Locality in the turbulent bubble breakup cascade

By W. H. R. Chan and P. L. Johnson

1. Motivation and objectives

Turbulent bubbly flows with a wide range of bubble sizes are ubiquitous in nature and engineering, including breaking waves in oceans (e.g., Blanchard & Woodcock 1957; Melville 1970; Melvin 1996). This wide range of bubble sizes contributes richly to various transport phenomena. Experiments such as those by Deane & Stokes (2002) and Blenkinsopp & Chaplin (2010) for breaking waves suggest that several physical mechanisms are in play at different length and time-scales in the generation and evolution of these bubbles. Among these mechanisms—many of which are not well understood—the turbulent fragmentation of bubbles has garnered significant interest.

Kolmogorov (1949) and Hinze (1955) suggested that turbulent eddies break up sufficiently large air cavities into bubbles of various sizes. The breakup frequency of bubbles of size $D$ fragmenting via this mechanism has been postulated to scale as $\epsilon^{1/3}D^{-2/3}$, where $\epsilon$ is the local and instantaneous rate of turbulent kinetic energy dissipation per unit mass. The concept behind this postulate is that the breakup of a bubble is facilitated by an eddy of a comparable size in its neighborhood (Hinze 1955; Chan et al. 2018a). Breakup frequencies for various turbulent bubbly flows in the experiments described by Martinez-Bazán et al. (1999a) and Rodríguez-Rodríguez et al. (2006), and the simulations by Chan et al. (2018b), corroborate this frequency scaling. Garrett et al. (2000) further proposed a quasi-steady forward bubble breakup cascade, where large volumes of air entrained approximately steadily—relative to the characteristic breakup rates of most bubbles in the system—are broken up by turbulence. They suggested that this cascade yields a quasi-stationary bubble size distribution with a $D^{-10/3}$ power-law scaling. A similar scaling was observed in the ensemble-averaged size distributions from breaking waves, as obtained in the experiments by Loewen et al. (1996), Deane & Stokes (2002), Rojas & Loewen (2007), Blenkinsopp & Chaplin (2010) and Na et al. (2016) [see also Figure 1 of Deike et al. (2016)], and the simulations by Deike et al. (2016) and Chan et al. (2018a,b).

In this work, the population balance equation (Smoluchowski 1916, 1918; Williams 1958; Friedlander 1960a,b; Hulburt & Katz 1964; Randolph 1964) is extended to demonstrate that the proposed and observed $D^{-2/3}$ and $D^{-10/3}$ power-law scalings for the bubble breakup frequency and size distribution, respectively, are compatible with a quasi-steady bubble breakup cascade mechanism for turbulent bubbly flows. In particular, this brief quantifies the degree to which bubble-mass transfer due to binary breakup events in these flows is local in bubble-size space, given that the accompanying energy cascade is self-similar. Locality of the bubble-mass transfer, together with self-similarity of the corresponding energy transfer, is shown to be compatible with the physical picture of self-similar bubble breakup. An ideal bubble breakup cascade—where bubble mass is transferred on average from large to successively smaller bubble sizes, and the dynamics of the transfer at intermediate sizes are independent of the largest and smallest bubbles and flow scales—should be both local and self-similar in nature. It also implies universality of
the breakup process at small bubble sizes across various turbulent bubbly flows. Locality is achieved when the net air transfer rate in bubble-size space at a certain size primarily depends on the breakup statistics of bubbles of similar sizes. Self-similarity is achieved when this rate is size invariant at intermediate sizes. This work attempts to construct analogies between this picture of turbulent bubble breakup and the ideas underlying the celebrated concept of the turbulent energy cascade (Richardson 1922; Kolmogorov 1941; Onsager 1945), drawing inspiration from the eddy-viscosity-based spectral energy transfer models of Obukhov (1941) and Heisenberg (1948a,b), as well as the quasi-local spectral energy transfer models of Kovasznay (1948) and Pao (1965, 1968).

With these objectives in mind, this brief is organized as follows. In Section 2, similarity hypotheses for turbulent bubble breakup are proposed, in parallel with the traditional similarity hypotheses for the turbulent energy cascade. Locality is argued to be crucial for the validity of these hypotheses. In Section 3, the mathematical formalism required to quantify this locality is introduced. This includes the distribution of bubble sizes, the population balance equation describing the dynamics of the bubble size distribution, and the model binary breakup kernel in the population balance equation and the corresponding breakup flux in bubble-size space. The locality of this flux is analyzed in Section 4 in the context of self-similar energy and bubble-mass transfer. Finally, conclusions are drawn in Section 5.

2. Universality, locality, and self-similarity of turbulent bubble breakup

The universality of a forward cascade requires the small-scale dynamics of some process, such as energy or bubble-mass transfer, to be independent of the large-scale flow geometry in which this process occurs. This demands a separation of scales so that the large-scale dynamics do not directly influence the small-scale dynamics. This decoupling between scales suggests that the small-scale dynamics are scale local. With sufficient scale separation, this locality further implies that the dynamics in an intermediate range of scales are independent of the largest and smallest scales. Because no characteristic scale can be present in this intermediate range, the corresponding dynamics must be self-similar with some degree of scale invariance. Kolmogorov (1941) advanced a number of similarity hypotheses to convey these ideas for turbulent energy transfer in wavenumber space. This work proposes a corresponding set of similarity hypotheses for turbulent bubble-mass transfer in bubble-size space. Note that these hypotheses hold in two scenarios: either the flow of interest and the accompanying entrainment of gas are statistically stationary, or they are quasi-steady over time-scales longer than those associated with turnover and breakup of most of the relevant eddies and bubbles, respectively. Quasi-steadiness may be assumed in a flow with sufficient scale separation. Given the importance of locality in enabling the aforementioned trinity of universality, locality, and self-similarity in a cascade, the end of this section discusses how this locality may be quantified.

