

## Multifractal spectra in homogeneous shear flow

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Employing numerical simulations of three-dimensional homogeneous shear flow we have calculated the associated multifractal spectra of the energy dissipation, scalar dissipation and vorticity fields. Our results for  $(128)^3$  simulations of this flow, and those obtained in recent experiments that analyzed 1- and 2-dimensional intersections of atmospheric and laboratory flows, are in some agreement. A two-scale Cantor set model of the energy cascade process which describes the experimental results from one-dimensional intersections quite well, describes the three-dimensional results only marginally.

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### 1. Introduction

Despite the usefulness of traditional statistical methods (*pdf*'s, spectra etc.) for describing many features of turbulent flow, they do poorly when applied to phenomena such as energy dissipation that are intermittent in space or time. Then many higher moments of the *pdf* must be known for an adequate description. Traditional measures also fail to make full use of the self-similarity over length scales known to be characteristic of processes associated with the inertial range. Description of such phenomena as multifractals combines both their intermittent and self-similar character in a unified way. In this paper we go beyond the one- and two-dimensional limitations of experiments and obtain, for the first time, multifractal spectra that make full use of the three-dimensional character of a flow field of engineering interest.

Kolmogorov (1941) introduced the notion of inertial range by making powerful use of self-similarity, and later introduced a correction to account for intermittency of the energy dissipation (Kolmogorov 1962). Mandelbrot (1974) was the first to suggest that the dissipation field might be a fractal — a set of noninteger (Hausdorff) dimension. Frisch and Parisi (1985) elaborated this idea, introducing the term multifractal to denote a set upon which the scaling properties (such as exponents) of measures are not uniform over their support. Subsets of the support over which scaling properties are uniform are homogeneous fractals, and a multifractal may be thought of as a union of interleaved homogeneous fractals, each of which possesses different scaling properties and fractal dimension. The relation between the dimension of these homogeneous fractals and their scaling exponents is termed the multifractal spectrum. Later works by others (Halsey *et al.* 1986) showed practical methods of obtaining the multifractal spectrum from a particular distribution. A

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number of recent experiments, and the results of this paper, support a multifractal model of energy dissipation. In addition, the scalar dissipation and vorticity fields are also found to be multifractal.

In the high Reynolds number limit the locally averaged energy dissipation,  $\epsilon$ , can be shown to scale as (Meneveau and Sreenivasan 1987)

$$\epsilon \sim r^{\alpha-1} \quad (1)$$

where  $r$  is the extent of the averaging domain (Kolmogorov's original formulation assumes  $\alpha = 1$ ). In the multifractal model each iso- $\alpha$  set has a different fractal dimension, denoted by  $f(\alpha)$ , whose determination is the objective of this work. A theoretical idea, supported by some experimental evidence, is that the multifractal spectrum,  $f(\alpha)$ , is a universal property of a given turbulent quantity, and so remains unchanged between different flows.

In a number of recent experiments Sreenivasan and his co-workers have obtained multifractal spectra of energy dissipation and passive scalar dissipation from different flows. While experimental measures of the dissipation field are likely to remain far more highly resolved than in numerically calculated fields, their three-dimensional character is inaccessible in experiments. This necessitates the use of one-(hot-wire), or two-dimensional(laser sheet visualization) "cross-sections" as the base data, and, consequently, assumptions regarding the sub-space structure of the fractals becomes unavoidable. In this paper we obtain spectra from a numerically calculated flow which is three-dimensional, and thus free from the experimental limitations. The field is from a homogeneous shear flow, a choice dictated by the  $(128)^3$  resolution of the data and the fact that its homogeneity allows us to analyze the entire field, unlike wall-bounded flows where near-wall regions must be excluded.

We organize this paper as follows. In §II we describe the computations which produced the data set we have analysed. In §III we describe the multifractal model and our methodology for obtaining the spectra. The results are described in §IV and we conclude the paper in §V.

