \Gamma_f^n & \mathrm{i}\beta \mathsf{I} \\ \mathrm{i}(n\alpha+\gamma)\mathsf{I} & \mathcal{D}^\eta & \mathrm{i}\beta\mathsf{I} & 0 \end{bmatrix}, \tag{2.47}$$

$$\mathsf{A}_{\mathsf{f(n,n-m)}} = \begin{bmatrix} \Gamma_{f,m}^n - \rho_f^* \mathcal{D}^\eta V_{f,m}^I & \rho_f^* \mathcal{D}^\eta U_{f,m}^I & 0 & -\Xi_m \\ \mathrm{i} m\alpha \rho_f^* V_{f,m}^I - \sum_{k=-M}^{M} \rho_f^* \Xi_k V_{f,m-k}^I & \Gamma_{f,m}^n + \rho_f^* \mathcal{D}^\eta V_{f,m}^I & 0 & 0 \\ 0 & 0 & \Gamma_{f,m}^n & 0 \\ -\Xi_m & 0 & 0 & 0 \end{bmatrix}, \tag{2.48}$$

where

$$\Gamma_f^n = -\frac{\mu_f^*}{\mathrm{Re}} \left[\mathcal{D}^{\eta 2} - (n\alpha+\gamma)^2 \mathsf{I} - \beta^2 \mathsf{I} \right] + \rho_f^* \left[\mathrm{i}(n\alpha+\gamma)(U_f^O - C^I \mathsf{I}) + V_f^O \mathcal{D}^\eta \right], \tag{2.49}$$

and

$$\Gamma_{f,m}^n = \mathrm{i}[(n-m)\alpha+\gamma]\rho_f^* U_{f,m}^I - \rho_f^*(U_f^O - C^I \mathsf{I})\Xi_m - \sum_{k=-M}^{M} \rho_f^* U_{f,m-k}^I \Xi_k$$
$$+ \rho_f^* V_{f,m}^I \mathcal{D}^\eta + \frac{2\mathrm{i}(n\alpha+\gamma)\mu_f^*}{\mathrm{Re}} \Xi_m - \frac{\mu_f^*}{\mathrm{Re}} \sum_{k=-M}^{M} \Xi_k \Xi_{m-k}. \tag{2.50}$$

Also,

$$\Xi_m = \mathrm{i} m\alpha h_0^{(m)} \mathcal{D}^\eta, \tag{2.51}$$

where $\mathcal{D}^\eta = \partial/\partial\eta$. Here, it is assumed that the summation associated with $\partial\eta/\partial\xi$ is also truncated to $[-M, M]$. The boundary conditions are unaffected by the transformation above provided the physical domain is infinite. When taken at $\eta = 0$, the coupling equation simplifies to

$$\left[\mathrm{i}(n\alpha+\gamma)\left(U^O - C^I\right) \right] h'^{(n)} + \sum_{m=-M}^{M} \left\{ \mathrm{i}[(n-m)\alpha+\gamma] U_m^I \right\} h'^{(n-m)} - v'^{(n)} = \mathrm{i}\sigma h'^{(n)}, \tag{2.52}$$

and the coupling conditions simplify to

$$u_l' + \frac{\partial U_l}{\partial \eta}(h - h_0) = u_g' + \frac{\partial U_g}{\partial \eta}(h - h_0), \tag{2.53}$$

$$v_l' + \frac{\partial V_l}{\partial \eta}(h - h_0) = v_g' + \frac{\partial V_g}{\partial \eta}(h - h_0), \tag{2.54}$$

