Toward evaluating contributions to skin friction enhancement by transition and turbulence in boundary layer flows

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1. Motivation and objectives

The viscous skin friction of laminar and turbulent boundary layers is the dominant source of drag for streamlined bodies. Transition to turbulence in boundary layers results in a dramatic increase of the skin friction associated with enhanced momentum transport in the wall-normal direction due to turbulent fluctuations (Schlichting 1960). In some scenarios, transition may be accompanied by an overshoot in skin friction and heating (Sayadi *et al.* 2013; Franko & Lele 2013; Fu *et al.* 2018), as well as more peculiar effects such as a blip in wall shear stress late in the transition process (Sayadi *et al.* 2011) or oscillator heat transfer patterns which remain even after significant time averaging (Franko & Lele 2013). A better understanding of the physical mechanisms driving the enhanced skin friction and heat transfer in transitional and turbulent boundary layers may inform the development and use of control schemes aimed at reducing skin friction losses (Choi *et al.* 1994) or aerodynamic heating.

To this end, Fukagata *et al.* (2002) introduced a decomposition of contributions to the friction coefficient in internal flows derived from a triple integration of the averaged momentum equation. Most notably, their result expressed the skin friction as a combination of laminar and turbulent contributions, showing that Reynolds stresses nearer the wall play a larger role in increasing the friction coefficient. The extension of this triple integration technique to boundary layer flows was less successful for various reasons. First, the integration had to be arbitrarily cut off at the edge of the boundary layer (which is always somewhat vaguely defined). Second, the spatial development term does not vanish in the laminar case, and thus the effect of turbulence is difficult to disentangle. Other more detailed shortcomings are discussed by Renard & Deck (2016). Applications of Fukagata *et al.* (2002) to control strategies have been reported by Iwamoto *et al.* (2005); Kametani & Fukagata (2011); Kametani *et al.* (2015); and Stroh *et al.* (2015).

More recently, Renard & Deck (2016) introduced an alternative decomposition of the wall shear stress for boundary layers using the conservation of kinetic energy in a reference frame moving with the freestream, as contrasted with the conservation of momentum used by Fukagata *et al.* (2002). This kinetic energy budget approach to decomposing the skin friction leads to more well-posed integrals which could be safely integrated through the freestream (to infinity). However, it does not separate laminar and turbulent contributions, but rather separates the dissipation of mean kinetic energy by viscous action on the mean velocity profile from that of turbulent kinetic energy production. In other words, turbulence alters each term in the decomposition, and the laminar friction coefficient is not isolated to one particular term. Moreover, the decomposition of Renard & Deck (2016) does not lead to a unified framework for understanding the friction losses in boundary layers and internal flows, which certainly have much in common. Nonetheless, this approach has been extended to compressible flows (Li *et al.* 2019; Fan *et al.* 2019).

It is argued in this research brief that extending the work of Fukagata *et al.* (2002) to boundary layers is properly done by double integration of the momentum deficit conservation equations rather than by the triple integration of Fukagata *et al.* (2002) or by resorting to the kinetic energy budget as in Renard & Deck (2016). This change is driven by the difference in the definition of the friction coefficient relevant to engineers in each case. That is, the friction coefficient relates the wall stress for internal flows to the flow rate but for boundary layers to the freestream velocity. This difference directly impacts how many integrations should be performed on the viscous force term, $\nu\partial^2 u/\partial y^2$, to obtain the laminar contribution to the skin friction.

A relationship is derived in this brief relating the friction coefficient in incompressible boundary layers to the laminar contributions as well as to turbulent contributions of stress and altered streamwise shape development. This new decomposition, taken together with the original work of Fukagata *et al.* (2002), provides a more unified framework for understanding turbulence-related losses in internal and external wall-bounded flows. Moreover, the new decomposition introduced here is intimately connected with the classical momentum integral equation typically used to analyze boundary layers (Schlichting 1960). In this view, the present approach can be more intuitively understood in terms of angular momentum, ultimately providing a fresh perspective on how to interpret the meaning of the friction coefficient in boundary layer flows.

The rest of the brief is organized as follows. First, the formulation of Fukagata *et al.* (2002) is briefly reviewed in Section 2. Then, the novel decomposition of contributions to skin friction in boundary layer flows is derived in Section 3, underscoring its most notable aspects and advantages over previous related work. Conclusions are drawn in Section 4.

2. The friction coefficient for incompressible internal flows

The friction coefficient for an incompressible internal flow is defined as

$$c_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U_b^2}, \quad \text{where} \quad U_b \equiv \frac{1}{A_\perp}\int_\perp u(\mathbf{x})dA \tag{2.1}$$

is the bulk velocity. Fukagata *et al.* (2002) showed that integrating the streamwise momentum equation successively three times with normalization by the bulk velocity yields

$$\frac{c_f}{6} = \frac{1}{Re_b} - \int_0^1 \left(1-\frac{y}{h}\right)\frac{\overline{u'v'}}{U_b^2}\, d\left(\frac{y}{h}\right) - \int_0^1 \frac{1}{2}\left(1-\frac{y}{h}\right)^2 I_x \, d\left(\frac{y}{h}\right) \tag{2.2}$$

for a channel flow with half height h. The bulk Reynolds number is $Re_b = U_b h/\nu$. The I_x term includes unsteady and spatial development terms that are negligible for steady, fully developed internal flows. The relation for the pipe flow is similar in nature.