## 2. Homogeneous shear flow

Approximately homogeneous shear flow has been studied extensively. The imposition of uniform shear causes homogeneous isotropic turbulence to lose its isotropy and allows turbulence to be maintained by the transfer of energy from the mean-field. Thus homogeneous shear flow turbulence is non-decaying. Rose (1966) and Champagne *et al.* (1970) performed experiments on homogeneous shear flow, and Tavoularis and Corrsin (1981) added a passive scalar (temperature) by heating the generating rods. The experiments find a continuous growth of turbulent kinetic energy and its components, as well as a growth in the integral- and micro-scales. There is also a continuous departure from isotropy. The numerical simulations of Rogallo (1981) are true homogeneous shear, but are subject to problems associated with limitations on resolving the flow. As time evolves the energy transferred from the mean-field tends to accumulate in the small wavenumbers. This produces

anomalous behavior in the evolution of integral scales and other measures based on the large-scales of motion. Small-scale measures such as microscales are unaffected. The simulations show a continuous growth in turbulent kinetic energy, but a decrease in component energies in the transverse and spanwise directions, counter to experimental finding.

Rogers *et al.* (1986) and Rogers *et al.* (1987) have used the numerical scheme developed by Rogallo to obtain a clearer understanding of the dynamics of the flow. They find that the flow possesses coherent structures in the form of hairpin-like vortices. While we are unable here to make a connection between these organized structures and the multifractal spectrum of the vorticity, we note that phase coherent vorticity, which is often used as identifier of coherent structures, appears to be a feature of all high Reynolds number flow.

For the study described in this paper we have used a  $(128)^3$  realization of this flow due to Rogers *et al.* (1986), designated C128U12 in the NASA Ames data base. The parameters of the simulation are: Taylor microscale Reynolds number,  $Re_\lambda = 108$ ; viscosity,  $\nu = 0.01$ ; and Prandtl number  $Pr = 0.7$ . The simulation began from a field which had constant energy for  $16k_0 \leq k \leq 32k_0$ , where  $k_0$  is the fundamental streamwise wavenumber. The chosen realization is at a time when a power law decay in the spectrum at high  $k$  can be observed, but the degradation at low-wavenumbers has not begun. The power law exponent is  $-3$ , not  $-5/3$ , so the flow does not attain an inertial range.

### 3. Multifractal model

Our analysis is based on the works of Hentschel and Proccacia (1983), Benzi *et al.* (1984), Frisch and Parisi (1985), Halsey *et al.* (1986) and Meneveau and Sreenivasan (1987a). Consider the kinetic energy dissipation field

$$\epsilon = \nu(\partial u_i/\partial x_j + \partial u_j/\partial x_i)^2 \quad (2)$$

Here  $u_i$  is the  $i$ -th component of the velocity,  $i, j = 1, 2, 3$  such that  $x_i$  are the Cartesian coordinates and  $\nu$  is the kinematic viscosity. The energy dissipation in a box of size  $r$  is

$$E_r \sim \epsilon r^3 \quad (3)$$

The theory of generalized dimensions considers the dimensions  $D_q$  of a set to be defined by

$$D_q = \frac{1}{(q-1)} \lim_{r \rightarrow 0} \frac{\log \sum p_m^q}{\log r} \quad (4)$$

where the sum is over all the boxes and  $p_m$  is the probability of finding a member of the set in the  $m$ -th box. For integer values of positive  $q$  the moments  $p_m^q$  correspond to finding  $q$ -tuple members of the set within a distance  $r$ . Here the set is the dissipation field, and we make the assumption

$$Z = \sum E_r^q \sim r^{(q-1)D_q}. \quad (5)$$

We take  $r$  to range from the mesh size to the size of the computational domain. This relation determines  $D_q$  from  $E_r$ . If the scaling (1) holds, then the sum in (5) can be represented in terms of  $r$ . This leads to an integral relation between  $D_q$  and  $\alpha$ , which in the limit  $r \rightarrow 0$  can be shown to lead to the relations (Meneveau and Sreenivasan 1987a, Halsey et al. 1986)

$$\alpha = \frac{d}{dq} [(q-1)(D_q + 1 - d)] \quad (6)$$

$$f(\alpha) = q\alpha - (q-1)(D_q + 1 - d) + d - 1. \quad (7)$$

Here  $d$  is the dimension of the embedding space and for our 3-dimensional field  $d = 3$ .