2.1. Kolmogorov’s similarity hypotheses for single-phase high-Re turbulent flows

The universality of the turbulent energy cascade is first revisited with reference to the Reynolds number associated with some length scale \( L_n \)

\[
Re_{L_n} = \frac{\rho u_L L_n}{\mu},
\]

(2.1)
Turbulent bubble breakup cascade in an incompressible turbulent flow, and the corresponding Kolmogorov length scale

\[ L_K \sim \left( \frac{\mu_l}{\rho_l} \right)^{3/4} \varepsilon^{-1/4}. \]  

(2.2)

Here, \( \rho_l \) and \( \mu_l \) refer to the density and dynamic viscosity of the bulk phase, respectively, assuming the bulk flow involves a liquid. \( \varepsilon \) refers to the turbulent kinetic energy cascade rate, while \( u_{Ln} \) refers to the magnitude of the characteristic velocity fluctuations associated with the length scale \( L_n \), defined in Eq. (2.2) of the brief by Chan et al. (2018b). First, \( Re_L \sim O(1) \). Second, \( L_K \) is a function of only \( \nu_l = \mu_l/\rho_l \) and \( \varepsilon \). Third

\[ \frac{L_K}{L} \sim Re_L^{-3/4}, \]  

(2.3)

where \( L \) is the integral length scale. These relations should be viewed in tandem with Kolmogorov’s hypotheses for the local structure of turbulence in high-\( Re \) flows, which were paraphrased by Pope (2000, Section 6.1.2) and are further paraphrased here for reference:

**HYPOTHESIS 1.** (Local isotropy.) In flows with sufficiently high Reynolds number, the statistics of the small-scale turbulent motions \( (L_n \ll L) \) are isotropic.

**HYPOTHESIS 2.** (First similarity hypothesis.) For locally isotropic turbulence, the statistics of the small-scale turbulent motions \( (L_n \ll L) \) have a universal form that is uniquely determined by \( \varepsilon \) and \( \nu_l \).

**HYPOTHESIS 3.** (Second similarity hypothesis.) At scales in the range \( L_K \ll L_n \ll L \) in locally isotropic turbulence, the statistics of the turbulent motions have a universal form that is uniquely determined by \( \varepsilon \) and independent of \( \nu_l \).

In the original hypotheses, the statistics of the turbulent motions refer specifically to the statistics of the second-order velocity structure functions.

### 2.2. Proposed similarity hypotheses for high-\( Re \) and high-\( We \) turbulent bubbly flows

The universality of the turbulent bubble breakup cascade in high-Reynolds-number and high-Weber-number two-phase flows is now introduced. Recall the discussion in Section 2 of the brief by Chan et al. (2018b), in particular Eqs. (2.1)–(2.4), which present the relation of the Weber number of the carrier liquid phase at some length scale \( L_n \)

\[ We_{L_n} = \frac{\rho_l u_{L_n}^2 L_n}{\sigma}, \]  

(2.4)

in an incompressible, immiscible, and turbulent two-phase flow, to the Hinze scale

\[ L_H \sim \left( \frac{\sigma}{\rho_l} \right)^{3/5} \varepsilon^{-2/5}, \]  

(2.5)

where \( \sigma \) refers to the liquid–gas surface tension coefficient. To summarize, first, \( We_{L_H} \sim O(1) \). Second, \( L_H \) is a function of only \( \sigma/\rho_l \) and \( \varepsilon \). Third

\[ \frac{L_H}{L} \sim We_L^{-3/5}. \]  

(2.6)

Observe the parallels between these statements and the corresponding statements in Section 2.1, and between Eqs. (2.1)–(2.3) and (2.4)–(2.6). One might surmise that in a
high-$Re_L$ and high-$We_L$ bubbly flow, a forward bubble-mass cascade transferring bubble mass from large to small bubble sizes occurs in parallel with the forward energy cascade transferring turbulent kinetic energy from large to small scales, provided the bubble-mass transfer is driven by turbulent eddies. If this parallel exists, then similarity hypotheses describing turbulent bubble-mass transfer may be put forward in a similar fashion to the similarity hypotheses in Section 2.1 advanced by Kolmogorov (1941) for turbulent kinetic energy transfer. A similar procedure was performed in the context of coalescence by Friedlander (1960a,b).

To determine when the proposed hypotheses are relevant, consider the physical limit where the following holds in the listed order. First, large pockets of gas ($L_n \sim L$) are injected into a bulk volume of liquid to facilitate the transfer of bubble mass from large to small bubble sizes. Second, turbulence of sufficiently high $Re_L$ and $We_L$ is present so that there is a sufficient separation of scales. Third, $L_K \ll L_H$, so that the smallest eddies are much smaller than the smallest bubbles formed by turbulent breakup, and forces of viscous origin have a limited effect on bubble fragmentation. Fourth, the bubble volume fraction is sufficiently low that coalescence between gas bubbles and cavities may be neglected, and the turbulence statistics are not significantly modified. Fifth, buoyancy may be neglected in the bubble dynamics. Sixth, a mechanism for the removal of small bubbles, such as complete and instantaneous dissolution of bubbles of sizes smaller than $L_H$, exists to prevent their accumulation. This limit holds when the time-scales of the secondary effects neglected above, such as coalescence, buoyancy, and the accumulation of small bubbles, exceed the flow and entrainment time scales of interest. At times where this limit is appropriate, one may formulate hypotheses for the resulting averaged bubble size distribution due to turbulent breakup in the vein of the hypotheses in Section 2.1:

Hypothesis 4. (Single-size approximation.) In bubbly flows with sufficiently high Weber number, the statistics of sufficiently small bubbles of volumes $L_n^3 \ll L^3$ may be analyzed by describing each of the bubbles by a single length scale $L_n$.