The first term on the right-hand side of Eq. (2.2) is the laminar contribution. It is equal to $c_f/6$ for the laminar channel flow. Thus, the second term, which is a weighted integral of the Reynolds stress, represents the increase in friction coefficient for turbulent channel flows. Of particular note is the weighting factor $1-(y/h)$, which shows that Reynolds stresses nearer the wall are more important in causing the friction losses to rise when internal flows become turbulent.

A straightforward extension of this triple integration approach to boundary layers, as first tried by Fukagata *et al.* (2002), is not promising. First, the fixed upper bound needs to be replaced by infinity for boundary layer analysis. More importantly, the friction factor for boundary layers is based on the freestream velocity. Therefore, the triple integration technique which produces the relationship of wall shear stress to bulk velocity is not optimal. Finally, some of the spatial variation terms included in I_x by Fukagata *et al.* (2002) are non-negligible and even important in boundary layers, which grow as they develop downstream. For these reasons, it is necessary to adjust the approach of Fukagata *et al.* (2002) to produce a relationship more suited to analyze boundary layer flows.

3. Decomposing the incompressible boundary layer skin friction

For the derivation for the wall shear stress decomposition, consider a statistically two-dimensional incompressible boundary layer characterized by mean streamwise velocity $\bar{u}(x, y, t)$, wall-normal velocity $\bar{v}(x, y, t)$, and mean pressure $\bar{p}(x, y, t)$. The boundary layer flow satisfies no-slip, no-penetration conditions at the wall, $y = 0$. In the freestream, $y \gg \delta$, the streamwise velocity becomes $U_\infty(x, t)$ and the freestream pressure is $P_\infty(x, t)$. The flow is not constrained to be statistically stationary. The overbar $\overline{(\cdot)}$ denotes Reynolds averaging, which in the present case can be taken as ensemble and/or spanwise averaging. While the present derivation is performed for incompressible flows, extension of this approach to compressible flows could be a rewarding future direction, if any potential complications which arise from such a generalization can be satisfactorily overcome.

3.1. *Governing equations*

The incompressible Reynolds-averaged Navier-Stokes (RANS) equations governing a statistically two-dimensional flow are

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \tag{3.1}$$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} = -\frac{\partial \bar{p}}{\partial x} + \nu \frac{\partial^2 \bar{u}}{\partial x^2} + \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y}. \tag{3.2}$$

In the freestream, Eq. (3.2) becomes

$$\frac{\partial U_\infty}{\partial t} + U_\infty \frac{\partial U_\infty}{\partial x} = -\frac{\partial P_\infty}{\partial x}. \tag{3.3}$$

Subtracting Eq. (3.2) from Eq. (3.3) yields

$$-\frac{\partial \bar{u}^2}{\partial x} - \frac{\partial \bar{u}\bar{v}}{\partial y} + U_\infty \frac{\partial U_\infty}{\partial x} = -\nu \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial \overline{u'v'}}{\partial y} - I_x, \tag{3.4}$$

where the terms typically negligible for steady, high Reynolds number boundary layers are combined into

$$I_x \equiv \frac{\partial (U_\infty - \bar{u})}{\partial t} + \frac{\partial (P_\infty - \bar{p})}{\partial x} + \nu \frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial \overline{u'^2}}{\partial x}. \tag{3.5}$$

Note that we do not explicitly neglect these terms, but rather de-emphasize them by agglomerating them into a single term. Unlike previous approaches (Fukagata *et al.* 2002; Renard & Deck 2016), this term does not include the dominant spatial development terms, which will be treated more explicitly here. Next, it may be shown using Eq. (3.1) that the left-hand side of Eq. (3.4) may be rewritten as

$$-\frac{\partial\bar{u}^2}{\partial x}-\frac{\partial\bar{u}\bar{v}}{\partial y}+U_\infty\frac{\partial U_\infty}{\partial x}=\frac{\partial(U_\infty-\bar{u})\bar{u}}{\partial x}+\frac{\partial(U_\infty-\bar{u})\bar{v}}{\partial y}+(U_\infty-\bar{u})\frac{\partial U_\infty}{\partial x}. \quad (3.6)$$

The substitution of Eq. (3.6) into Eq. (3.4) results in

$$\frac{\partial(U_\infty-\bar{u})\bar{u}}{\partial x}+\frac{\partial(U_\infty-\bar{u})\bar{v}}{\partial y}+(U_\infty-\bar{u})\frac{\partial U_\infty}{\partial x}=-\nu\frac{\partial^2\bar{u}}{\partial y^2}+\frac{\partial\overline{u'v'}}{\partial y}-I_x. \quad (3.7)$$

This equation is the starting point for the integrations to be performed in the wall-normal direction. Note that this is the same starting point as the derivation for the integral momentum equation classically employed to analyze laminar and turbulent boundary layers.