The dissipation of passive scalars

$$\chi_i = \gamma(\partial\theta/\partial x_i)^2, \quad (8)$$

where  $\theta$  is concentration or temperature, and  $\gamma$  is molecular diffusivity, can be treated analogously. In experiments this quantity is easier to determine, since it involves no cross terms. The homogeneous shear flow that we have analyzed includes three scalar fields corresponding to imposed gradients in the three directions. We have performed the multifractal analysis on all three of these fields.

#### 4. Results

Figure 1 shows the energy dissipation field from the simulation and serves to focus our ideas. For purposes of illustration only ( $32 \times 32 \times 32$ ) points, corresponding to a 1/64th corner portion of the entire computational domain is shown. The intermittent nature of this quantity is evident and we wish to determine if it is self-similar over a range of length scales. By examining the data we have determined that the dissipation field tends to align itself along the direction of the mean shear much in the way that the vortices identified by Rogers (1986) do. While this does not affect our present analysis, an account of nonisotropic effects might be an extension of this and other works.

Given this field we calculate the generalized dimensions by obtaining  $Z$  as a function of  $r$  from Eqn. 5. Figure 2 shows the result where  $Z^{1/q-1}$  versus  $r$  is plotted for different values of  $q$ . Nonextremal points appear to lie very nearly on a straight line. Hence we find least squares fit for each  $q$  value and determine  $D_q(q)$  as the slopes of these lines. In calculating the slopes we have excluded the smallest value and the largest two values of  $r$ . The former we exclude because linear behavior is not evident here. We exclude the largest values of  $r$  because these scales are the size of the computational domain and we are interested in probing behavior for  $r \rightarrow 0$ . Note that the Kolmogorov microscale  $\eta = 0.012$ , which is smaller than the smallest value of  $r = 0.049$ . This means that in the numerical simulation the smallest scales are not quite resolved. Some of the departure from linearity observed in Fig. 2 is likely due to this lack of resolution.

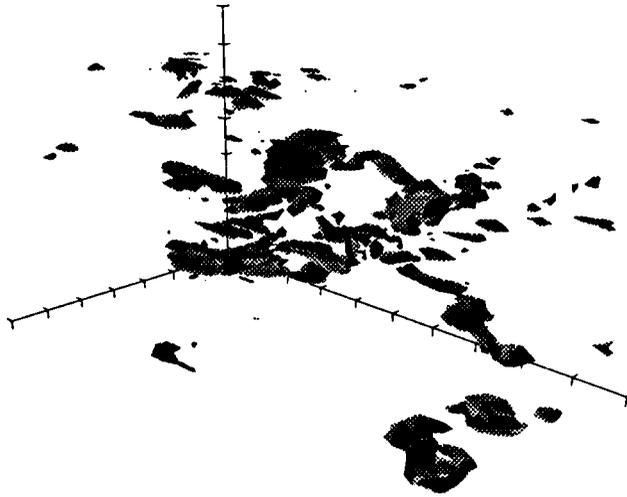


FIGURE 1. The three-dimensional contour surface  $\epsilon = 200$ , of the energy dissipation field.

