Instabilities of supersonic jets

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1. Motivation and objectives

Supersonic jets are ubiquitous in many practical applications, including jet engines and rockets. Jet flows are usually unsteady and accompanied by structural vibrations as well as noise. The phenomenon of jet screech, an intense tone associated with interactions of instability waves and shocks, may pose severe limitations on engines due to structural fatigue and significant noise. Understanding the mechanism of unsteadiness may aid in developing quieter and more capable jet engines.

Local stability theory has been commonly applied to perfectly expanded jets because they vary slowly with respect to the axial coordinate with the exception of the region in the immediate vicinity of the nozzle. The instability mechanism identified by local stability theory is associated with the inflectional shear layer and describes a convectively unstable Kelvin-Helmholtz mode. Several investigators have considered the effects of non-parallelism (e.g., Malik & Chang 2000) and obtained reasonably good agreement with experiments. Nevertheless, these studies exclude the region close to the nozzle from the analysis. Global stability theory avoids any a priori assumptions on the base state, thus offering a natural path towards analyzing the role of the nozzle in the instability mechanism.

Huerre & Monkewitz (1990) review local and global instabilities in spatially developing flows. They show that absolute instabilities, which may be triggered spontaneously in the flow, correspond to unstable modes in the global analysis. On the other hand, convective instabilities, which need a constant source of excitation, are associated with stable modes in the global analysis. The first application of global stability analysis to supersonic jets has been performed by Nichols & Lele (2011) who investigated a perfectly expanded non-parallel jet, albeit excluding the nozzle from the computational domain. They show that, in line with local stability theory, all the global modes are stable and the least stable mode corresponds to an upstream-traveling acoustic mode.

Non-perfectly expanded jets are associated with the shock diamonds phenomenon, a structure consisting of interactions of shocks and expansion fans. As such flows vary significantly along the axial coordinate, they are not amenable to local stability analysis and their stability properties must be analyzed in a global setting. One phenomenon of particular interest, associated with non-perfectly expanded jets, is jet screech (see, e.g., Tam 1995; Raman 1999, for a review). The mechanism of screech generation consists of the following cycle: (1) Kelvin-Helmholtz instability waves amplify in the shear layer; (2) The instability waves interact with shocks impinging on the shear layers, leading to the generation of acoustic waves; (3) The acoustic waves travel upstream to the nozzle lip; and (4) The cycle is closed by conversion of the acoustic waves to instability waves via a receptivity process in the vicinity of the nozzle lip.

The elements of the screech cycle have been analyzed separately in simplified settings. The generation of sound has been studied by Manning & Lele (2000) and Suzuki & Lele (2003) who considered the interaction of an instability wave, growing within
the shear layer, and oblique shocks. The generation of noise is attributed to the shock leakage phenomenon, which describes the escape of acoustic waves in the core region through distortions introduced by the instability wave. The receptivity stage has been investigated by Barone & Lele (2005) who considered a compressible mixing layer in the presence of a splitter plate. Combining direct and adjoint analyses, they determined the mechanism which transfers energy from upstream boundary-layer modes to downstream Kelvin-Helmholtz modes. More recently, Edgington-Mitchell et al. (2019) performed a global stability analysis of a screeching jet. The analysis was conducted on the mean of the experimentally measured turbulent flow. Remarkably good agreement between the theoretical and experimental results was observed. Both setups showed three distinct structures: an upstream-traveling mode, a downstream-propagating Kelvin-Helmholtz mode, and another component with a radial structure that differs from that of the other two waves.

The purpose of this study is to explore the mechanisms of instability of supersonic jets by means of global linear stability theory. In particular, we are interested in analyzing the effect of the nozzle on the instability mechanism and the role of the shock cells.

2. Methodology

Global stability analysis of axisymmetric supersonic jets is considered herein, with the aim of assessing the role of the nozzle and the shocks on the instability mechanism. In order to isolate the effect of the nozzle lip and the shock cells we consider the following three setups, described schematically in Figure 1. The first case corresponds to a plain weakly non-parallel jet, as indicated by the sketch in Figure 1(a). This setup serves as a reference case to compare with literature (e.g., Nichols & Lele 2011). The next case, sketched in Figure 1(b), corresponds to a fully expanded jet and includes the nozzle in the computational domain. The third case corresponds to an under-expanded jet, as indicated by the sketch in Figure 1(c).

