On the assessment of symmetries in large-eddy simulation subgrid-scale models

By J. Guo AND O. Shende

1. Motivation and objectives

For any mathematical expression that constitutes a certain relationship among variables, a symmetry is present if a group of transformation operators can be applied to the original expression and the resultant expression reads the same in the new variables as in the original (Cantwell 2002); thus, this defines a valid invariance of the expression under the transformation group. The incompressible Navier-Stokes equations govern the evolution of solenoidal velocity fields. For the Navier-Stokes equations, finding inherent symmetries can lead to the formulation of closed-form solutions and guide intuition toward the physical laws inherent in subregimes of complex, turbulent flows. Such solutions are valued because solving the fully resolved Navier-Stokes equations numerically is insurmountably expensive at the high Reynolds numbers relevant to engineering applications.

A desire to mitigate this computational cost has led to the creation of paradigms such as large-eddy simulation (LES), which seeks to resolve only the largest energy-containing scales of turbulent motion and to model the effects of the smallest scales that occur below the scale of the computational grid or applied filter using subgrid-scale (SGS) models. These models are meant to satisfy a modified, filtered version of the Navier-Stokes equations; thus, the models must be formulated and used carefully to respect the same symmetries as the underlying equations. However, many of even the most commonly used SGS models do not respect all symmetries of the governing equations.

Oberlack (1997), in particular, describes the known symmetries of the incompressible Navier-Stokes equations based on physical principles and then extends that analysis to whether certain proposed SGS models, and the filters needed to perform LES, violate or satisfy said symmetries. Our goals in this brief are as follows:

(*a*) Review known symmetries of the incompressible Navier-Stokes equations and the physical interpretation of each symmetry, describe the implications of each symmetry for solutions, and examine how those symmetries apply to other forms of the Navier-Stokes equations.

(*b*) Review the analysis of SGS models provided by Oberlack (1997), focusing on using known mathematical methods to assess whether they satisfy or violate the symmetry conditions of the Navier-Stokes equations.

(*c*) Apply such an analysis to more recent (since 1997) SGS models and ascertain whether they satisfy or violate the correct symmetry conditions.

2. Governing equations and associated symmetries

First, let us examine the symmetries of the Navier-Stokes equations that SGS models must satisfy with regard to these invariant conditions. In the Navier-Stokes equations, the momentum equation is

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial u_k \partial u_k}, \tag{2.1}$$

and the continuity equation is

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.2}$$

where $u_i = \{u, v, w\}$ is a velocity component in 3-dimensional space, $x_i = \{x, y, z\}$ is a coordinate in a Cartesian reference frame, t is a temporal coordinate, p is pressure, ρ is the constant fluid density, and ν is the constant fluid kinematic viscosity.

2.1. *Symmetries, via Lie group operators*

The continuous symmetries of the incompressible Navier-Stokes equations, following the notation of Cantwell (2002) and the accompanying software package for finding the symmetries of the differential equations, are as follows:

(*a*) Invariance under translation in time

$$X^1 = \frac{\partial}{\partial t}.$$

(*b*) Invariance with arbitrary oscillations in pressure magnitude

$$X^2 = g(t)\frac{\partial}{\partial p}.$$

(*c*) Invariance to rotation about the z axis

$$X^3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}.$$

(*d*) Invariance to rotation about the x axis

$$X^4 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}.$$

(*e*) Invariance to rotation about the y axis

$$X^5 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w}.$$

(*f*) Invariance to non-uniform translation in the x direction (this includes Galilean invariance)

$$X^6 = D(t)\frac{\partial}{\partial x} + \frac{\partial D}{\partial t}\frac{\partial}{\partial u} - x\frac{\partial^2 D}{\partial t^2}\frac{\partial}{\partial p}.$$

(*g*) Invariance to non-uniform translation in the y direction (this includes Galilean invariance)

$$X^7 = E(t)\frac{\partial}{\partial y} + \frac{\partial E}{\partial t}\frac{\partial}{\partial v} - y\frac{\partial^2 E}{\partial t^2}\frac{\partial}{\partial p}$$

(*h*) Invariance to non-uniform translation in the z direction (this includes Galilean invariance)

$$X^8 = F(t)\frac{\partial}{\partial z} + \frac{\partial F}{\partial t}\frac{\partial}{\partial w} - z\frac{\partial^2 F}{\partial t^2}\frac{\partial}{\partial p}$$

(*i*) Invariance to dilation

$$X^9 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2p\frac{\partial}{\partial p}.$$

In these groups, $D(t), E(t)$, and $F(t)$ are arbitrary, twice-differentiable functions of time.