$$w_l' = w_g', \tag{2.55}$$

$$\begin{aligned} &\frac{1}{\mathrm{Re}}\left\{\frac{\partial u_l'}{\partial \eta} + \frac{\partial v_l'}{\partial \xi} + \frac{\partial}{\partial \eta}\left(\frac{\partial U_l}{\partial \eta} + \frac{\partial V_l}{\partial \xi}\right)(h - h_0) - \sum_{m=-M}^{M}\left[\Xi_m v_l' + (h - h_0)\Xi_m \mathcal{D}^\eta V_l\right]\right\} = \\ &\frac{\mu_g^*}{\mathrm{Re}}\left\{\frac{\partial u_g'}{\partial \eta} + \frac{\partial v_g'}{\partial \xi} + \frac{\partial}{\partial \eta}\left(\frac{\partial U_g}{\partial \eta} + \frac{\partial V_g}{\partial \xi}\right)(h - h_0) - \sum_{m=-M}^{M}\left[\Xi_m v_g' + (h - h_0)\Xi_m \mathcal{D}^\eta V_g\right]\right\}, \end{aligned} \tag{2.56}$$

$$\frac{1}{\mathrm{Re}}\left(\frac{\partial w_l'}{\partial \eta} + \frac{\partial v_l'}{\partial \zeta}\right) = \frac{\mu_g^*}{\mathrm{Re}}\left(\frac{\partial w_g'}{\partial \eta} + \frac{\partial v_g'}{\partial \zeta}\right), \tag{2.57}$$

$$\begin{aligned} &\left\{-\left(p_l' + \frac{1}{\mathrm{Fr}}h\right) + \frac{2}{\mathrm{Re}}\left[\frac{\partial v_l'}{\partial \eta} + \frac{\partial^2 V_l}{\partial \eta^2}(h - h_0)\right]\right\} - \\ &\left\{-\left(p_g' + \frac{\rho_g^*}{\mathrm{Fr}}h\right) + \frac{2\mu_g^*}{\mathrm{Re}}\left[\frac{\partial v_g'}{\partial \eta} + \frac{\partial^2 V_g}{\partial \eta^2}(h - h_0)\right]\right\} = -\frac{1}{\mathrm{We}}\left[(n\alpha + \gamma)^2 + \beta^2\right] h. \end{aligned} \tag{2.58}$$

Note that the Taylor approximation now incurs an error $O[h'^{(n)2}]$ instead of $O(h_0^2)$ since the η-coordinate now follows the unperturbed interface.

3. Preliminary test cases

In the first two test cases that follow in this brief, h_0 is neglected because the input parameters were chosen such that h_0 is small, and in turn the coordinate transformation is not performed. The first test case is then repeated with the coordinate transformation in the final subsection following a brief explanation of further changes to the system of equations necessary to carry out this test case.

3.1. *Poiseuille flow*

Following Herbert (1983), an instability analysis of plane channel flow is performed where $y \in [-1, 1]$, $U_f^O = 1 - y^2$, $V_f^O = 0$, $\mathrm{Re} = 5000$, $\mathrm{We} \to \infty$, $\alpha = 1.12$, $\gamma = 0.5\alpha$, $\rho^* = 1$ and $\mu^* = 1$. (Some of the physical parameters were chosen such that the two-phase solver is effectively solving a single-phase flow.) All sums are truncated at $M = 1$ and $N = 1, 2, 3$, and $U_{f,\pm1}^I$, $V_{f,\pm1}^I$ and C_I are obtained from the solution to the Orr-Sommerfeld equation for the above flow except with $\beta = 0$. $U_{f,\pm1}^I$ and $V_{f,\pm1}^I$ are normalized such that

$$A = \max_y \sqrt{2U_{f,1}^I(y)U_{f,-1}^I(y)} = 0.0248. \tag{3.1}$$

In Figure 1, the results obtained with the solver corresponding to this work are compared with those reported by Herbert (1983) for 36 Chebyshev collocation points. Note that Herbert (1983) uses only 18 points but assumes symmetry in y. The two solvers generally agree for $N = 1$. The slight discrepancy may be attributed to slight modifications in Herbert's linearized equations (Herbert 1984) to account for the truncation at