If the phase space of the bubble size distribution contains no other dimensions, then the single-size approximation enables the treatment of the distribution as a one-dimensional probability distribution in bubble-size space.

Hypothesis 5. (First similarity hypothesis for gas transfer in bubble-size space due to turbulent breakup.) The statistics of sufficiently small bubbles of sizes $L_n \ll L$ have a universal form that is uniquely determined by $\varepsilon$ and $\sigma/\rho_l$.

Hypothesis 6. (Second similarity hypothesis for gas transfer in bubble-size space due to turbulent breakup.) The statistics of bubbles of sizes $L_H \ll L_n \ll L$ have a universal form that is uniquely determined by $\varepsilon$ and independent of $\sigma/\rho_l$.

These hypotheses implicitly assume that the kinematic viscosity of the dispersed gaseous phase $\nu_g$ is less than $\nu_l$, so that the corresponding Kolmogorov length scale in the gaseous phase is less than $L_K$ (Kolmogorov 1949). In addition, it is assumed that the density of the dispersed gaseous phase $\rho_g$ is smaller than $\rho_l$, so that inertial mechanisms involving the dispersed phase may be neglected.

2.3. Locality in a self-similar framework for universal turbulent bubble breakup

Hypothesis 6 implies the presence of an intermediate bubble-size subrange for bubble-mass transfer. Locality of the bubble breakup flux in this subrange embodies the trinity of universality, locality, and self-similarity that constitutes a bubble-mass cascade. It
should be emphasized that locality of the averaged breakup dynamics—not the locality of individual breakup events—is the quantity of interest since turbulent cascades should always be interpreted in a statistical manner. To enable this interpretation, the breakup flux should involve the averaged system dynamics. The formalism to be introduced in Section 3 yields a breakup flux $W_b(D)$ that describes, on average, the rate at which bubble mass is transferred from bubbles of sizes larger than $D$ to bubbles of sizes smaller than $D$.

Locality in $W_b$ is quantified using measures inspired by the concepts of infrared and ultraviolet locality introduced by L’vov & Falkovich (1992) and Eyink (2005) for energy transfer. First, one is interested in the degree to which contributions to $W_b(D)$ from all larger bubble sizes arise primarily from bubble sizes just larger than $D$. This metric is termed infrared locality, since infrared radiation has a longer wavelength than visible light. If the differential rate at which parent bubbles of sizes between $D_p > D$ and $D_p + dD_p$ transfer bubble mass to bubbles of sizes smaller than $D$ is $I_p(D_p|D) dD_p$, then $W_b(D)$ is the integral of this differential rate over all $D_p > D$, and infrared locality may be quantified by considering how quickly $I_p(D_p|D)$ decays with increasing $D_p$:

**Definition 1.** (Infrared locality.) If $W_b(D)$ may be written as

$$W_b(D) = \int_D^{\infty} dD_p I_p(D_p|D),$$

(2.7)

then infrared locality describes the rate at which $I_p$ decays from $D_p \sim D$ to $D_p \to \infty$.

Infrared locality describes the proximity of the origins of $W_b(D)$ to $D$. Second, one is interested in the degree to which contributions to $W_b(D)$ from all smaller bubble sizes are due primarily to bubble sizes just smaller than $D$. This is correspondingly termed ultraviolet locality. If the differential rate at which child bubbles of sizes between $D_c$ and $D_c + dD_c < D$ receive bubble mass from bubbles of sizes larger than $D$ is $I_c(D_c|D) dD_c$, then $W_b(D)$ is the integral of this differential rate over all $D_c < D$, and ultraviolet locality may be quantified by determining how quickly $I_c(D_c|D)$ decays with decreasing $D_c$:

**Definition 2.** (Ultraviolet locality.) If $W_b(D)$ may be written as

$$W_b(D) = \int_0^D dD_c I_c(D_c|D),$$

(2.8)

then ultraviolet locality describes the rate at which $I_c$ decays from $D_c \sim D$ to $D_c \to 0$.

Ultraviolet locality describes the proximity of the termini of $W_b(D)$ to $D$. The notions of infrared and ultraviolet locality are illustrated in Figure 1. In the next section, the
bubble size distribution and its corresponding population balance equation are introduced in order to derive a suitable expression for the breakup flux of interest, $W_b$. This expression is subsequently analyzed in Section 4 for the presence and strength of locality in order to determine the extent of validity of the proposed similarity hypotheses.