We can, following Oberlack (1997), group these nine individual symmetries into two finite forms that cover dilation and rotation, which are the most critical symmetries of interest. Since most SGS models are defined almost solely in terms of gradients of the velocity field, finite transformations for time and pressure are not studied here, but it is important to note that an SGS model that uses these quantities must satisfy said symmetries.

The simplified finite rotation group for the velocity and position, combining X^3, X^4, and X^5, for the incompressible Navier-Stokes equation is given by

$$\tilde{x}_i = B_{ij}x_j, \quad \tilde{u}_i = B_{ij}u_j + \dot{B}_{ij}x_j, \tag{2.3}$$

where B can be a time-invariant or constant-rate rotation tensor. Here, the tilde overbar $\tilde{\phi}$ denotes a transformed variable, and the overdot $\dot{\phi}$ denotes the time derivative $\partial\phi/\partial t$ for some variable ϕ. In particular, the expression signifies that B_{ij} is orthogonal and $\dot{B}_{ik}B_{jk} = \epsilon_{3ij}M_3$, with M_3 being a constant rotation rate and ε_{ijk} being the Levi-Civita cross-product tensor. We also know that the finite dilation groups for the incompressible Navier-Stokes equations for the position and velocity are given by

$$\tilde{x}_i = e^a x_i, \quad \tilde{u}_i = e^{-a}u_i, \tag{2.4}$$

where a is a real scalar-valued parameter. This expression corresponds to the scaling invariance property presented in Section 2.1(i).

2.2. *The filtered incompressible Navier-Stokes equations*

The filtered Navier-Stokes equations are solved in the LES context. For some variable ϕ, the overbar $\overline{\phi}$ denotes a filtered quantity. Following Oberlack (1997), the filtered Navier-Stokes equations are

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{2.5}$$

and

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu\frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}. \tag{2.6}$$

The purpose of this brief is to assess the invariance of models for the τ_{ij} term as a whole for the so-called SGS closure models, in the context of tools for symmetry analysis.

Following Eq. (55) in Oberlack (1997), the SGS models have the form

$$\tau_{ij} = \mathcal{F}_{ij}\left[\mathbf{u};\mathbf{x}\right], \tag{2.7}$$

with bolded terms denoting vector quantities. For all transformations allowed by the Navier-Stokes equations, invariance of the SGS model requires that

$$\tilde{\tau}_{ij} = \mathcal{F}_{ij}\left[\tilde{\mathbf{u}};\tilde{\mathbf{x}}\right]. \tag{2.8}$$

More specifically, usually only the deviatoric part of the stress tensor is modeled. This quantity should thus be invariant under applied transformations, such that

$$X\Omega = 0,\ \text{with}\ \Omega_{ij} = \tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk}. \tag{2.9}$$

While the implications of the invariance properties of the Navier–Stokes equations on the types of filter kernel used is a key point of Oberlack (1997), in this brief, we do not focus on the filter operator and its effect but rather on the invariance of the SGS models under the finite-form invariants outlined in Section 2.1. Following Oberlack (1997), the finite invariants given at the end of Section 2.1 for the total velocity field and spatial coordinates apply to the mean filtered fields as well. These would be equivalent to invariance via the Lie group operators. In other words,

$$\tilde{\tau}_{ij} = \mathcal{F}_{ij}\left[\tilde{\mathbf{u}};\tilde{\mathbf{x}}\right] \quad \underset{\longleftrightarrow}{\text{corresponds to}} \quad X\Omega = 0. \tag{2.10}$$

In particular, we note that $\tau_{ij} = \overline{u_i' u_j'}$, so it follows that

$$\tilde{\tau}_{ij} = e^{-2a}\tau_{ij} \tag{2.11}$$

for a dilation group and

$$\tilde{\tau}_{ij} = B_{im}B_{jn}\tau_{mn} \tag{2.12}$$

for a rotation group. An SGS model that respects invariance of the governing equations must show that

$$\tilde{\Omega}_{ij} = e^{-2a}\Omega_{ij} \tag{2.13}$$

for a dilation group and

$$\tilde{\Omega}_{ij} = B_{im}B_{jn}\Omega_{mn} \tag{2.14}$$

for a rotation group. This may seem overly restrictive, as it is only the gradient of the Reynolds stress tensor that plays a role in the LES equations, but the stresses themselves are important characterization tools for measures like flow anisotropy.