Our aim is to obtain and compare the most unstable eigenfunctions for each of the setups. A prerequisite for conducting the stability analysis is the base flow, which is obtained as a steady laminar solution of the nonlinear compressible Navier-Stokes equations. In the following, the nonlinear equations are presented, followed by the linearized equations and a description of the numerical framework.

2.1. Nonlinear compressible Navier-Stokes

We consider the compressible three-dimensional Navier-Stokes equations in conserved form

$$\frac{\partial p}{\partial t} + \frac{\partial m_j}{\partial x_j} = 0, \quad (2.1a)$$
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\[
\frac{\partial m_i}{\partial t} + \frac{\partial m_i u_j}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \frac{\mu}{Re} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \right],
\]

(2.1b)

\[
\frac{\partial e}{\partial t} + \frac{\partial (e + p) u_j}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\mu}{Re} \frac{\partial T}{\partial x_j} + \frac{\mu}{Re} Pr \frac{\partial p}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left[ \frac{\mu}{Re} \left( \frac{\partial u_k}{\partial x_j} - \frac{2}{3} \frac{\partial u_i}{\partial x_j} \delta_{ij} \right) u_k \right],
\]

(2.1c)

with the conserved variables given by

\[
q \equiv \begin{pmatrix} \rho \\ m_i \\ e \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u_i \\ \gamma - 1 + \frac{\rho u_i u_k}{2} \end{pmatrix},
\]

(2.2)

All variables are made non-dimensional by \( \rho^*, a^*, T_{ref} = (a^*)^2/c_p^*, \) \( p_{ref} = \rho^* (a^*)^2, \mu^*, \) and \( R_e^* \) where * implies dimensional quantities and the subscript e refers to the nozzle exit. The acoustic Reynolds number is defined as

\[ Re = \rho^* a^* R_e^*/\mu^* \]

The velocity components \((u, v, w)\) correspond to velocities along \((x, r, \theta)\), the streamwise, radial, and azimuthal dimensions, respectively.

As mentioned above, the nonlinear equations are used to obtain the laminar base flows for the stability analysis. For the cases shown in Figure 1(b) and Figure 1(c), the nozzle is included within the computational domain. The subsonic Mach number inside the nozzle, \( M_{in} \), is calculated based on one-dimensional isentropic flow relations to achieve a given nozzle exit Mach number, \( M_e \). To avoid recirculations in the computational domain due to entrainment, a co-flow with magnitude 1% \( M_e \) is imposed at the inflow outside of the nozzle. The ambient flow conditions are

\[
T_a = T_e \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right), \quad p_a = \frac{p_e}{r_p},
\]

(2.3)

where \( r_p \) is the ratio of the nozzle exit pressure and the ambient pressure, and the nozzle exit conditions are \( T_e = (\gamma - 1)^{-1} \) and \( p_e = \gamma^{-1} \). The wall is assumed to be isothermal with temperature matching the ambient, \( T_w = T_a \). Ambient conditions are imposed at the top of the computational domain and the quantities at the outflow are extrapolated from inside the domain to avoid streamwise gradients at that region. A power-law is assumed for the viscosity, \( \mu \sim T^{0.7} \), and the Prandtl number, \( Pr = c_p \mu^*/k^* \), is assumed to have a constant value of 0.7.

The nozzle in the current study is chosen to achieve an exit Mach number of \( M_e = 1 \). It consists of a convergent part with a length of 3.4, and the ratio of the radii at the inlet and exit is set to 1.5. Thus, the inflow Mach number in the nozzle is \( M_{in} \approx 0.27 \). The convergent part of the nozzle follows a hyperbolic tangent, with a maximal slope angle of 30°. The wall thickness of the nozzle is 0.1, contracting to 0.03 at the lip, with the slope at the outer side of the lip set to 10°. The base flows are obtained for a computational domain of dimensions \( L_x = 47 \) and \( L_r = 10 \).