Figure 3 shows the result of finding the slopes of the straight line fits to the data shown in the previous figure. Shown is  $D_q$  as a function of  $q$ . The nature of the curve is such that for large negative values of the moment  $q$  the curve asymptotes to a value that is larger than the value obtained asymptotically for large positive  $q$ . The intermediate values represent the range of (generalized) dimensions of the energy dissipation field. We find that  $D_{-\infty} = 3.85$  and  $D_{\infty} = 2.25$ . We note that  $D_0 = 3$ , from Fig. 3, implying that the dissipation is space-filling.

Having obtained the  $D_q$  variation with  $q$  we can now use Eqns. 6,7 to determine the multifractal spectrum. We have used central differences to approximate the slope of the  $D_q$  curve in Fig. 3. The result of applying the transformations are shown in Fig. 4., which shows  $f(\alpha)$  vs.  $\alpha$  for the energy dissipation. Since we had found  $D_0 = 3$ , which from Eqn. 8 gives  $f(\alpha_0)$ , the data has a maximum at this value. We find that  $f(1) = 2.97$  and  $\alpha_0 = 1.12$ .

We have carried out similar calculations for the scalar dissipation fields  $\chi_1, \chi_2$  and  $\chi_3$ . We omit presentation of intermediate results and show in Fig. 5 the  $f(\alpha)$  curves for the three scalar fields. While there are differences between the spectra of these fields, these differences are small when compared to those with the  $f(\alpha)$  for energy dissipation. The scalar field spectra are all broader. This is in accord with experimental finding. The result implies that the so-called intermittency exponent (see Meneveau and Sreenivasan 1987a) is larger for scalar dissipation fields.

The multifractal approach currently applies only to positive quantities. In order to apply the method to the vorticity field we consider the magnitude of the vorticity

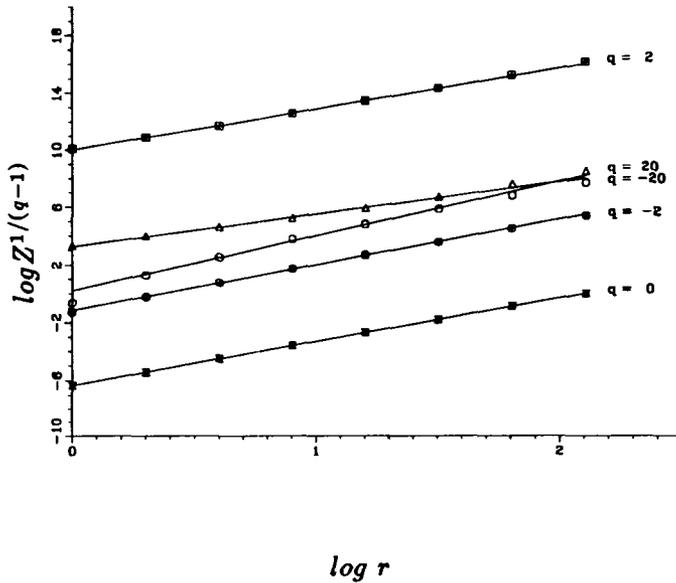


FIGURE 2. Results of applying eq. 5 to the dissipation field. The symbols are the calculated values. Lines are least-squares fits to the data with first point and last two points excluded.

$|\omega|$ . Fig. 6 shows the  $f(\alpha)$  curve for this field. Of more general interest than  $\omega$  is  $\omega^2$ , the enstrophy density. We have not calculated its  $f(\alpha)$  curve, but it can be obtained from that of  $\omega$  (Meneveau and Sreenivasan 1988).

Numerical simulations require initial conditions on velocity, and this implies a certain dissipation and vorticity field as well. The initial conditions used for the shear flow calculation did have a multifractal structure, as can be seen from the  $f(\alpha)$  curves for the initial energy dissipation and initial vorticity magnitude fields in Figure 7. A purely random field  $\zeta(x, y, z)$  yields the single point  $f(0) = 3$  in Fig. 7. While the initial dissipation multifractal structure covers relatively more scales than the vorticity magnitude, it covers relatively fewer when the flow is developed (Fig. 3). For both quantities the expected increase in scaling range under the action of the Navier-Stokes equations is found. A strict inertial range may not have developed yet, but self-similarity is increasing at higher wavenumbers.