Note, however, that there are two unique rotational invariances being studied; there is an invariance to a single rotation of the coordinate axes, which is when B_{ij} is a constant, and invariance to being solved in a rotating, non-inertial frame, which comes from B_{ij} being a function of time. The second invariance is also known as material frame invariance or material indifference.

3. Subgrid-scale models

The SGS models discussed by Oberlack (1997) are detailed below.

3.1. *Smagorinsky (1963) model*

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -C_s\Delta^2|\overline{\mathbf{S}}|\overline{S}_{ij},\ \overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial\overline{u}_i}{\partial x_j} + \frac{\partial\overline{u}_j}{\partial x_i}\right), \tag{3.1}$$

where C_s is referred to as the Smagorinsky constant and S_{ij} is the strain-rate tensor.

3.2. *Métais & Lesieur (1992) structure-function model*

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = C^{SF}\Delta\langle(\overline{\mathbf{u}}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}))^2\rangle^{1/2}\overline{S}_{ij}, \tag{3.2}$$

where C^{SF} is a model constant and $\langle\cdot\rangle$ is a spatial average.

3.3. *Germano* et al. *(1991) dynamic Smagorinsky model*

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = \frac{\left(\widetilde{\overline{u}_m\overline{u}_n} - \tilde{\overline{u}}_m\tilde{\overline{u}}_n\right)\overline{S}_{mn}}{\left(\frac{\tilde{\Delta}}{\Delta}\right)^2 |\tilde{\overline{\mathbf{S}}}|\tilde{\overline{S}}_{mn}\tilde{\overline{S}}_{mn} - \widetilde{|\overline{\mathbf{S}}|\overline{S}_{pq}}\overline{S}_{pq}} |\overline{\mathbf{S}}|\overline{S}_{ij}. \tag{3.3}$$

Here, the tilded quantities refer to the test filter

$$\tilde{h}(\mathbf{x}) = \int_V \tilde{G}(\mathbf{x}, \mathbf{y}) h(\mathbf{y}) d^3y, \tag{3.4}$$

where the filter length is $\tilde{\Delta}$ and $\tilde{\Delta} > \Delta$.

In addition, we extend the analysis by Oberlack (1997) to the following SGS models, which have been proposed in the years since.

3.4. *Nicoud & Ducros (1999) wall-adapting local eddy-viscosity model*

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2\nu_T\overline{S}_{ij}, \tag{3.5}$$

with

$$\nu_T = (C_w\Delta)^2 \frac{\left(S^d_{ij}S^d_{ij}\right)^{3/2}}{\left(\overline{S}_{ij}\overline{S}_{ij}\right)^{5/2} + \left(S^d_{ij}S^d_{ij}\right)^{5/4}}. \tag{3.6}$$

$C_w\Delta$ is a numerical constant within each calculation. Here,

$$S^d_{ij} = \left(\overline{g}^2_{ij} + \overline{g}^2_{ji}\right)/2 - \delta_{ij}\overline{g}^2_{kk}/3, \tag{3.7}$$

with

$$\overline{g}_{ij} = \frac{\partial \overline{u}_i}{\partial x_j},\ \overline{g}^2_{ij} = \overline{g}_{ik}\overline{g}_{kj}. \tag{3.8}$$

3.5. *Vreman (2004) model*

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2\nu_e\overline{S}_{ij} \tag{3.9}$$

where

$$\nu_e = c\sqrt{\frac{B_\beta}{\alpha_{ij}\alpha_{ij}}}, \tag{3.10}$$

with

$$\alpha_{ij} = \frac{\partial \overline{u}_j}{\partial x_i}, \beta_{ij} = \Delta^2_m\alpha_{mi}\alpha_{mj}, \tag{3.11}$$

$$B_\beta = \beta_{11}\beta_{22} - \beta^2_{12} + \beta_{11}\beta_{33} - \beta^2_{13} + \beta_{22}\beta_{33} - \beta^2_{23}, \tag{3.12}$$

and where $c \approx 2.5C^2_s$, a constant.