3.2. *First integration*

By integrating Eq. (3.7) from the wall to a point y above the boundary, the result is

$$\frac{\partial}{\partial x}\left(\int_0^y\ [U_\infty-\bar{u}(\hat{y})]\ \bar{u}(\hat{y})\ d\hat{y}\right)+[U_\infty-\bar{u}(y)]\bar{v}(y)+\frac{\partial U_\infty}{\partial x}\int_0^y\ [U_\infty-\bar{u}(\hat{y})]\ d\hat{y}$$
$$=\frac{\tau_w}{\rho}-\nu\frac{\partial\bar{u}}{\partial y}+\overline{u'v'}(y)-\int_0^y I_x(\hat{y})d\hat{y}, \quad (3.8)$$

where $\hat{y}$ is the dummy integration variable in the wall-normal coordinate. Equation (3.8) has explicitly made use of the fact that $\bar{v}(0)=\overline{u'v'}(0)=0$ and the definition of the skin friction,

$$\frac{\tau_w}{\rho}\equiv\nu\left.\frac{\partial\bar{u}}{\partial y}\right|_0. \quad (3.9)$$

The reader may recognize that the integral momentum equation is recovered from Eq. (3.8) by taking the limit $y\to\infty$ and neglecting the integral of I_x,

$$\frac{d}{dx}\left(U_\infty^2\theta\right)+\delta^*U_\infty\frac{dU_\infty}{dx}=\frac{\tau_w}{\rho}. \quad (3.10)$$

The displacement and momentum thicknesses, respectively, are

$$\delta^*\equiv\int_0^\infty\ \left[1-\frac{\bar{u}(y)}{U_\infty}\right]\ dy, \quad (3.11)$$

$$\theta\equiv\int_0^\infty\ \left[1-\frac{\bar{u}(y)}{U_\infty}\right]\ \frac{\bar{u}(y)}{U_\infty}\ dy, \quad (3.12)$$

according to their standard definitions.

3.3. *Second integration*

Returning to Eq. (3.8) and integrating again from the wall to a point y above the wall,

$$\frac{\partial}{\partial x}\left(\int_0^y(y-\hat{y})\ [U_\infty-\bar{u}(\hat{y})]\ \bar{u}(\hat{y})\ d\hat{y}\right)+\int_0^y\ [U_\infty-\bar{u}(\hat{y})]\ \bar{v}(\hat{y})\ d\hat{y}$$
$$+\frac{\partial U_\infty}{\partial x}\int_0^y(y-\hat{y})\ [U_\infty-\bar{u}(\hat{y})]\ d\hat{y}=\frac{y\tau_w}{\rho}-\nu\bar{u}(y)+\int_0^y\ \overline{u'v'}\ d\hat{y}-\int_0^y\ (y-\hat{y})\ I_x(\hat{y})\ d\hat{y}. \quad (3.13)$$

Note that the limit $y \to \infty$ cannot be directly taken for Eq. (3.13) because terms on both sides would increases in an unbounded way. The culprit is the explicit y inside the various integrals, as well as the y pre-multiplying the wall shear stress. To remove this divergence in the $y \to \infty$ limit, the singly integrated form in Eq. (3.8) may be multiplied by $y - \ell$ to form

$$\begin{aligned}\frac{\partial}{\partial x}\left(\int_0^y (y-\ell)\ [U_\infty - \bar{u}(\hat{y})]\ \bar{u}(\hat{y})\ d\hat{y}\right) + \left(\int_0^y [U_\infty - \bar{u}(\hat{y})]\ \bar{u}(\hat{y})\ d\hat{y}\right)\frac{d\ell}{dx} \\ + (y-\ell)[U_\infty - \bar{u}(y)]\bar{v}(y) + \frac{\partial U_\infty}{\partial x}\int_0^y (y-\ell)\ [U_\infty - \bar{u}(\hat{y})]\ d\hat{y} + \int_0^y (y-\ell)\ I_x(\hat{y})d\hat{y} \\ = (y-\ell)\ \frac{\tau_w}{\rho} - \nu(y-\ell)\frac{\partial \bar{u}}{\partial y} + (y-\ell)\overline{u'v'}. \quad (3.14)\end{aligned}$$