Having obtained  $f(\alpha)$  curves for the various dissipation fields we now compare our findings with those obtained experimentally by Sreenivasan and his co-workers (Meneveau and Sreenivasan 1987a, Prasad *et al.* 1988). Shown in Fig. 3 as a dashed curve is their data for  $D_q$  vs.  $q$ . For  $-10 < q < 10$  our data falls close to their curve, obtained by averaging over a number of developed laboratory and atmospheric flows. Our data is within their error bars. For large  $|q|$  there are significant differences between our finding and theirs, some of them attributable to

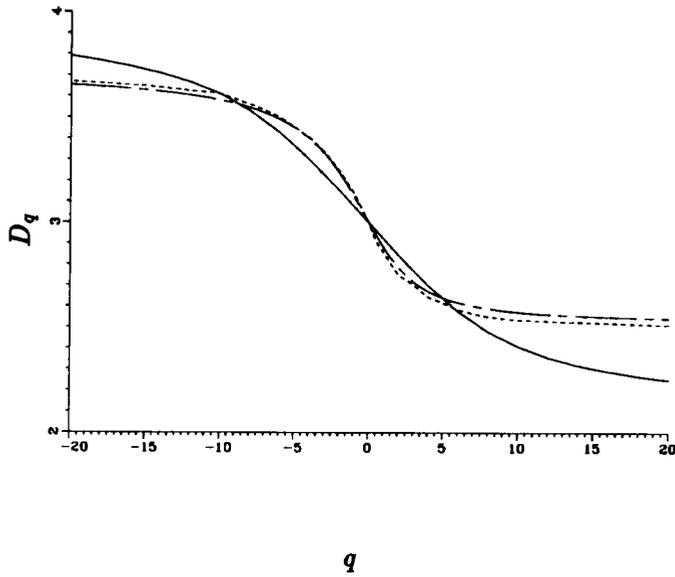


FIGURE 3. Variation of  $D_q$  with  $q$ . Solid line: our calculations; Dashed line: data of Meneveau and Sreenivasan(1987a); Chain dash:  $p$ -model with  $p_1 = .7$ .

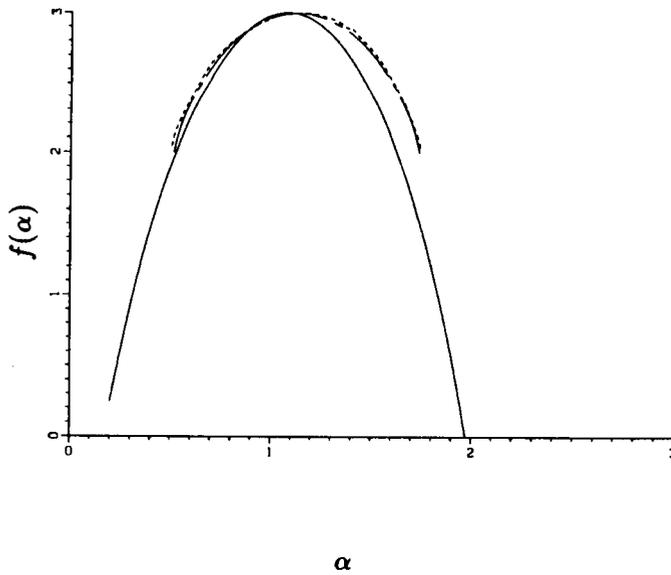


FIGURE 4. The multifractal spectrum  $f(\alpha)$  of the dissipation field. Lines as in Figure 3.