3. Mathematical formalism

3.1. The bubble size distribution

At every location $x$, for every bubble size $D$, and at some time $t$, the number density function for a bubble population $f$ may be constructed by adding a contribution from each bubble $i = 1, \ldots, N_b(t)$ having a centroid location $x_i$ and an equivalent size $D_i$:

$$f(x, D; t) = \sum_{i=1}^{N_b(t)} \delta(x - x_i(t)) \delta(D - D_i(t)),$$

where $\delta$ is the Dirac delta function. In statistically stationary and homogeneous flows, the probability distribution of bubble sizes $f(D)$ may be obtained by ensemble averaging $\langle \cdot \rangle$ over statistically independent but similar runs, volume averaging over some volume $\int_{\Omega} d\mathbf{x} = V$ that always contains all $N_b(t)$ bubbles, and time averaging over the interval $T$:

$$f(D) = \left\langle \frac{1}{VT} \int_{\Omega} d\mathbf{x} \int_0^T dt f(x, D; t) \right\rangle.$$

The averaging operations commute if $T$ and $V$ are identical in each run. The assumption of stationarity and homogeneity provides an approximate universal picture of local regions in turbulent bubbly flows, in the spirit of universal local isotropy in turbulent flows.

3.2. The population balance equation

The population balance equation for $f$ should be a physically reasonable time evolution equation that respects the conservation laws governing the underlying bubble population. The phase space for this equation comprises the bubble-size dimension $D$, assuming a single parameter suitably describes the size of the bubbles (Williams 1958). As suggested in Hypothesis 4, this is appropriate in a flow with a sufficiently high $W_{el}$. In this phase space, the total mass of gas in all the bubbles is conserved, except for the influx of gas due to entrainment and the assumed dissolution of gas. Given these definitions, choices, and observations, a time evolution equation for $f$ resembling the classical Liouville equation may be phenomenologically constructed (Hulburt & Katz 1964; Randolph 1964)

$$\frac{\partial \left[ f(D) D^3 \right]}{\partial t} + \frac{\partial \left[ v_D(D) f(D) D^3 \right]}{\partial D} = \mathcal{P}(D),$$

for some appropriately averaged source or sink term $\mathcal{P}$ [see Eq. (3.2) for the averaging operation] that includes the effects of breakup, coalescence, entrainment, the assumed dissolution, and possibly other effects. One may also interpret Eq. (3.3) as a generalized Boltzmann equation (Solsvik & Jakobsen 2015) where bubbles may split, dissolve, or be entrained in addition to colliding with one another. The first term is zero for statistically stationary flows, but is retained here for easier extension of the formalism to statistically nonstationary flows. In the second term, $v_D$ is the velocity along the $D$-axis. Note that Eq. (3.3) has been written in conservative form, as the function $f$ at some bubble size $D$ always appears with a factor $D^3$ proportional to the corresponding bubble
volume (Martínez-Bazán et al. 2010). The equation thus describes the movement of gas in bubble-size space, albeit in a probabilistic manner. Hence, by the conservation of total mass of gas and statistical stationarity, $\mathcal{T}$ must satisfy

$$\int_0^\infty dD \mathcal{T}(D) = \mathcal{T}_d/V + \mathcal{T}_e/V = 0,$$

(3.4)

where $\mathcal{T}_d$ and $\mathcal{T}_e$ are the appropriately averaged rates of dissolution and entrainment, respectively. In other words, the left-hand side of Eq. (3.3) must be balanced by the normalized dissolution and entrainment rates after integration over $D$-space. The second term on the left-hand side of Eq. (3.3), $\partial (\nu D \mathcal{T} D^3) / \partial D$, represents the local transport of $\mathcal{T} D^3$ along the $D$-axis from some bubble size $D$ to infinitesimally larger and smaller bubble sizes $D \pm \delta D$. $\mathcal{T}$, less the contributions from dissolution and entrainment, represents the nonlocal transport of $\mathcal{T} D^3$ along the $D$-axis from some bubble size $D$ to any other arbitrarily distant bubble size. The population balance equation, Eq. (3.3), is often alternatively written, in the absence of mass-transfer processes other than the assumed dissolution, as

$$\frac{\partial \mathcal{T}(D) D^3}{\partial t} = \mathcal{T}_b(D) + \mathcal{T}_c(D) + \mathcal{T}_d(D) + \mathcal{T}_e(D),$$

(3.5)

for some source/sink terms $\mathcal{T}_b$, $\mathcal{T}_c$, $\mathcal{T}_d$, and $\mathcal{T}_e$ for breakup, coalescence, dissolution, and entrainment, respectively. Comparing Eqs. (3.3) and (3.5) suggests that the source/sink terms $\mathcal{T}_b$ and $\mathcal{T}_e$ may comprise both local and nonlocal net transport in bubble-size space. $\mathcal{T}_b$ and $\mathcal{T}_e$ must individually satisfy the conservation of bubble mass; for example

$$\int_0^\infty dD \mathcal{T}_b(D) = 0.$$

(3.6)

Also, one may define $\mathcal{T}_d$ and $\mathcal{T}_e$ such that $\int_0^\infty dD \mathcal{T}_d(D) = \mathcal{T}_d/V$ and $\int_0^\infty dD \mathcal{T}_e(D) = \mathcal{T}_e/V$. As noted in Section 2.2, $\mathcal{T}_e$ is assumed to be negligible in the model problem considered here. Common model kernels used for $\mathcal{T}_b$ are the subject of the next subsection.