2.2. Linearized compressible Navier-Stokes

The conserved variables are decomposed into a steady base state, denoted by an overbar, and a disturbance, indicated by prime, i.e.,

\[
q = \overline{q}(x) + q'(t, x).
\]

(2.4)
Substituting the above expression into the governing equations and retaining only linear terms with respect to the perturbations yields the linearized compressible three-dimensional Navier-Stokes Equations, which can be written in the following form

$$\frac{\partial q'}{\partial t} + \frac{\partial F'_j(q')}{\partial x_j},$$

where $F'$ corresponds to the linearized operator and the explicit expressions are given in the Appendix. Since the base state is homogeneous in time, the disturbance can be further decomposed as

$$q'(t, x) = \hat{q}(x) \exp(-i\omega t),$$

where $\omega = \omega_r + i\omega_i$ is the eigenvalue and $\hat{q}(x)$ is the eigenfunction. The real part of the eigenvalue gives the disturbance frequency and the imaginary part gives the growth rate.

The boundary conditions for the perturbations are homogeneous Dirichlet at the inflow, top, and outflow of the computational domain, along with numerical sponges which ensure dampening of the perturbations towards the boundaries of the computational domain. The perturbation boundary conditions at the isothermal wall are vanishing velocity and temperature, with the remaining quantities determined as part of the solution. The disturbances are obtained for a computational domain size of $L_x = 80$ and $L_r = 20$ which encapsulates the domain used to obtain the base flows, with the added parts of the domain being utilized by the numerical sponges.

### 2.3. Numerical framework

An identical numerical framework is employed for the nonlinear and linearized equations to ensure consistency. The spatial discretization uses a fourth-order finite-difference formulation based on summation-by-parts (Strand 1994). Time integration is facilitated by a fourth-order explicit Runge-Kutta scheme. The solver supports complex geometries via a curvilinear formulation consistent with geometric conservation properties (Thomas & Lombard 1979). Boundary conditions are weakly enforced via the simultaneous approximation terms (Svärd & Nordström 2014). Capturing of the shocks in the nonlinear calculations is achieved by using the artificial bulk viscosity (Kawai & Lele 2008). The reader is referred to Flint & Hack (2018) for a more detailed description of the numerical implementation.

### 3. Laminar base states

In the following, the laminar base states upon which the global stability analysis is performed are described. The exit mach number is set to $M_e = 1$ and the acoustic Reynolds number to $Re = 10000$. Since the laminar base state may be globally unstable, the underlying steady laminar flow is recovered by solving the axisymmetric two-dimensional equations and application of selective frequency damping (Åkervik et al. 2006). The computational mesh contains $(N_x, N_r) = (8192, 1024)$ points along the streamwise and radial directions, respectively, clustered around the nozzle lip. The mesh for the cases that explicitly represent the nozzle is presented in Figure 2, where every 20th point is shown. The inset shows the actual mesh in the vicinity of the nozzle lip. In the radial dimension, the mesh contains 480 points between the axis and the inner part of the nozzle and 38 points across the nozzle lip.
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Figure 2. Computational mesh for the base flows having a nozzle; every 20th point is presented. The actual grid near the nozzle lip is shown at the inset.

<table>
<thead>
<tr>
<th>Case</th>
<th>NPR</th>
<th>NTR</th>
<th>$M_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without nozzle</td>
<td>1.89</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Fully expanded nozzle</td>
<td>1.89</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Under-expanded nozzle</td>
<td>4.54</td>
<td>1</td>
<td>1.64</td>
</tr>
</tbody>
</table>

Table 1. Nozzle pressure ratio (NPR), nozzle temperature ratio (NTR) and fully expanded jet Mach number ($M_j$) for the three considered cases.

To ensure consistency between the base flows of the cases without the nozzle (Figure 1(a)) and with the nozzle (Figure 1(b)), the base flow from the latter case is used to generate the former. Consequently, the base flow for the fully expanded nozzle is calculated first and then the base flow for the case without the nozzle is obtained by discarding the upstream part of the solution which includes the nozzle and replacing it with values obtained by linear extrapolation.