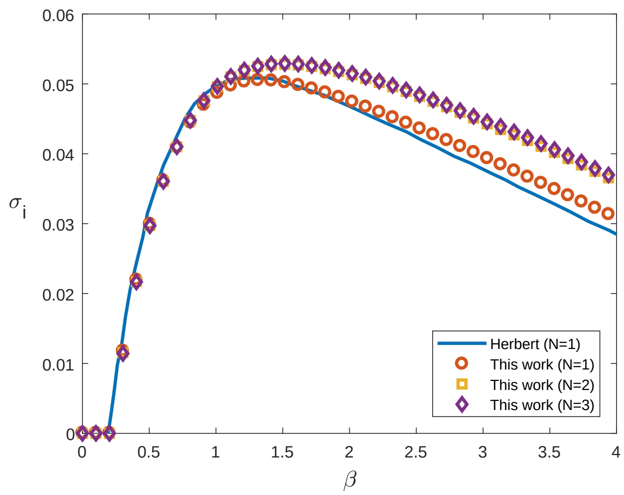

FIGURE 1. Variation of the imaginary part of the most unstable eigenvalue with β for the physical parameters listed in the main text (plane channel flow). All open symbols refer to results from the solver described in this work.

$N = 1$. Results from the solver corresponding to this work at $N = 2$ and $N = 3$ are also reported, and the solver appears to converge at $N \geq 3$.

3.2. *Blasius boundary layer*

The capability of the proposed instability formulation is also demonstrated by analyzing a base flow comprising a Blasius velocity profile in the gas which goes to zero at the interface and zero flow in the liquid. Here, L is chosen as the dimensional Blasius boundary layer thickness. The numerical and physical parameters for this flow are $y \in [-500, 500]$, $V_f^O = 0$, $\mathrm{Re} = 606$, $\mathrm{We} \to \infty$, $\alpha = 0.2034$, $\beta = 0.1091$, $\gamma = 0.5\alpha$, $\rho^* = 1 \times 10^{-3}$ and $\mu^* = 1 \times 10^{-2}$, the latter two mimicking the air-water density and viscosity ratios. (These parameters correspond to the parameters used to construct Figure 9 in Herbert (1984).) Again, all sums are truncated at $M = 1$ and $N = 1$, and $U_{f,\pm1}^I$, $V_{f,\pm1}^I$ and C_I are obtained from the solution to the Orr-Sommerfeld equation for the above flow except that $\beta = 0$. $U_{f,\pm1}^I$ and $V_{f,\pm1}^I$ are normalized such that $A = 0.025$. 200 Chebyshev collocation points were employed per phase.

In Figure 2, some of the eigenvalues of the temporal stability eigenproblem formulated above are plotted. In Figure 3, the absolute values of the velocity and pressure fluctuation eigenfunctions associated with the most unstable eigenmode are plotted. (The absolute values are normalized by the maximum magnitude of the x-velocity perturbation eigenfunction.) The eigenvalues and mode shapes resemble those from the single-phase formulation, except that the eigenvectors now satisfy the prescribed coupling conditions at the interface.

3.3. *Modifying the coordinate transformation formulation for primary disturbances derived from a numerical instability analysis*

In Section 2.3, it was mentioned that the coordinate transformation approach is suitable for cases where the primary disturbance is obtained from a DNS. The coefficients $U_{f,m}^I$ and $V_{f,m}^I$ for the transformed coordinates can then be obtained through a Fourier