3.3. The model binary breakup kernel and the corresponding breakup flux

Assuming that all breakup events are independent of one another, and that only binary breakup events occur, a model form for $\mathcal{T}_b$ may be constructed as follows (Valentas et al. 1966; Coulaloglou & Tavlarides 1977; Ramkrishna 1985; Martínez-Bazán et al. 1999a,b)

$$\mathcal{T}_b(D) = \int_D^\infty dD_p q_b(D|D_p) g_b(D_p) \mathcal{T}(D_p) D^3 - g_b(D) \mathcal{T}(D) D^3.$$

(3.7)

The constituent terms were briefly discussed by Chan et al. (2018b) and are revisited here. The first term on the right-hand side is a source (birth) term due to the breakup of bubbles of sizes larger than $D$, while the second term is a sink (death) term due to the breakup of bubbles of size $D$ into smaller bubbles. The breakup rate $g_b(D) \mathcal{T}(D)$ is the expected rate of breakup events for bubbles of size $D$, which is modeled as being proportional to the bubble size distribution $\mathcal{T}(D)$ given by Eq. (3.2). Then, $g_b(D)$ is the characteristic breakup frequency of a bubble of size $D$. In addition, $q_b(D|D_p)$ is the probability that a bubble of size $D_p$ breaks into a bubble of size $D$ and another bubble of complementary volume such that the total gaseous volume remains constant through the breakup event. It is appropriate at this point to introduce several properties of $q_b(D_c|D_p)$ (Ramkrishna 1985; Martínez-Bazán et al. 2010) in order to facilitate
subsequent derivations. The mechanics of breakup imply that (Valentas et al. 1966)
\[ q_b(D_c|D_p) = 0 \quad \text{if } D_c \geq D_p, \] (3.8)
since a bubble cannot break to form bubbles larger than itself. Then, \( q_b \) may be normalized such that
\[ \int_0^{D_p} dD_c q_b(D_c|D_p) = \int_0^\infty dD_c q_b(D_c|D_p) = 2, \] (3.9)
where the factor of 2 arises from the assumption of binary breakup. As a result of this normalization, as well as the conservation of bubble mass, \( q_b \) will also need to satisfy
\[ \int_0^{D_p} dD_c q_b(D_c|D_p) D_c^3 = \int_0^\infty dD_c q_b(D_c|D_p) D_c^3 = D_p^3. \] (3.10)

Also, if a bubble of size \( D_p \) breaks into bubbles of sizes \( D_1 \) and \( D_2 \), then \( q_b(D_1|D_p)/D_1^2 = q_b(D_2|D_p)/D_2^2 \) by symmetry. Using the properties of \( q_b \) described above, one may verify that the model breakup kernel, Eq. (3.7), satisfies Eq. (3.6) by direct substitution.

The breakup flux \( \overline{W}_b(D) \) describes the net loss of mass from bubbles of sizes larger than \( D \), or the net gain in mass in bubbles of sizes smaller than \( D \), due to the breakup process modeled by \( \overline{I}_b \). From Eq. (3.6), it is evident that these two quantities are equal in magnitude, leading to the following equivalent definitions for the breakup flux
\[ \overline{W}_b(D) = \int_0^D dD_c \overline{I}_b(D_c) = -\int_D^\infty dD_p \overline{I}_b(D_p). \] (3.11)

One may show, via the properties of \( q_b \) outlined earlier, that \( \overline{W}_b \) then satisfies
\[ \overline{W}_b(D) = \int_0^D dD_c \int_D^\infty dD_p q_b(D_c|D_p) g_b(D_p) \overline{I}(D_p) D_c^3 - \int_0^D dD_c g_b(D_c) \overline{I}(D_c) D_c^3 \nonumber \\
= \int_0^D dD_c D_c^3 \int_0^\infty dD_p q_b(D_c|D_p) g_b(D_p) \overline{I}(D_p) - \int_0^D dD_p g_b(D_p) \overline{I}(D_p) D_p^3 \nonumber \\
= \int_0^D dD_c D_c^3 \int_0^\infty dD_p q_b(D_c|D_p) g_b(D_p) \overline{I}(D_p) - \int_0^D dD_c D_c^3 \int_D^\infty dD_p q_b(D_c|D_p) g_b(D_p) \overline{I}(D_p) \nonumber \\
+ \int_0^D dD_c D_c^3 \int_0^\infty dD_p q_b(D_c|D_p) g_b(D_p) \overline{I}(D_p). \] (3.12)

Note that \( \overline{W}_b(D) \) has been expressed in terms of integrals with limits involving \( D \), similar to the expressions in Eqs. (2.7)–(2.8). One may, then, directly infer that
\[ I_p(D_p|D) = \int_0^D dD_c D_c^3 q_b(D_c|D_p) g_b(D_p) \overline{I}(D_p), \] (3.13)
\[ I_c(D_c|D) = \int_D^\infty dD_p D_p^3 q_b(D_c|D_p) g_b(D_p) \overline{I}(D_p). \] (3.14)

The analysis of the constituent terms in these quantities is the subject of the next section.
4. Locality in bubble-mass transfer across bubble-size space

4.1. Locality of $W_b$ in turbulent bubble breakup

The presence of locality in the breakup flux $W_b(D)$ that arises from the action of turbulent eddies, and thus the presence of a turbulent bubble breakup cascade, may be analyzed using simplified expressions for the constituent model terms $g_b(D_p)\overline{f}(D_p)$ and $q_b(D_c|D_p)$ suitable for turbulent bubble fragmentation. Consider, first, the variation of the breakup rate $g_b(D_p)\overline{f}(D_p)$ with the parent bubble size $D_p$. As discussed in Section 1 and as presented by Chan et al. (2018a,b), the bubble size distribution has been theoretically, experimentally, and numerically demonstrated to scale as $D_p^{-10/3}$ for parent bubbles of a set of intermediate sizes where fragmentation occurs due to turbulence in the carrier phase. Assuming