3.6. *Rozema* et al. *(2015) anisotropic minimum-dissipation model*

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2\nu_e\overline{S}_{ij}, \tag{3.13}$$

with

$$\nu_e = \frac{-\left(\hat{\partial}_k\overline{u}_i\right)\left(\hat{\partial}_k\overline{u}_j\right)\overline{S}_{ij}}{\left(\partial_l\overline{u}_m\right)\left(\partial_l\overline{u}_m\right)}. \tag{3.14}$$

Here, $\hat{\partial}_i = C_i \delta_i \partial_i$ is the scaled gradient, where δ_i are the dimensions (edge lengths) of the rectangular box filter used and C_i is the modified Poincaré constant.

4. Symmetries of subgrid-scale models, with mathematical justification

For convenience, we drop the overbarred and filter-tilded notation and assume all quantities represent their filtered or mean counterparts as shown in Section 3. Oberlack (1997) showed that the invariance conditions on filtered quantities follow from the invariance of the total fields. Thus, tilded quantities are used only for transformation notation. If only gradients of the velocity vector are used, then simple Galilean invariance is automatically satisfied. We assume an isotropic filter with constant filter width, Δ.

4.1. *Smagorinsky model*

Assuming a constant, isotropic filter, let us examine the finite group of Eq. (2.4), given by

$$\tilde{\Omega} = -C_s \Delta^2 \sqrt{\tilde{S}_{ij} \tilde{S}_{ij}} \tilde{S}_{ik} = -C_s \Delta^2 \sqrt{S_{ij} S_{ij}} S_{ik} e^{-4a} = e^{-4a} \Omega.$$

This is not the invariance criterion required for dilation, Eq. (2.13); this means

$$X^9 \Omega \neq 0.$$

Note that we assume here that the filtering is explicit. That is, a length scale unrelated to the discretization is introduced into the problem. If the filtering is implicit, that is, if the filter length scale also scales with the dilation, then the model will scale correctly and demonstrate invariance.

However, if we examine the finite rotational group, and we note that

$$\tilde{S}_{ij} = \frac{B_{il} B_{jm}}{2} \frac{\partial u_l}{\partial x_m} + \epsilon_{3ij} M_3/2 + \frac{B_{im} B_{jl}}{2} \frac{\partial u_m}{\partial x_l} - \epsilon_{3ij} M_3/2 = B_{il} B_{jm} S_{lm}, \tag{4.1}$$

then we see that the Smagorinsky model is rotationally invariant by examining the form of the model. In $\tilde{\Omega}$ in the Smagorinsky formulation, Eq. (3.1), the magnitude of the strain rate and the values of the constants are unaffected by change in rotation. Thus, the only term that is modified by rotation is the strain-rate tensor, which satisfies the both requirements for rotational invariance, as detailed by Eq. (2.14).

4.2. *Structure-function model*

If we again follow the simple dilation group,

$$\begin{aligned}\tilde{\Omega} &= C^{SF} \Delta \langle (\tilde{\mathbf{u}}(\tilde{\mathbf{x}} + \mathbf{r}) - \tilde{\mathbf{u}}(\tilde{\mathbf{x}}))^2 \rangle^{1/2} \tilde{S}_{ij} \\ &= C^{SF} \Delta \langle (\mathbf{u}(e^a \mathbf{x} + \mathbf{r}) - \mathbf{u}(e^a \mathbf{x}))^2 \rangle^{1/2} e^{-3a} S_{ij} = e^{-3a} \Omega,\end{aligned}$$

we see that the dilational invariance condition, Eq. (2.13), does not hold.

If we instead look at the rotational group, we can follow Oberlack (1997) and note that the spatial average will remove any dependence on frame orientation but will retain the inertial rotation term inside the spatial average as

$$\begin{aligned}\tilde{\Omega} &= C^{SF} \Delta B_{il} B_{jm} \langle (\mathbf{u}(B_{ij}\mathbf{x} + \mathbf{r}) - \mathbf{u}(B_{ij}\mathbf{x}) - \epsilon_{3lm} M_3 (x_l + r_l) + \epsilon_{3lm} M_3 (x_l))^2 \rangle^{1/2} S_{lm} \\ &= C^{SF} \Delta B_{il} B_{jm} \langle (\mathbf{u}(B_{ij}\mathbf{x} + \mathbf{r}) - \mathbf{u}(B_{ij}\mathbf{x}) - \epsilon_{3lm} M_3 r_l)^2 \rangle^{1/2} S_{lm}.\end{aligned}$$

This is not invariant for a rotating frame, but is invariant under a single imposed coordinate rotation.