Then, subtracting Eq. (3.14) from Eq. (3.13) and taking the $y \to \infty$ limit,

$$\begin{aligned}\frac{\partial}{\partial x}\left(\int_0^\infty (\ell-\hat{y})\ [U_\infty - \bar{u}(\hat{y})]\ \bar{u}(\hat{y})\ d\hat{y}\right) - \left(\int_0^\infty [U_\infty - \bar{u}(\hat{y})]\ \bar{u}(\hat{y})\ d\hat{y}\right)\frac{d\ell}{dx} \\ + \int_0^\infty\ [U_\infty - \bar{u}(\hat{y})]\ \bar{v}(\hat{y})\ d\hat{y} + \frac{\partial U_\infty}{\partial x}\int_0^\infty (\ell-\hat{y})\ [U_\infty - \bar{u}(\hat{y})]\ d\hat{y} \\ = \frac{\ell\tau_w}{\rho} - \nu U_\infty + \int_0^\infty \overline{u'v'}\ d\hat{y} - \int_0^\infty\ (\ell-\hat{y})\ I_x(\hat{y})\ d\hat{y}. \quad (3.15)\end{aligned}$$

Note that an unspecified length scale ℓ has been introduced for dimensional consistency in the effort to regularize the $y \to \infty$ limit. Before considering how to choose ℓ, the derivation is completed by nondimensionalizing Eq. (3.15) by dividing by $U_\infty^2\ell$, which forms the wall shear stress term into the skin friction coefficient,

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}. \quad (3.16)$$

Solving the resulting equation in terms of the skin friction coefficient yields

$$\frac{C_f}{2} = \frac{1}{Re_\ell} + \int_0^\infty \frac{-\overline{u'v'}}{U_\infty^2\ell}\ dy + \frac{\partial\theta_\ell}{\partial x} + \frac{\theta_\ell - \theta}{\ell}\frac{d\ell}{dx} + \frac{\theta_v}{\ell} + \frac{\delta_\ell^* + 2\theta_\ell}{U_\infty}\frac{\partial U_\infty}{\partial x} + \mathcal{I}_{x,\ell}, \quad (3.17)$$

where the Reynolds number is based on ℓ,

$$Re_\ell \equiv \frac{U_\infty \ell}{\nu}, \quad (3.18)$$

and the terms typically negligible in boundary layer analysis are represented by

$$\mathcal{I}_{x,\ell} \equiv \int_0^\infty \left(1 - \frac{y}{\ell}\right)\frac{I_x}{U_\infty^2}\ dy. \quad (3.19)$$

The following length scales have been introduced in Eq. (3.17),

$$\delta_\ell^* \equiv \int_0^\infty \left(1 - \frac{y}{\ell}\right)\ \left[1 - \frac{\bar{u}(y)}{U_\infty}\right]\ dy, \quad (3.20)$$

$$\theta_\ell \equiv \int_0^\infty \left(1 - \frac{y}{\ell}\right)\ \left[1 - \frac{\bar{u}(y)}{U_\infty}\right]\ \frac{\bar{u}(y)}{U_\infty}\ dy, \quad (3.21)$$

and

$$\theta_v \equiv \int_0^\infty\ \left[1 - \frac{\bar{u}(y)}{U_\infty}\right]\ \frac{\bar{v}(y)}{U_\infty}\ dy. \quad (3.22)$$

These three length scales can be intuitively understood as follows. The last one, Eq. (3.22), is the most straightforward. It represents the integral of the vertical flux of streamwise momentum deficit, $1-\bar{u}/U_\infty$. This is analogous to the momentum thickness, θ, given by Eq. (3.12), which represents the streamwise flux of the streamwise momentum deficit.

The first two are first-order moments about ℓ of the momentum deficit and momentum deficit fluxes, respectively. Note that the zeroth-order moment of these two are the displacement and momentum thicknesses, respectively, hence the chosen notation. The displacement thickness, δ^*, given by Eq. (3.11), is the unweighted integral of the streamwise momentum deficit. The new length δ^*_ℓ is the weighted integral of streamwise momentum deficit where momentum deficit at locations $y > \ell$ is given a negative weight. Likewise, θ_ℓ is the weighted integral of the streamwise flux of streamwise momentum deficit, just as the momentum thickness, θ, given by Eq. (3.12), is the unweighted integral of this quantity.

3.4. *Why not triple integration?*

Before proceeding to the choice of ℓ, it is worthwhile to comment on one aspect of procedural difference between Fukagata *et al.* (2002) and this present work. In Fukagata *et al.* (2002), the mean streamwise momentum equation was integrated three times successively to obtain a friction coefficient decomposition for a channel flow which isolated the laminar contribution. In the present case dealing with boundary layers, only two successive integrations have been performed. The reason for this difference is quite straightforward. In the case of internal flows such as the channel flow, the engineering friction coefficient must relate the wall shear stress (i.e., pressure drop) to the flow rate, which is the velocity integrated over the cross section. To transform the viscous force, $\nu\partial^2\bar{u}/\partial y^2$, into the flow rate, three successive integrations must be performed, with the last integration carried over the full extent of the flow. In contrast, the engineering friction coefficient for external flows such as the boundary layers considered here must relate the wall shear stress to the freestream velocity – not the flow rate. The freestream velocity is obtained from the viscous term using only two successive integrations with the $y \to \infty$ upper bound on the last integration. Thus, the double integration performed here to obtain Eq. (3.17) is conceptually consistent with the triple integration performed by Fukagata *et al.* (2002) for channel flows. Both are targeted to isolate the laminar contribution from contributions due to transitional and turbulent flow, the laminar contribution being obtained by successive integration of the viscous term. The number of integrations is simply determined by the engineering needs in these scenarios.