$$\overline{f}(D_p) \sim D_p^{-10/3}$$

in this range of bubble sizes, it remains to examine the scaling of the characteristic breakup frequency $g_b$ with $D_p$. This may be estimated by observing that at some length scale $D_p$ in the inertial subrange $L \gg D_p \gg L_K$, turbulent velocity fluctuations scale as $u_{D_p} \sim D_p^{1/3}$. The characteristic breakup frequency for super-Hinze-scale bubbles of size $D_p$ may then be estimated as the inverse of the corresponding eddy turnover time; in other words

$$g_b(D_p) \sim u_{D_p}/D_p \sim D_p^{-2/3},$$

assuming that the breakup of a bubble occurs primarily due to a turbulent eddy of a comparable size (Hinze 1955; Chan et al. 2018a). This yields

$$g_b(D_p)\overline{f}(D_p) \sim D_p^{-4}.$$ 

It may then be shown, using the scaling of Garrett et al. (2000) for $\overline{f}$, that $g_b\overline{f}$ has no explicit dependence on $\varepsilon$. The frequency scaling $g_b \sim D_p^{-2/3}$ has been alluded to in other studies, including the breakup models of Coulaloglou & Tavlarides (1977), Lee et al. (1987a,b), and Martínez-Bazán et al. (1999a). In addition, Martínez-Bazán et al. (2010) demonstrated that several other models in the literature that do not exhibit an explicit $D_p^{-2/3}$ scaling do in fact predict a very similar scaling at sufficiently large $D_p$. As mentioned in Section 1, this frequency scaling was also observed in related experiments discussed by Martínez-Bazán et al. (1999a) and Rodríguez-Rodríguez et al. (2006), as well as the breaking-wave simulations discussed by Chan et al. (2018b).

A complete characterization of locality requires knowledge of the breakup probability $q_b(D_c|D_p)$ as well. Unlike the quantities above, there is less convergence between analytical, experimental, and numerical studies on the appropriate scaling of $q_b$ with $D_c$ and $D_p$ in the context of turbulent breakup. Various model forms for $q_b$ have been developed from statistical ansatz, phenomenological arguments, and empirical data, as reviewed in detail by Lasheras et al. (2002), Liao & Lucas (2009), Martínez-Bazán et al. (2010), and Solsvik et al. (2013). Due to the general lack of consensus on a suitable model form, two canonical distributions in bubble-volume space are used as surrogate models: the uniform distribution and the beta distribution. Note that the uniform distribution is a special case of the beta distribution, but is discussed separately for clarity.
4.1.1. Uniform distribution in volume space

Consider, first, the uniform distribution in bubble-volume space \( (D^3\text{-space}) \)

\[
q_b(D^3_c|D_p^3) = \begin{cases} \frac{1}{D_p^3}, & 0 \leq D^3_c \leq D^3_p, \\ 0, & D^3_p < D^3_c. \end{cases}
\]

From the properties of \( q_b \) discussed in Section 3.3, this is equivalent to the following distribution in bubble-size space \( (D\text{-space}) \)

\[
q_b(D_c|D_p) = \begin{cases} \frac{D^2}{D_p^3}, & 0 \leq D_c \leq D_p, \\ 0, & D_p < D_c. \end{cases}
\]

With the available scalings for \( q_b \) and \( q_b{\overline{f}} \), Eqs. (3.13)–(3.14) yield

\[
I_p(D_p|D) \sim \int_0^D dD_c D^5_c D_p^{-7} \sim D^6 D_p^{-7}, \quad I_c(D_c|D) \sim \int_D^\infty dD_p D^5_c D_p^{-7} \sim D^4_c D^{-6}.
\]

Observe that \( I_p \) and \( I_c \) decrease as \( D_p \to \infty \) and \( D_c \to 0 \), respectively, indicating that \( \overline{W}_b \) may be reasonably approximated as local. More specifically, the limits

\[
I_p(D_p|D) \sim D^5_p \sim D^7_p^{-2}, \quad I_c(D_c|D) \sim D^5_c \sim D^5_c^{-2}
\]

hold as \( D_p \to \infty \) and \( D_c \to 0 \), respectively. The exponents \( \gamma_p \) and \( \gamma_c \) were referenced earlier in Figure 1. Note that these relations hold even at \( D_p \sim D \) and \( D_c \sim D \), respectively, because \( q_b \) is separable in \( D_p \) and \( D_c \). Thus, a stronger statement on locality may be made in the case of the uniform distribution: since

\[
\overline{W}_b(D) \sim \int_0^D dD_c D^5_c \int_D^\infty dD_p D_p^{-7},
\]

may be expressed as the separable product of two integrals, one may further conclude that \( \overline{W}_b(D) \) may be directly approximated by a movement of bubble mass in bubble-size space from some bubble size just larger than \( D \) to some bubble size just smaller than \( D \). Finally, as a self-consistency check, one may obtain the scaling of \( \overline{W}_b(D) \) itself with \( D \)

\[
\overline{W}_b(D) \sim \int_0^D dD_c D^5_c D^{-6} \sim \int_D^\infty dD_p D^5_c D_p^{-7} \sim D^4_c D^{-6} \sim \text{constant}.
\]

If the underlying energy flux in the surrounding turbulence is scale invariant within an inertial scale subrange, and the breakup probability is chosen to be size invariant in a corresponding intermediate range of bubble sizes, then the resulting bubble-mass breakup flux is size invariant, revealing the presence of an intermediate bubble-size subrange.