4.3. *Dynamic Smagorinsky model*

For the dynamic Smagorinsky model, using the dilation group, we note that the numerator and denominator of the model have the same corresponding terms of the strain-rate tensor, so the only contribution to dilation will come from the terms corresponding to the product of the velocity vectors. This means that

$$\tilde{\Omega} = e^{-2a}\Omega,$$

which is the correct condition for scale invariance.

If we examine the rotational group, we again note that the strain-rate tensor transforms as required for invariance under rotation. Again, we conclude that the only terms that contribute in net to rotation of the dynamic model are the velocity vector terms, leading to

$$\tilde{\Omega} = B_{im}B_{jn}\Omega,$$

which is the correct invariance condition for invariance in a rotating frame and for a single imposed coordinate rotation.

4.4. *Wall-adapting local eddy-viscosity model*

The dilational symmetries of the strain-rate tensor have previously been shown to scale as $\tilde{S}_{ij} = e^{-2a}S_{ij}$. The wall-adapting local eddy-viscosity model additionally involves S^d_{ij} terms, which each had scaling as $\widetilde{S^d_{ij}} = e^{-4a}S^d_{ij}$. Thus, we can define

$$\tilde{\nu}_e = \frac{\left(e^{-4a}e^{-4a}\right)^{3/2}}{\left(e^{-2a}e^{-2a}\right)^{5/2}}\nu_e = e^{-2a}\nu_e. \tag{4.2}$$

Plugging this into the expression for Ω yields

$$\tilde{\Omega} = -\widetilde{2\nu_e S_{ij}} = e^{-4a}\Omega, \tag{4.3}$$

which is an incorrect scaling, showing a lack of dilational invariance.

Examining the rotational group, we note that Ω is composed of S^d_{ij}, S_{ij}, and constant terms. S^d_{ij} is composed of the sum of two products of the velocity gradient $g_{ij} = \partial u_i/\partial x_k$, which is expressed as $g^2_{ij}+g^2_{ji}$, with $g^2_{ij} = g_{ik}g_{kj}$. After imposing the transformed variables, the product of velocity gradients becomes

$$\widetilde{g^2_{ij}} = B_{il}B_{km}\frac{\partial u_l}{\partial x_m}B_{kn}B_{jp}\frac{\partial u_n}{\partial x_p} + B_{il}B_{km}\frac{\partial u_l}{\partial x_m}\varepsilon_{3kj}M_3 + \\ B_{kn}B_{jp}\frac{\partial u_n}{\partial x_p}\varepsilon_{3kj}M_3 + \varepsilon_{3ik}\varepsilon_{3kj}M_3^2,$$

summed with the corresponding expression for $\widetilde{g^2_{ji}}$. When added together, the resultant expression yields terms that remain and that include the rotation rate M_3. As a result, after transformation, the desired expression

$$\widetilde{g^2_{ij} + g^2_{ji}} = B_{il}B_{jm}\left(g^2_{ij} + g^2_{ji}\right)$$

is not achieved; thus,

$$\widetilde{S^d_{ij}} \neq B_{il}B_{jm}S^d_{lm}. \tag{4.4}$$

As a result, unlike the subgrid stress tensor S_{ij}, this S_{ij}^d tensor is not invariant to an imposed rotation. Thus, we show that

$$\tilde{\Omega} \neq B_{il} B_{jm} \Omega, \tag{4.5}$$

demonstrating lack of invariance under an imposed rotation. However, invariance under a single imposed coordinate rotation still holds.

4.5. *Vreman model*

Analyzing the terms of the Vreman model, we see that they scale as

$$\alpha_{ij} = \frac{\partial u_j}{\partial x_i} \rightarrow \tilde{\alpha}_{ij} = e^{-2a} \alpha_{ij}, \quad \beta_{ij} = \Delta_m^2 \alpha_{mi} \alpha_{mj} \rightarrow \tilde{\beta}_{ij} = e^{-4a} \beta_{ij}. \tag{4.6}$$

We also know that B_β has terms involving the product of β and $\beta \rightarrow \tilde{B_\beta} = e^{-8a} B_\beta$. With this knowledge, we see that

$$\tilde{\nu_e} = e^{\frac{1}{2}(-8a-(-4a))} \nu_e = e^{-2a} \nu_e. \tag{4.7}$$

Plugging this into the expression for Ω yields

$$\tilde{\Omega} = -\widetilde{2\nu_e S_{ij}} = e^{-4a} \Omega, \tag{4.8}$$

which is an incorrect scaling, showing a lack of dilational invariance.