The difference in the number of integrations necessary to obtain the desired engineering relations does have a concrete consequence on the spatial weighting of the Reynolds stress contribution term. In the case of internal flows, triple integration leads to weighting the Reynolds stress by $1-y/h$, where h is the channel half height. This weighting emphasizes the near-wall Reynolds stress over that of the region near the centerline compared with the boundary layer case here, where no weighting is observed. This absence of weighting highlights the fact that the direct contribution of the Reynolds stress at various wall-normal locations depends on whether the flow rate or the freestream velocity is used in defining the friction coefficient.

3.5. *Alternative derivation and interpretation*

The double integration procedure for deriving Eq. (3.17) above was chosen to highlight the similarity of the present approach to boundary layers with the triple integration approach of Fukagata *et al.* (2002) for internal flows. A simpler derivation with more intuitive interpretation is as follows. The integral of the viscous force across the boundary layer is the wall shear stress,

$$F_\nu = \int_0^\infty \nu \frac{\partial^2 u}{\partial y^2} dy = -\frac{\tau_w}{\rho}. \tag{3.23}$$

On the other hand, if one considers angular momentum, the integral of the viscous torque (about $y = 0$) across the boundary layer is the freestream velocity,

$$T_{\nu,0} = \int_0^\infty \nu y \frac{\partial^2 u}{\partial y^2} dy = -\nu U_\infty. \tag{3.24}$$

The friction coefficient, then, may be interpreted as the following ratio,

$$\frac{C_f}{2} Re_\ell = \frac{\ell \tau_w}{\mu U_\infty} = \frac{\ell F_\nu}{T_{\nu,0}} = \frac{\ell \int_0^\infty \nu \frac{\partial^2 u}{\partial y^2} dy}{\int_0^\infty \nu y \frac{\partial^2 u}{\partial y^2} dy} = \frac{\text{viscous force acting at } y = \ell}{\text{viscous torque about the wall}}. \tag{3.25}$$

By prudent choice of ℓ to be the moment of the viscous force, this ratio may be set to unity in the baseline laminar, zero pressure gradient boundary layer. The details will be discussed later in this report. Alternatively, the choice to set the ratio in Eq. (3.25) to unity may also be phrased as

$$\frac{U_\infty}{\ell} = \left.\frac{\partial u}{\partial y}\right|_0, \tag{3.26}$$

which means that ℓ may be interpreted as a gradient length or slip length for the reference laminar, zero pressure gradient boundary layer.

The competition between numerator and denominator of Eq. (3.25), i.e., skin friction versus freestream velocity, can be directly laid out by considering the torque about $y = \ell$,

$$T_{\nu,\ell} = \int_0^\infty \nu(y-\ell) \frac{\partial^2 u}{\partial y^2} dy = \frac{\ell \tau_w}{\rho} - \nu U_\infty = \nu U_\infty \left(\frac{C_f}{2} Re_\ell - 1 \right). \tag{3.27}$$

It is straightforward to show that multiplying Eq. (3.7) by $y - \ell$ and integrating from 0 to ∞ in y leads to Eq. (3.17). The new thicknesses introduced in Eq. (3.20) and Eq. (3.21) can then be interpreted as

$$\delta_\ell^* \equiv \text{displacement thickness of angular momentum about } y = \ell, \tag{3.28}$$

$$\theta_\ell \equiv \text{angular momentum thickness about } y = \ell, \tag{3.29}$$

and

$$\theta - \theta_\ell \equiv \text{angular momentum thickness about } y = 0. \tag{3.30}$$

The relation in Eq. (3.17) can be seen as an integral angular momentum equation. This interpretation requires some appreciation of nuance, since the angular momentum must be taken about the same x location at the profile, so this is not a straightforward application of the conservation of angular momentum. Rather, this demonstration was to provide some intuition as to the meaning of the terms involved. Prior to the widespread use of RANS codes for computing turbulent boundary layers, several methods were developed for marching a 'moment of momentum' integral equation, essentially the same as what is considered here, along with the standard momentum integral equation (Kline *et al.* 1968).