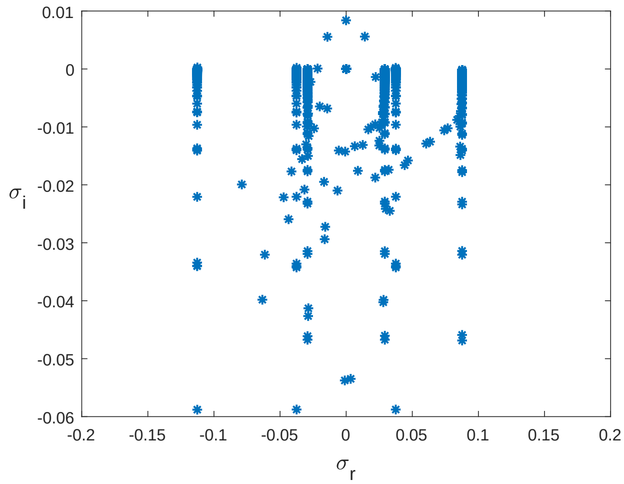

FIGURE 2. Eigenvalues for the temporal stability eigenproblem with the physical parameters listed in the main text (two-phase flow with quiescent water and Blasius profile in air).

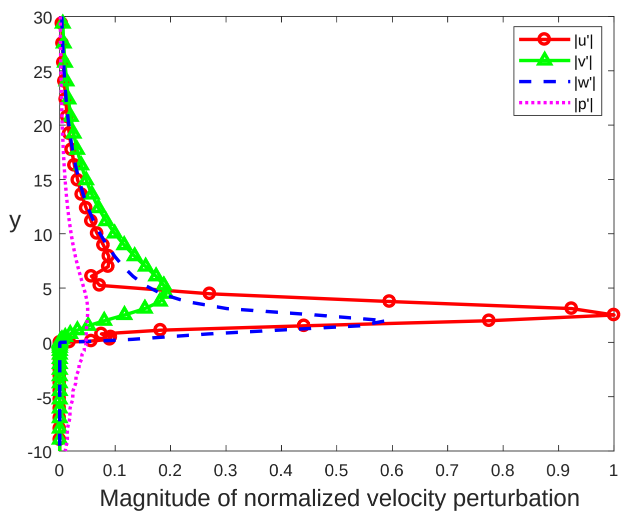

FIGURE 3. Magnitudes of the eigenvectors for the temporal stability eigenproblem with the physical parameters listed in the main text (two-phase flow with quiescent water and Blasius profile in air).

transformation of the numerical data sampled on a predefined grid in the transformed coordinates. The coordinate transformation approach can also be performed for cases where the primary disturbance is obtained from a primary instability analysis using, for example, a numerical Orr-Sommerfeld solver. In this case, the coefficients are not directly accessible from the output of the solver due to the mismatched y and η grids. In order to directly validate the transformed equations, the second approach is used so that the results can be directly compared with those obtained in the above subsections. In this setting, the input to the Floquet solver from the primary instability analysis has to be adjusted to account for the mismatched grids. The following development is carried out for the sole purpose of validation, and does not enter into the forthcoming Floquet analysis for wind-driven periodic water waves.

Recall that $\eta = y - h_0$. For a generic flow with $V_f^O = 0$, it immediately follows that

$$U_f^O(\eta = \eta_C) = U_f^O(\eta = y_C) - h_0 \frac{\partial U_f^O(\eta)}{\partial \eta} + O(h_0^2), \tag{3.2}$$

$$X_{f,m}^I(\eta = \eta_C) = X_{f,m}^I(\eta = y_C) - h_0 \frac{\partial X_{f,m}^I(\eta)}{\partial \eta} + O(h_0^2), \tag{3.3}$$

where y_C and η_C are the corresponding y and η collocation points for a given number of modes, and $X = \{U, V\}$. Assuming the primary disturbance only contains a fundamental wavelength, $h_0 = h_0^{(1)} \exp(\mathrm{i}\alpha x) + h_0^{(-1)} \exp(-\mathrm{i}\alpha x)$, and the following rules follow if $M = 1$: firstly, the cross terms $X_{f,m-k}\Xi_k$, $\Xi_k X_{f,m-k}$ and $\Xi_k \Xi_{m-k}$ vanish because $X_{f,0}$ and $X_{f,2}$ do not exist. Secondly, if a term in the (n,n)-th block matrix involves U_f^O, then the corresponding term in the $(n, n-1)$-th block matrix will need to be supplemented by $-h_0^{(1)} \mathcal{D}^\eta U_f^O$, and the term in the $(n, n+1)$-th block matrix by $-h_0^{(-1)} \mathcal{D}^\eta U_f^O$. Lastly, if a term in the $(n, n-m)$-th block matrix involves $X_{f,m}^I$, then the corresponding term in the $(n, n-m-1)$-th block matrix will need to be supplemented by $-h_0^{(1)} \mathcal{D}^\eta X_{f,m}^I$, and the term in the $(n, n-m+1)$-th block matrix by $-h_0^{(-1)} \mathcal{D}^\eta X_{f,m}^I$.