The base flows for the cases with the nozzle are calculated by setting $r_p = 1$ for the fully expanded jet (Figure 1(b)) and $r_p = 2.4$ for the under-expanded jet (Figure 1(c)). The nozzle pressure ratio, temperature ratio and fully expanded jet Mach number are given in Table 1. The laminar velocity fields are presented in Figure 3, where contours of the Mach number are shown. As detailed above, the base flows for the plain jet (Figure 3(a)) and fully expanded jet (Figure 3(b)) are identical with the exception of the nozzle region, with thin shear layers spreading slightly downstream. The base flow for the under-expanded jet (Figure 3(c)) contains shock cells which have an approximately equal axial extent of 3.5, with nearly self-similar cells downstream of the third shock cell. The Mach number distribution along the axis for the three cases is shown in Figure 4. A constant value equal to one is observed for the plain jet. Close agreement is observed inside the nozzle for the fully expanded jet and under-expanded jet. For the under-expanded jet, four normal shocks with decreasing strength are obtained ($x=6, 9, 12,$ and 16). A deceleration to nearly stagnant conditions is observed between the second and third shock cells ($x = 10$). The distributions of velocity and temperature at the nozzle exit for the fully expanded and under-expanded jets are shown in Figure 5. The boundary layer thickness based on 99% of the maximal velocity is approximately 6% for the fully expanded case and 3% for the under-expanded case, with similar values obtained for the thermal boundary layers. The number of grid points across the boundary layer is 67 and 40 for the fully expanded and under-expanded cases, respectively.
4. Linear stability analysis

Our aim is to determine the most unstable eigenmode for each of the base flows described above. The computational mesh contains \((N_x, N_r) = (16384, 2048)\) points along the streamwise and radial directions, respectively, and is obtained by adding points around the computational mesh utilized to obtain the base flow. The current study focuses
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Figure 5. Distributions of velocity (a) and temperature (b) at the nozzle exit for the fully expanded nozzle (dashed) and under-expanded nozzle (solid).

Table 2. Frequency and growth rate of the most unstable eigenvalue for the three considered cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\omega_r$</th>
<th>$\omega_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without nozzle</td>
<td>0.395</td>
<td>-0.004</td>
</tr>
<tr>
<td>Fully expanded nozzle</td>
<td>0.399</td>
<td>0.136</td>
</tr>
<tr>
<td>Under-expanded nozzle</td>
<td>0.687</td>
<td>0.090</td>
</tr>
</tbody>
</table>

on axisymmetric perturbations since local stability theory predicts that the mechanism extracting energy from the shear layer is most effective for axisymmetric disturbances.

The frequency and growth rate of the most unstable eigenmodes for each case are given in Table 2. The reference case without the nozzle is globally stable, whereas the fully expanded and under-expanded cases are unstable. As detailed in the review by Huerre & Monkewitz (1990), globally unstable modes are indicative of absolute instability, whereas globally stable modes correspond to convective instabilities. Thus, for the plain jet without explicit representation of the nozzle, we recover the classical result of convective instability. This finding is in line with the stability analysis of cold jets by Nichols & Lele (2011) who obtained only globally stable modes. The representation of the nozzle within the computational domain introduces an absolute instability, even in the absence of any shock cells. The under-expansion reduces the growth rate of the instability, although the configuration remains absolutely unstable.

Insight into the instability mechanism is obtained by an examination of the eigenfunctions. The pressure is chosen as a representative quantity of the eigenfunction, and the decimal logarithm of the magnitude of the real part is shown for the three considered cases. The eigenfunction for the plain jet is presented in Figure 6. The peak magnitudes of the eigenfunction are obtained in the shear layer, where downstream spatial amplification occurs. Outside of the core of the jet, an upstream-traveling acoustic wave is observed, qualitatively similar to the least stable mode reported by Nichols & Lele (2011).
The eigenfunction for the fully expanded nozzle jet is shown in Figure 7. Although spatial growth in the shear layer and an upstream-traveling acoustic wave are observed in this case as well, the major difference is a coupling between the two introduced by the nozzle. The conversion of the upstream-traveling acoustic wave to the shear layer mode in the vicinity of the nozzle lip, captured by the eigenfunction, leads to absolute instability.