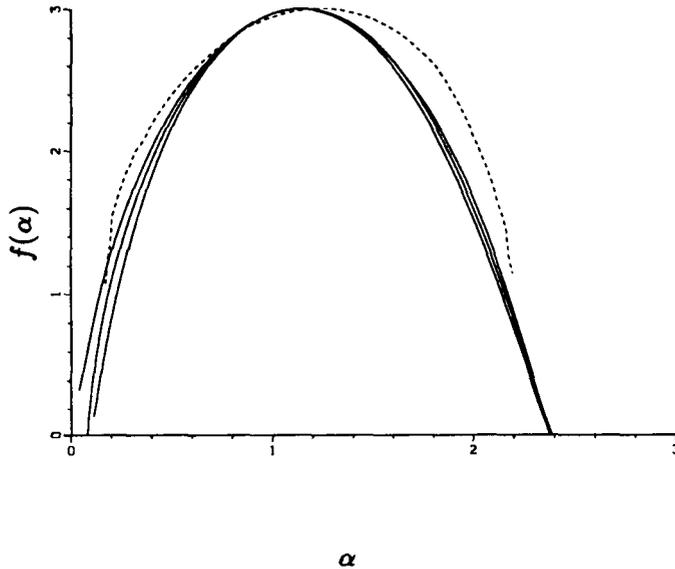


FIGURE 5. Multifractal spectra of the scalar dissipation fields,  $\chi_i$ . Solid lines are our data, the dashed line is data from Prasad, Meneveau, and Sreenivasan(1988).

the general inaccuracy of the curves at large  $q$ , where small errors are compounded. In Figs. 4 and 5 the dashed curve shows their  $f(\alpha)$  curves for energy and scalar dissipation. Although differences are exaggerated (by differentiation) when going from the  $D_q$  vs.  $q$  to the associated  $f(\alpha)$  curves, the agreement is still fair. Note that since their data considered a one-dimensional (two-dimensional) intersection of the flow their data does not extend below  $f(\alpha) < 2$  ( $f(\alpha) < 1$ ). We conclude that our findings are in broad agreement with the experimental results. Like Meneveau and Sreenivasan we find that the energy and scalar dissipation fields are space-filling and  $D_0 = 3$ ; that  $\alpha_0 > 1$ ; that the  $f(\alpha)$  curve is nearly symmetric and that spectrum for the scalar dissipation  $f(\alpha)$  is broader than that for energy dissipation.

Since the energy dissipation field possesses multifractal scaling properties it might be possible to determine the general class of multifractal models to which it belongs. This prompted Meneveau and Sreenivasan (1987b) to investigate a Cantor-set model of the energy cascade process. Their so-called  $p$ -model envisions that the energy cascade in a turbulent flow may be modelled by the breakdown of an eddy into two smaller eddies of the same size, but with energies distributed in the ratio  $p_1 : p_2$ . This process, *ad infinitum*, results in a Cantor-set like structure for the kinetic energy that is a multifractal. A closed form solution relates  $D_q$  to  $q$ .

$$D_q = \log_2 \{p_1^q + p_2^q\}^{1/(1-q)} \quad (9)$$

Meneveau and Sreenivasan found that their data was well approximated by taking  $p_1 = 1 - p_2 = 0.7$ . Following their work we perform a similar analysis to see if a