4.1.2. Beta distribution in volume space

Recall from Section 3.3 that the breakup probability is symmetric in bubble-volume space. The beta distribution that satisfies this constraint can take only a single shape parameter \( \alpha \), and may be expressed in bubble-volume space, or \( D^3\text{-space} \), as

\[
q_b(D^3_c|D^3_p) = \begin{cases} D^{3(\alpha-1)} (D^3_p - D^3_c)^{\alpha-1} D_p^{-6(\alpha-1)-3}/B(\alpha, \alpha), & 0 \leq D^3_c \leq D^3_p, \\ 0, & D^3_p < D^3_c. \end{cases}
\]
where $B(\alpha, \alpha)$ is the beta function. This distribution is plotted in Figure 2 for several values of $\alpha$. Note that the uniform distribution is recovered when $\alpha = 1$. The beta distribution is only defined for $\alpha > 0$. When $0 < \alpha < 1$, the distribution is U-shaped and goes to infinity at the endpoints of the domain. When $\alpha > 1$, the distribution is N-shaped and goes to zero at the endpoints. The beta distribution is thus a reasonable surrogate model for most observed and modeled breakup distributions, except for the class of M-shaped distributions—see the reviews cited in Section 4.1 for examples of these distributions. The corresponding distribution in bubble-size space, or $D$-space, is

$$q_b(D_c|D_p) = \begin{cases} D_c^{3\alpha-1} (D_p^3 - D_c^3)^{\alpha-1} D_p^{3-6\alpha} / B(\alpha, \alpha), & 0 \leq D_c^3 \leq D_p^3, \\
0, & D_p^3 < D_c^3. \end{cases}$$

With the available scalings for $q_b$ and $g_b$, Eqs. (3.13)–(3.14) yield

$$I_p(D_p|D) \sim \int_0^D dD_p \frac{D_p^{3\alpha+2}}{(D_p^3 - D_c^3)^{1-\alpha}} D_p^{-1-6\alpha} \sim D_p^{-1} \int_0^{D^3/D_p^3} dx x^\alpha (1 - x)^{\alpha-1}, \quad (4.4)$$

$$I_c(D_c|D) \sim \int_D^\infty dD_p \frac{D_p^{-1-6\alpha}}{(D_p^3 - D_c^3)^{1-\alpha}} D_c^{3\alpha+2} \sim D_c^{-1} \int_0^{D_c^3/D^3} dx x^\alpha (1 - x)^{\alpha-1}. \quad (4.5)$$

The final integrals in Eqs. (4.4) and (4.5) are the incomplete beta functions $B_{D_p^3/D_c^3}(\alpha + 1, \alpha)$ and $B_{D_c^3/D^3}(\alpha + 1, \alpha)$, respectively (Abramowitz & Stegun 1964, Sections 6.6.1 and 26.5.3). The results discussed in Section 4.1.1 are exactly recovered when $\alpha = 1$. Equations (4.4) and (4.5) are plotted in arbitrary units as functions of $D_p/D$ and $D_c/D$, respectively, in Figure 3. As $\alpha$ increases, the rates of decay of $I_p$ and $I_c$ as $D_p \to \infty$ and $D_c \to 0$, respectively, increase, indicating that as breakup events involving child bubbles of similar sizes are increasingly favored, locality correspondingly increases. At small $\alpha$, where the most likely breakup events involve child bubbles of differing sizes, the cascade is diffuse, or leaky, and the bubble breakup flux is better described as quasi-local. Also, for sufficiently large $D_p$ and sufficiently small $D_c$, Eqs. (4.4)–(4.5) may be approximated...
as

\[ I_p(D_p|D) \approx \frac{D_p^{-1-6\alpha}}{D_p^{1-\alpha}} \sim D_p^{-4-3\alpha} \sim D_p^{\gamma_p}, \quad (4.6) \]

\[ I_c(D_c|D) \approx D_c^{3\alpha+2} \sim D_c^{\gamma_c}. \quad (4.7) \]

Note that the exponents \( \gamma_p \) and \( \gamma_c \) were referenced earlier in Figure 1. Note, also, that in these limits, \( I_p \) decays at least as quickly as \( D_p^{-4} \), and \( I_c \) grows at least as quickly as \( D_c^2 \), so the bubble breakup flux is always at least quasi-local regardless of \( \alpha \). Once again, the results of Section 4.1.1 are recovered—in an exact fashion—for \( \alpha = 1 \). One may also examine the dependence of \( \overline{W_b}(D) \) on \( D \) as a self-consistency check

\[
\overline{W_b}(D) \sim \int_0^D dD_c D_c^{-1} \int_0^{D_c^2/D^3} dx x^\alpha (1 - x)^{\alpha-1} \\
\sim \int_0^\infty dD_p D_p^{-1} \int_0^{D_p^3/D^3} dx x^\alpha (1 - x)^{\alpha-1} \sim \int_0^1 dy y^{-1} \int_0^y dx x^\alpha (1 - x)^{\alpha-1} \\
\sim \text{constant},
\]

which reveals, as expected, an intermediate bubble-size subrange for the bubble breakup flux if \( \alpha \) is constant over the bubble-size subrange.