The terms in Eq. (3.17) may be interpreted as

$$\frac{1}{Re_\ell} \equiv \text{baseline contribution (e.g., laminar zero pressure gradient)}, \tag{3.31}$$

$$\int_0^\infty \frac{-\overline{u'v'}}{U_\infty^2 \ell}\, dy \equiv \text{torque of turbulent fluctuations (Reynolds stress)}, \tag{3.32}$$

$$\frac{\partial \theta_\ell}{\partial x} + \frac{\theta_\ell - \theta}{\ell}\frac{d\ell}{dx} \equiv \text{streamwise development about } \ell \text{ of momentum deficit}, \tag{3.33}$$

$$\frac{\theta_v}{\ell} \equiv \text{torque due to mean vertical flux of momentum deficit}, \tag{3.34}$$

$$\frac{\delta_\ell^* + 2\theta_\ell}{U_\infty}\frac{\partial U_\infty}{\partial x} \equiv \text{torque of favorable or adverse pressure gradient}, \tag{3.35}$$

and

$$\mathcal{I}_{x,\ell} \equiv \text{effect of departure from the boundary layer approximation.} \tag{3.36}$$

3.6. *Conditions for convergence*

Before discussing how to choose the length scale ℓ, it is of interest to consider whether or not the integrals in Eqs. (3.20) and (3.21) converge, because the upper limit of integration being ∞ requires that the integrand vanish sufficiently quickly for large y values. In particular, seeing that $\bar{u}/U_\infty$ converges to a constant (1) at large y, convergence of δ_ℓ^* and θ_ℓ requires that

$$\lim_{y\to\infty} y^2 \left(1 - \frac{\bar{u}}{U_\infty}\right) = 0. \tag{3.37}$$

Note that this is simply an extension to the requirements for the standard thicknesses, Eqs. (3.11) and (3.12), to converge, namely

$$\lim_{y\to\infty} y \left(1 - \frac{\bar{u}}{U_\infty}\right) = 0. \tag{3.38}$$

In both cases, the presumed exponential decay of the velocity deficit (as is the case for the Blasius equation) at the edge of the boundary layer is more than sufficient for these integrals to be finite.

Besides these integrals, other terms in Eq. (3.14) must vanish to form Eq. (3.15), in particular

$$\lim_{y\to 0}\ y\ [U_\infty - \bar{u}(y)]\bar{v}(y) = 0, \tag{3.39}$$

$$\lim_{y\to 0}\ y\ \nu \frac{\partial \bar{u}}{\partial y} = 0, \tag{3.40}$$

and

$$\lim_{y\to 0}\ y\ \overline{u'v'} = 0. \tag{3.41}$$

As with the integrands above, these conditions are not particularly worrisome in the context of boundary layer analysis.

3.7. *The $\ell \to \infty$ limit*

Next the choice of ℓ is considered. To begin, the behavior of Eq. (3.15) in the two limiting cases is briefly considered, helping to frame the need to choose a finite ℓ.

In the limit $\ell \to \infty$, it may be seen that the new integral lengths δ_ℓ^* and θ_ℓ, as defined in Eqs. (3.20) and (3.21), converge to their classical counterparts, i.e.,

$$\lim_{\ell\to\infty} \delta_\ell^* = \delta^* \qquad \text{and} \qquad \lim_{\ell\to\infty} \theta_\ell = \theta. \tag{3.42}$$

It may be straightforwardly seen, therefore, that taking the $\ell \to \infty$ limit of Eq. (3.17) leads back to the classical momentum integral equation

$$\frac{C_f}{2} = \frac{\partial \theta}{\partial x} + \frac{\delta^* + 2\theta}{U_\infty}\frac{\partial U_\infty}{\partial x} + \int_0^\infty \frac{I_x}{U_\infty^2}\, dy. \tag{3.43}$$

Therefore, in this limit, nothing new is obtained by double integration compared with classical results. In the zero-pressure gradient boundary layer, the skin friction is due to only one term, i.e., the streamwise growth in the momentum thickness.

3.8. *The $\ell \to 0$ limit*

In the opposite limit, $\ell \to 0$, δ_ℓ^* and θ_ℓ diverge to $-\infty$. This behavior can be characterized by the following limits,

$$\lim_{\ell \to 0} \ell \delta_\ell^* = -\int_0^\infty y\, \left[1 - \frac{\bar{u}(y)}{U_\infty}\right]\, dy \equiv -\Delta^2 \tag{3.44}$$

and

$$\lim_{\ell \to 0} \ell \theta_\ell = -\int_0^\infty y\, \left[1 - \frac{\bar{u}(y)}{U_\infty}\right]\, \frac{\bar{u}(y)}{U_\infty}\, dy \equiv -\Theta^2. \tag{3.45}$$

Multiplying Eq. (3.17) by ℓ, then taking the $\ell \to \infty$ limit leads to,

$$0 = \frac{\nu}{U_\infty} + \int_0^\infty \frac{-\overline{u'v'}}{U_\infty^2}\, dy - \frac{\partial \Theta^2}{\partial x} + \theta_v - \frac{\Delta^2 + 2\Theta^2}{U_\infty}\frac{\partial U_\infty}{\partial x} - \int_0^\infty \frac{y I_x}{U_\infty^2}\, dy. \tag{3.46}$$

The skin friction does not appear in the integral balance in this limit because this is essentially an angular momentum equation about $y = 0$, to which the wall shear stress does not contribute.