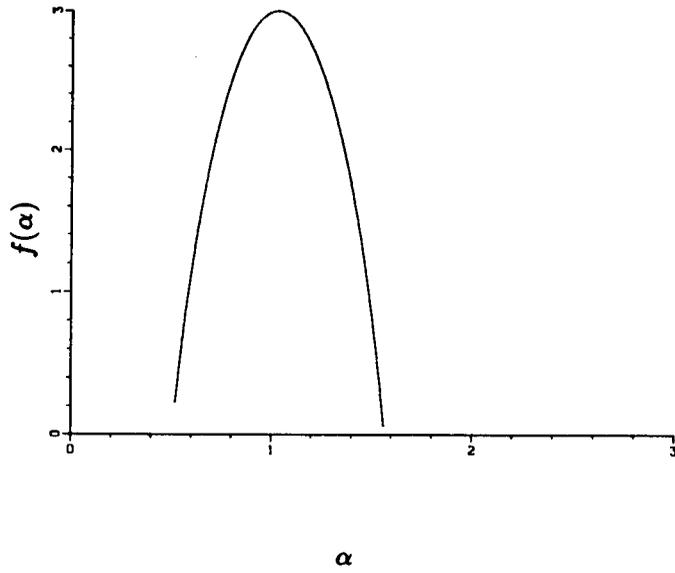


FIGURE 6. Multifractal spectra of the vorticity magnitude field,  $|\omega|$ .

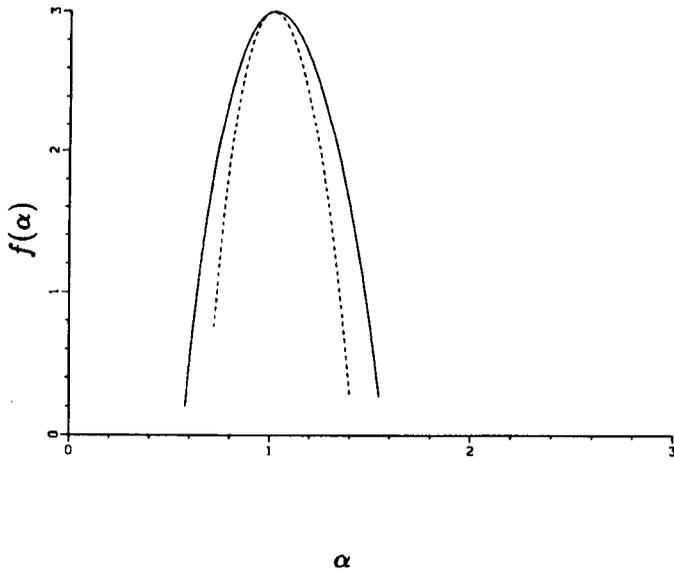


FIGURE 7. Multifractal spectra of the initial  $\epsilon$  and  $|\omega|$  fields. Solid line:  $\epsilon$ ; Dashed line:  $|\omega|$ .

two-scale Cantor set model fits our data. We do not find a satisfactory combination of values that fits our data even with nonequal eddies and energy partitions. The chain-dashed curve shown in Figs. 2-5 are from the  $p$ -model of Meneveau and Sreenivasan (1987b).

## 5. Conclusion

We have obtained first results for the multifractal spectra of several three-dimensional fields. Although these have been made for one realization of a specific flow — homogeneous shear flow — the multifractal spectra for a given field are thought to be a universal property for sufficiently high Reynolds number flows. Even though the flow we have examined is not at the asymptotically large Reynolds number where the multifractal formulation applies strictly, we find that a range of scales can be found in the flow that are self-similar. This self-similarity cannot be ascribed to the presence of an inertial range, but the multifractal model takes into account both the intermittent nature of the fields and the self-similarity in a unified way. Other available flow-fields are currently being examined in an on-going effort to understand multifractal structure in turbulent flow. The results are of value in modeling turbulent flow and energy cascade processes. Models which possess multifractal structure for the appropriate variables might more successfully predict quantities of engineering interest. Our study is a step in that direction.

We have made comparisons with results obtained from experimental studies. The conditions of the experiments differ from our conditions in three ways: the experiments are at higher Reynolds number; they use intersections of the fields; and the results represent an averaging over a number of realizations and flows. Thus, while we find broad agreement with experimental results, there are numerous differences in detail. It is not possible to ascribe the differences in the results to a particular difference in condition. However, such an evaluation will be possible with the availability of more data. The concurrence of experimental and numerical findings is of great importance in the justification of key assumptions made in the experiments. Conversely the analysis of intersections of flows, if justified, greatly increases the geometries and Reynolds numbers of flows which can be analysed for multifractal structure through experiments, but which are beyond practical numerical simulation.

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