The resultant matrices are

$$\mathsf{A}_{\mathsf{f(n,n)}} = \begin{bmatrix} \Gamma_f^n + \sum\limits_{k=-1}^{1} \left(\Omega_k V_{f,k}^I\right) & \rho_f^* \mathcal{D}^\eta U_f^O - \sum\limits_{k=-1}^{1} \left(\Omega_k U_{f,k}^I\right) & 0 & \mathrm{i}(n\alpha + \gamma)\mathsf{I} \\ -\sum\limits_{k=-1}^{1} \left(\Upsilon_k V_{f,k}^I\right) & \Gamma_f^n - \sum\limits_{k=-1}^{1} \left(\Omega_k V_{f,k}^I\right) & 0 & \mathcal{D}^\eta \\ 0 & 0 & \Gamma_f^n & \mathrm{i}\beta\mathsf{I} \\ \mathrm{i}(n\alpha+\gamma)\mathsf{I} & \mathcal{D}^\eta & \mathrm{i}\beta\mathsf{I} & 0 \end{bmatrix}, \tag{3.4}$$

$$\mathsf{A}_{\mathsf{f(n,n-m)}} = \begin{bmatrix} \Gamma_{f,m}^n - \rho_f^* \mathcal{D}^\eta V_{f,m}^I & \rho_f^* \mathcal{D}^\eta \left(U_{f,m}^I - h_0^{(m)} \mathcal{D}^\eta U_f^O\right) & 0 & -\Xi_m \\ \mathrm{i}m\alpha\rho_f^* V_{f,m}^I & \Gamma_{f,m}^n + \rho_f^* \mathcal{D}^\eta V_{f,m}^I & 0 & 0 \\ 0 & 0 & \Gamma_{f,m}^n & 0 \\ -\Xi_m & 0 & 0 & 0 \end{bmatrix}, \tag{3.5}$$

where

$$\Gamma_f^n = -\frac{\mu_f^*}{\mathrm{Re}} \left[\mathcal{D}^{\eta 2} - (n\alpha+\gamma)^2\mathsf{I} - \beta^2\mathsf{I}\right] + \rho_f^* \left[\mathrm{i}(n\alpha+\gamma)(U_f^O - C^I\mathsf{I}) + V_f^O \mathcal{D}^\eta\right]$$
$$- \sum_{k=-1}^{1} \left\{\mathrm{i}[(n-k)\alpha+\gamma]\rho_f^* h_0^{(-k)} \mathcal{D}^\eta U_{f,k}^I + \rho_f^* h_0^{(-k)} \mathcal{D}^\eta V_{f,k}^I \mathcal{D}^\eta - \rho_f^* h_0^{(-k)} \mathcal{D}^\eta U_f^O \Xi_k\right\}, \tag{3.6}$$

and

$$\Gamma_{f,m}^n = \mathrm{i}[(n-m)\alpha+\gamma]\rho_f^* U_{f,m}^I - \rho_f^*(U_f^O - C^I\mathsf{I})\Xi_m + \rho_f^* V_{f,m}^I \mathcal{D}^\eta$$
$$+ \frac{2\mathrm{i}(n\alpha+\gamma)\mu_f^*}{\mathrm{Re}} \Xi_m - \rho_f^* \left[\mathrm{i}(n\alpha+\gamma) h_0^{(m)} \mathcal{D}^\eta U_f^O\right]. \tag{3.7}$$