3.9. *Choosing the length scale*

In view of these two limits, a finite choice of ℓ is necessary in order to obtain a boundary layer relationship with properties similar to that of the channel flow formulation of Fukagata *et al.* (2002). In terms of the angular momentum interpretation, the reference origin must be set at some point between the wall and the freestream to observe the competition between the numerator and denominator in Eq. (3.25). In particular, the relationship sought should isolate a laminar (or other baseline) contribution to the skin friction, and thus facilitate analysis of the turbulence-induced enhancement. This is easily facilitated by Eq. (3.17) since ℓ can be chosen such that the laminar, zero pressure gradient boundary layer skin friction is $C_{f,lam} = 2Re_\ell^{-1}$. The reality of making this choice is more nuanced than perhaps evident in this paragraph, and as such, a full discussion is presently delayed until Section 3.10. In this next section, the use of Eq. (3.17) is demonstrated in laminar flows for comparing non-zero pressure gradients boundary layers to the zero pressure gradient case.

One possible motivation for choosing a different ℓ is to facilitate a comparison of a transitional boundary layer to a fully turbulent one. For instance, one might want to investigate the causes for skin friction in a transitional boundary layer exceeding that of a fully turbulent correlation. In that case, choosing ℓ such that the fully turbulent friction coefficient is the sum of the first two terms in Eq. (3.17) may be more insightful. The details for this choice are discussed in Section 3.11.

3.10. *Using the zero pressure gradient laminar boundary layer to choose ℓ*

Laminar boundary layers are described by solutions to the similarity equation (Blasius 1908; Falkner & Skan 1930),

$$\phi'''(\eta) + \frac{1}{2}\phi(\eta)\phi''(\eta) = m\left(\phi'^2(\eta) - \frac{1}{2}\phi(\eta)\phi''(\eta) - 1\right), \tag{3.47}$$

where

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}, \quad \phi(\eta) = \frac{\psi(x,y)}{\sqrt{\nu x U_\infty(x)}}, \quad \phi'(\eta) = \frac{u(x,y)}{U_\infty(x)} \quad \text{and} \quad m = \frac{x}{U_\infty(x)}\frac{dU_\infty}{dx}. \tag{3.48}$$

Here, $\psi(x,y)$ is the stream function, and the streamwise variation of $U_\infty \sim x^m$ defines the pressure gradient via Eq. (3.3). For $m > 0$, the freestream is accelerating in the streamwise direction, which imposes a favorable pressure gradient. If $m < 0$, the freestream is decelerating and requires an adverse pressure gradient. Although no analytical solution exists, a numerical solutions may be easily obtained using a standard ordinary differential equation (ODE) solver.

The friction coefficient and boundary layer thicknesses can be written as

$$\frac{C_f}{2} = \frac{\phi''(0)}{\sqrt{Re_x}}, \quad \theta = \alpha\sqrt{\frac{\nu x}{U_\infty}}, \delta^* = \beta\sqrt{\frac{\nu x}{U_\infty}}, \quad \text{and} \quad \delta_{99} = \gamma\sqrt{\frac{\nu x}{U_\infty}}, \tag{3.49}$$

where

$$\alpha \equiv \int_0^\infty [1 - \phi'(\eta)]\,\phi'(\eta)d\eta, \quad \text{and} \quad \beta \equiv \int_0^\infty [1 - \phi'(\eta)]\,d\eta = \lim_{h\to\infty}[h - \phi(h)], \tag{3.50}$$

and $\phi'(\gamma) = 0.99$. For zero pressure gradient ($m = 0$), the numerical solution yields $\phi_0''(0) \approx 0.332$, $\alpha_0 \approx 0.664$, $\beta_0 \approx 1.72$ and $\gamma_0 \approx 4.91$. In general, for favorable or adverse pressure gradient boundary layers, the coefficients $\phi''(0)$, α, β and γ depend on m.

As discussed in Section 3.9, a desirable choice of ℓ is to make $C_f = 2Re_\ell^{-1}$ for the zero pressure gradient (ZPG) laminar boundary layer, which facilitates comparison to other flows. For the similarity solution, this choice can be written as

$$\frac{1}{Re_\ell} = \frac{\phi_0''(0)}{\sqrt{Re_x}} = \frac{\alpha_0\phi_0''(0)}{Re_\theta} = \frac{\beta_0\phi_0''(0)}{Re_{\delta^*}} = \frac{\gamma_0\phi_0''(0)}{Re_{\delta_{99}}}. \tag{3.51}$$

This implies multiple options for ℓ, namely,

$$\ell = \frac{1}{\phi_0''(0)}\sqrt{\frac{\nu x}{U_\infty}} = \frac{\theta}{\alpha_0\phi_0''(0)} = \frac{\delta^*}{\beta_0\phi_0''(0)} = \frac{\delta_{99}}{\gamma_0\phi_0''(0)}, \tag{3.52}$$

or

$$\ell \approx 3.01\sqrt{\frac{\nu x}{U_\infty}} \approx 4.54\theta \approx 1.75\delta^* \approx 0.613\delta_{99}.$$

Now, for the ZPG laminar boundary layer, all of these definitions of ℓ coincide. However, the point of using Eq. (3.17) is to analyze something other than a ZPG laminar boundary layer, where Eq. (3.52) implies distinct options. Therefore, when comparing a given flow to the ZPG boundary layer, one must choose which Reynolds number is matched between the two flows. This choice is inherent in the exercise of comparing two boundary layers, because matching Re_x or Re_θ, for example, leads to two different comparisons. For instance, if one compares a turbulent boundary layer with a laminar one, the comparison is only fully specified when the particular Re to be matched between the two flows is chosen. The fact that one must choose ℓ in Eq. (3.17) only reflects this fact.