### 4.2. Relation to spectral energy flux models

The breakup flux \( \overline{W_b} \) describes the averaged movement of bubble mass \( \sim \overline{T} D^3 \) in bubble-size space as governed by the population balance equation, Eq. (3.5). This is analogous to how the transfer flux \( W(k) = \int_0^k dk' T(k') \) describes the movement of turbulent kinetic energy \( E(k) \) in wavenumber space based on the spectral turbulent kinetic energy equation, where the rate of change of \( E(k) \) due to inter-scale transfer is \( T(k) \). In particular, the double-integral form of the breakup flux, Eq. (3.12), is reminiscent of the spectral energy transfer model of Heisenberg (1948a,b), where \( W(k) \) is modeled as a separable product of integrals

\[
W(k) \sim \int_0^k dk' E(k') k'^2 \int_k^\infty dk'' \sqrt{\frac{E(k'')}{k''^3}}. \quad (4.8)
\]

By substituting the inertial subrange scaling \( E(k) \sim k^{-5/3} \) into Eq. (4.8), one obtains \( k^{5/3} k'^2 \sim k'^{1/3} \) and \( \sqrt{k''^{-5/3} k''^{-3}} \sim k''^{-7/3} \) for the scalings of the two integrands, suggesting some degree of infrared and ultraviolet locality, respectively. Note that Eq. (4.8) has no dependence on \( k \), as one would expect for self-similar energy transfer.

If the true \( W(k) \) is quasi-local in \( k \)-space, then it may be well approximated by a wavenumber-local expression. This brings to mind the quasi-local models of Kovasznay (1948) and Pao (1965, 1968). Kovasznay (1948) argued that if \( W(k) \) is dependent only on \( E(k) \) and \( k \), then the only dimensionally consistent expression is

\[
W(k) \sim [E(k)]^{3/2} k^{5/2}.
\]

Subsequently, Pao (1965, 1968) allowed \( W(k) \) to depend on \( \varepsilon \) as well. If it is further assumed that \( W(k) \) is linear in \( E(k) \), then it follows from dimensional arguments that

\[
W(k) \sim \varepsilon^{1/3} k^{5/3} E(k).
\]

In a similar fashion, \( \overline{W_b}(D) \) may be justifiably modeled by an expression local in \( D- \)
Turbulent bubble breakup cascade

Figure 3. Integrands of $\overline{W_b}$ demonstrating the extent of (a) infrared locality using Eq. (4.4) and (b) ultraviolet locality using Eq. (4.5) after substituting the symmetric beta distribution with various shape factors $\alpha$ for the breakup probability $q_b$, as well as scalings for the bubble breakup rate corresponding to a turbulent breakup mechanism. Since the proportionality constants are dropped in Eqs. (4.4)–(4.5), the integrands here are plotted in arbitrary units, with the values at $D_p = D$ [for (a)] and $D_c = D$ [for (b)] fixed at unity. The power-law fits at large $D_p$ and small $D_c$ correspond to the scaling limits in Eqs. (4.6)–(4.7) for $\alpha = 1/5$ and $\alpha = 5$.

Returning to Eq. (3.3), one desires an appropriate model for $v_D$ such that $W_b \sim v_D T D^3$. If there exists an intermediate bubble-size subrange where $W_b$ is independent of $D$, and $T \sim D^{-10/3}$, then an appropriate model for $v_D$ should satisfy $v_D \sim D^{1/3}$. Note that this is similar to the scaling for the turbulent velocity fluctuations with eddy size $u_D \sim D^{1/3}$. The scaling for $v_D$ was previously postulated by Garrett et al. (2000) on the dimensional grounds that $v_D \sim u_D$, but one should be cognizant of the difference between bubble-size space and eddy-size space.
5. Conclusions

This brief explores the possibility of universal breakup behavior of small bubbles in high-Reynolds-number and high-Weber-number turbulent bubbly flows. A bubble breakup cascade, which requires the trinity of universality, locality, and self-similarity, is postulated to exist in these flows. The population balance equation for the bubble size distribution is extended to determine locality in the breakup process amid self-similar energy and bubble-mass transfer. Parallels are drawn between the bubble-mass cascade and the traditional turbulent energy cascade. The key ingredient for locality in the breakup flux is the adoption of turbulent scalings for the constituent model terms. The presence of locality is not too sensitive to the breakup probability distribution, but the shape of the distribution influences the strength of locality, and thus the degree to which the resulting breakup cascade is leaky. With locality in the breakup flux and a bubble-size-invariant breakup probability distribution, the presence of an intermediate bubble-size subrange in which the breakup flux is constant is confirmed. Quantifying this locality has implications for our fundamental understanding of bubble breakup, as well as modeling of the breakup of subgrid bubbles in large-eddy simulations of turbulent bubbly flows. The binary breakup assumption precludes the formation of satellite bubbles, which decrease this locality and may also disrupt self-similarity. These bubbles are under ongoing investigation.

Acknowledgments

W. H. R. Chan is funded by a National Science Scholarship from the Agency of Science, Technology and Research in Singapore. The authors acknowledge computational resources from the US Department of Energy’s INCITE Program, as well as from the Certainty cluster awarded by the National Science Foundation to CTR. The authors are grateful to P. Moin for his insightful direction in the investigation of key scientific questions, J. Urzay for his patient guidance on the interpretation of the governing equations, A. Mani for fruitful discussions on the bubble size distribution, D. Livescu and A. Lozano-Durán for exploratory discussions on locality in the model equations, and M. S. Dodd for joint work on algorithms to retrieve statistics from numerical simulations that inspired this work. The authors also thank H. Hwang for his comments on this manuscript.

REFERENCES

Chan, W. H. R., Dodd, M. S., Johnson, P. L., Urzay, J. & Moin, P. 2018b Formation and dynamics of bubbles in breaking waves. II. The evolution of the


Martínez-Bazán, C., Montañés, J. L. & Lasheras, J. C. 1999b On the breakup