Also,

$$\Omega_k = \rho_f^* h_0^{(-k)} (\mathcal{D}^\eta)^2 \qquad \text{and} \qquad \Upsilon_k = \mathrm{i}k\alpha\rho_f^* h_0^{(-k)} \mathcal{D}^\eta. \tag{3.8}$$

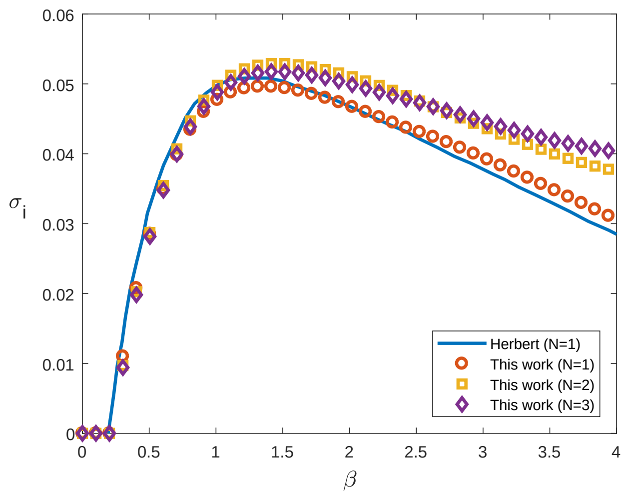

FIGURE 4. Variation of the imaginary part of the most unstable eigenvalue with β for the physical parameters listed in the main text (plane channel flow) and with the coordinate transformation of Section 2.3. All open symbols refer to results from the solver described in this work.

When taken at $\eta = 0$, the coupling equation simplifies to

$$\left\{ \mathrm{i}(n\alpha+\gamma)\left(U^O - C^I\right) - \sum_{k=-1}^{1} \left\{ \mathrm{i}[(n-k)\alpha+\gamma] h_0^{(-k)} \mathcal{D}^\eta U_k^I \right\} \right\} h'^{(n)}$$

$$+ \sum_{m=-1}^{1} \left\{ -\mathrm{i}(n\alpha+\gamma) h_0^{(m)} \mathcal{D}^\eta U^O + \mathrm{i}[(n-m)\alpha+\gamma] U_m^I \right\} h'^{(n-m)} - v'^{(n)} = \mathrm{i}\sigma h'^{(n)}. \quad (3.9)$$

The coupling conditions can also be adjusted accordingly. Note that the Taylor approximations above, and thus this procedure, incur an error $O(h_0^2)$.

3.4. *Poiseuille flow with coordinate transformation*

The test case in Section 3.1 is repeated with the coordinate transformation discussed in Section 2.3 using the modifications in Section 3.3. In Figure 4, the results obtained with these modifications are again compared with those reported by Herbert (1983) for 36 Chebyshev collocation points. Reasonable agreement is once again obtained for $N = 1$. The increased deviation is likely due to two reasons. Firstly, as previously mentioned, the procedure described above incurs an $O(h_0^2)$ error due to the Taylor approximation of the base velocities. Secondly, the preceding analysis is technically only accurate if the corresponding physical domain is infinite. Since plane channel flow involves a bounded physical domain, the error incurred by the incorrect boundaries is $O(h_0)$. While the second source of error has a stronger scaling with h_0, it appears only at the boundaries where the eigenfunctions have a small magnitude, and it is conceivable that both sources of error may be important. Both errors are expected to disappear in the forthcoming analysis of wind-driven periodic water waves, since the domain is effectively infinite in that case and the velocities are directly obtained from a DNS without any Taylor approximations.

4. Conclusions

In this work, the Floquet stability problem is formulated for a two-phase configuration with explicit consideration of the surface tension and without any assumptions on the Reynolds number of the flow. Solutions to the problem are presented for several preliminary cases. Work to apply this analysis to wind-driven periodic water waves is in progress.

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