The eigenfunction for the under-expanded nozzle jet is shown in Figure 8. Although the mode is qualitatively similar to that of the fully expanded case, the magnitude of the mode in the region outside of the jet core is significantly elevated for the under-expanded case. This suggests that the part of the mode associated with the shear layer is appreciably weaker than that in the other cases. This weakening may be attributed to the higher Mach numbers achieved along the high-speed side of the shear layer in the base state (see Figure 3(c)) compared to the other cases. The higher Mach numbers may decrease the spatial amplification of the shear layer part of the mode, thus reducing its contribution to the overall eigenmode. A more detailed analysis of the instability mechanisms is currently under way.

5. Conclusions

Global linear stability analysis of supersonic jets is conducted to determine the role of the nozzle lip and the shock cells in the instability mechanism. Three setups are considered to isolate the effect of the nozzle lip and the shock cells. The first setup corresponds to a plain weakly non-parallel jet, used as a reference to compare with the literature. The second case includes the nozzle in the computational domain and
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Figure 7. Pressure eigenfunction for the fully expanded nozzle. The contours correspond to the decimal logarithm of the magnitude of the real part.

Figure 8. Pressure eigenfunction for the under-expanded nozzle. The contours correspond to the decimal logarithm of the magnitude of the real part.
corresponds to a perfectly expanded jet, without shocks. The third setup corresponds to an under-expanded jet, achieved by increasing the ratio of the nozzle exit pressure and the ambient pressure.

Calculation of the least stable eigenmodes for the three cases indicates that while the plain non-parallel jet is only convectively unstable, the representation of the nozzle within the computational domain gives rise to absolute instability. Inspection of the pressure eigenfunction points to a coupling between the shear layer mode and the upstream-traveling acoustic wave in the vicinity of the nozzle. The effect of the under-expansion is a reduction in the instability growth rate, which is possibly related to the higher Mach numbers achieved in the high-speed side of the shear layer. Nevertheless, the mechanisms of instability are qualitatively similar for both fully expanded and under-expanded jets.

Future work will include further analysis of the instability mechanisms and calculation of adjoint instability modes. The adjoint analysis will enable the assessment of the receptivity and sensitivity of the instability modes and may open possible venues to their modeling and control.

Acknowledgments

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Appendix: Linearized compressible Navier-Stokes equations

The linearized compressible Navier-Stokes equations can be written in vector form as

$$\frac{\partial q'}{\partial t} + \frac{\partial F'_j}{\partial x_j} = 0,$$

where

$$F'_j \equiv \begin{pmatrix} \bar{\rho}u'_j + \rho' \bar{u}_j \\ \zeta_j' + \iota_j' - \xi_j' \\ \end{pmatrix}.$$

and

$$\sigma'_{ij} \equiv \frac{\bar{\mu}}{\text{Re}} \left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} - \frac{2}{3} \frac{\partial u'_k}{\partial x_k} \delta_{ij} \right) + \frac{\mu'}{\text{Re}} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} - \frac{2}{3} \frac{\partial \bar{u}_k}{\partial x_k} \delta_{ij} \right),$$

$$\zeta'_i \equiv \left( \rho u'_i + p' \bar{u}_i \right) \frac{\gamma}{\gamma - 1} + \frac{1}{2} \left( \rho \bar{u}_k \bar{u}_i u'_k + 2 \rho \bar{u}_k u'_i \bar{u}_k + \rho' \bar{u}_k \bar{u}_i \bar{u}_k \right),$$

$$\iota'_i \equiv -\frac{\bar{\mu}}{\text{Re} \text{Pr}} \frac{\partial T'}{\partial x_i} - \frac{\mu'}{\text{Re} \text{Pr}} \frac{\partial \bar{T}}{\partial x_i}.$$
with the viscosity perturbation given by

\[ \mu' = \frac{d\mu}{dT} T'. \] (5.4)

REFERENCES