3.11. *Using a fully turbulent boundary layer to choose ℓ*

The previous section demonstrated how Eq. (3.17) may be utilized to give a quantitative comparison of how various effects change the friction coefficient in zero-pressure gradient laminar boundary layers when the imposed pressure gradient is changed. In future work, it is hoped that this approach could be leveraged to yield insight into transitional and turbulent boundary layers. For zero-pressure gradient boundary layers, the impact of turbulence may be considered by removing the pressure gradient term from Eq. (3.17) to form

$$\frac{C_f}{2} = \frac{1}{Re_\ell} + \int_0^\infty \frac{-\overline{u'v'}}{U_\infty^2 \ell}\, dy + \frac{\partial \theta_\ell}{\partial x} + \frac{\theta_\ell - \theta}{\ell}\frac{d\ell}{dx} + \frac{\theta_v}{\ell}. \tag{3.53}$$

The first term on the right-hand side represents the friction coefficient of the equivalent laminar boundary layer. Following this term, the contributions of the Reynolds stress, streamwise development, and vertical flux are enumerated. This interpretation would be facilitated by choosing ℓ in accordance with the laminar friction coefficient, depending on which Reynolds number is to be kept fixed for the comparison, as done in the previous section.

An alternative approach could treat the fully turbulent boundary layer as the baseline for choosing ℓ. In this case, a turbulent correlation could be used to make the first two terms sum to equal the friction coefficient of the turbulent boundary layer at a given Reynolds number. Again, the particulars of this choice will vary depending on which particular Reynolds number is chosen. This approach could facilitative the comparison of wall shear stresses in the transitional regime to that of fully turbulent boundary layers. In some transition scenarios, the deviation from the boundary layer approximation may be significant and the term $\mathcal{I}_{x,\ell}$ may also be of interest.

4. Conclusions

In this report, a relationship is derived relating the friction coefficient of laminar and turbulent boundary layer flows to the composition of turbulent stresses, spatial growth terms, and non-zero pressure gradient effects. This decomposition of skin friction for external flows forms a unified framework in combination with the triple-decomposition approach of Fukagata *et al.* (2002) for internal flows. For boundary layers, only double integration is needed, since the friction factor of engineering interest relates the wall shear stress to the freestream velocity, not the flow rate (integral of velocity). This insight yields a considerable simplification over the boundary layer formulation of Fukagata *et al.* (2002), as well as later attempts such as that by Renard & Deck (2016). Additionally, the double integration approach introduced here for boundary layers allows a more intuitive interpretation in terms of angular momentum conservation, providing a new perspective on the meaning of the friction coefficient in boundary layer flows.

The impact of the Reynolds stress on the skin friction is weighted by $1-y/h$ for internal flows (Fukagata *et al.* 2002), and Renard & Deck (2016) argued for weighting the Reynolds stress with the mean velocity gradient for boundary layer flows. However, the results here show that the Reynolds stress should be unweighted when considering the direct effect on boundary layer skin friction. This illustrates an important difference in how the Reynolds stress impacts internal and external flows related to the reference velocity relevant to engineering analysis (bulk velocity versus freestream velocity, respectively).

Furthermore, the present approach naturally reflects the fact that a comparison of two boundary layers is contingent upon the specification of which particular Reynolds number (e.g., Re_x, Re_θ, etc.) is to be matched between the two. This specification manifests itself in the required choice of a length scale here denoted ℓ. While the length scale ℓ may seem somewhat arbitrary at first, viewing it from the perspective of choosing a Reynolds number to keep constant under comparison shows that the choice of ℓ actually enhances the interpretability of the proposed analysis. In the angular momentum view introduced here, this length scale sets the origin about which the relevant torques are to be computed. Also, an arbitrary freestream pressure gradient is naturally handled in this approach by utilizing the standard manipulations for the momentum integral equation.

Further work is currently underway to demonstrate this analysis approach for laminar boundary layers with various favorable and adverse pressure gradients, with particular attention to the imbued meaning of choosing a particular length scale for ℓ. In the near future, this approach will be leveraged to analyze turbulent and transitional boundary layers. In particular, insight will be sought regarding the behavior of the average wall shear stress in the later stages of transition, where peculiar effects and overshoots may be observed (Sayadi *et al.* 2013). Furthermore, the analysis introduced here may be extended to heat transfer and compressible boundary layers, for example, to seek understanding of transition-induced overshoots and peculiar behaviors in wall heat flux (Franko & Lele 2013).

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