In dicussing the invariance of the Vreman SGS model under rotation, we note that Ω is composed of $B_\beta, \alpha_{ij}\alpha_{ij}$, and S_{ij} terms (and model constants). As in the wall-adapting local eddy-viscosity model, we now show that each component term is rotationally invariant. We have previously shown in other models that S_{ij} is invariant to rotation. Instead of writing out the tensorial math to show the invariance of the other components, it is perhaps more intuitive to point out that $\alpha_{ij}\alpha_{ij}$ is the first invariant (trace) of the tensor $\alpha^T \alpha$ and that B_β is the second invariant of the matrix β. This property was also noted by Vreman (2004) during the model's original construction. This shows that Ω is invariant to a rotation in coordinate axes.

Regarding the invariance under an imposed rotation, we note that B_β is the second principal invariant of the tensor β, which is invariant to an imposed rotation. If we analyze the $\alpha_{ij}\alpha_{ij}$ term, plugging in the transformed variables yields

$$\widetilde{\alpha_{ij}\alpha_{ij}} = \widetilde{\frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i}} = \left(B_{jl} B_{im} \frac{\partial u_l}{\partial x_m} + \varepsilon_{3ji} M_3 \right) \left(B_{jn} B_{ip} \frac{\partial u_n}{\partial x_p} + \varepsilon_{3ji} M_3 \right),$$

which, similar to in the wall-adapting local eddy-viscosity, will involve terms that do not cancel. The desired expression

$$\tilde{\Omega} = B_{il} B_{jm} \Omega \tag{4.9}$$

does not hold, demonstrating lack of invariance under an imposed rotation.

4.6. *Anisotropic minimum-dissipation model*

Each scaled velocity gradient scales as $\widetilde{\hat{\partial}_k u_i} = \Delta e^{-2a} \partial_k u_i$, as the scaling by the modified Poincaré constant does not affect the dilational scaling because it is a global constant. Thus

$$\tilde{\nu}_e = \Delta^2 e^{[-6a-(-4a)]} \nu_e = \Delta^2 e^{-2a} \nu_e. \tag{4.10}$$

Plugging this into the expression for Ω yields

$$\tilde{\Omega} = -\widetilde{2\nu_e S_{ij}} = \Delta^2 e^{-4a}\Omega, \tag{4.11}$$

which is an incorrect scaling if the filter width is imposed explicitly, showing a lack of dilational invariance. If implicit filtering is used, the filter width Δ will scale like e^a, so $\tilde{\Omega} = e^{-2a}\Omega$, preserving invariance under dilation.

For rotational invariance, we plug in the transformed variables into the expression for ν_e, with $M_3 = 0$, yielding

$$\tilde{\nu}_e = \frac{\left(B_{iq}B_{kr}\hat{\partial}_r u_q\right)\left(B_{jn}B_{kp}\hat{\partial}_p u_n\right)}{\left(B_{ms}B_{lt}\partial_t u_s\right)\left(B_{ms}B_{lt}\partial_t u_s\right)} B_{ia}B_{jb}S_{ij}. \tag{4.12}$$

This provides the correct condition for coordinate rotation invariance; Silvis *et al.* (2017) previously stated the invariance of this model under an imposed rotation.

5. Conclusions

Using some of the lessons learned in this work, we can extend this analysis to assess more complicated turbulence models in contexts similar to LES, such as the Reynolds-averaged Navier-Stokes (RANS) sense (which can be thought of as LES with infinite filter width). For example, a commonly used one-equation RANS model is the Spalart & Allmaras (1992) model. The original version of this model has a defined length scale that corresponds to distance from a wall; thus the model is not scale invariant, following the results for the original LES Smagorinsky model, which also has a defined filter length scale. The transport equation for the eddy viscosity in the Spallart-Almaras model involves a rotation-rate tensor, so the model is not invariant to frame rotation, following the analysis for the structure-function model above. This deficiency is corrected in later versions of the Spalart-Allmaras model; the utility of simple symmetry analysis discussed in this brief in examining complex models is demonstrated.

In this work, we have conducted an assessment of several LES SGS models by assessing the validity of their conformance to the underlying symmetries of the Navier-Stokes equations, assuming that the filtered equations being solved are meant to asymptotically converge to the true governing equations. In particular, following Oberlack (1997), we only examined algebraic models, as these are among the most widely used in industry for their relative ease of application.

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