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# Modeling the turbulent kinetic energy equation for compressible, homogeneous turbulence

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The turbulent kinetic energy transport equation, which is the basis of turbulence models, is investigated for homogeneous, compressible turbulence using direct numerical simulations performed at CTR. It is shown that the partition between dilatational and solenoidal modes is very sensitive to initial conditions for isotropic decaying turbulence but not for sheared flows. The importance of the dilatational dissipation and of the pressure-dilatation term is evidenced from simulations and a transport equation is proposed to evaluate the pressure-dilatation term evolution. This transport equation seems to work well for sheared flows but does not account for initial condition sensitivity in isotropic decay. An improved model is proposed.

## 1. Introduction

### 1.1 Turbulent kinetic energy equation

Industrial turbulence models, i.e. one-point closures, commonly use transport equations for averaged quantities. The first equation to be considered in one- or two-equation models is the turbulent kinetic energy equation.

For compressible flows, it is convenient to use Favre averaging. We use  $\bar{\cdot}$  and  $\cdot'$  to denote respectively an ensemble average and the fluctuation with respect to the ensemble average,  $\tilde{\cdot}$  and  $\cdot''$ , a Favre average, and fluctuations with respect to the Favre average.

From the continuity and momentum equations, it is possible to deduce a transport equation for the turbulent kinetic energy per unit mass,  $k = \frac{1}{2} \overline{u_i'' u_i''}$ ,

$$\underbrace{\frac{\partial \bar{\rho} k}{\partial t} + \frac{\partial \bar{\rho} \tilde{u}_j k}{\partial x_j}}_{(1)} = \underbrace{-\bar{\rho} \tilde{u}_i'' \tilde{u}_j'' \frac{\partial \tilde{u}_i}{\partial x_j}}_{(2)} - \underbrace{\overline{\tau_{ij} \frac{\partial u_i''}{\partial x_j}}}_{(3)} + \underbrace{\overline{p' \frac{\partial u_i''}{\partial x_i}}}_{(4)} + \underbrace{\overline{u_i'' \frac{\partial \bar{p}}{\partial x_i}}}_{(5)} \quad (1.1)$$

$$+ \underbrace{\frac{\partial}{\partial x_k} \left[ -\frac{1}{2} \overline{\rho u_i'' u_i'' u_k''} + \overline{u_i'' \tau_{ik}} - \overline{p' u_k''} \right]}_{(6)}$$

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Term (1) is the advection term, while term (2) represents the turbulent kinetic energy production due to the action of the mean velocity gradients upon the turbulent stresses. These terms are exact and do not require modeling, besides a way to compute the turbulent stresses.

The terms to be modelled are (3), which is the dissipation due to the work of the viscous stress

$$\tau_{ij} = \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] ; \quad (1.2)$$

the pressure-dilatation term, (4); the mean pressure gradient term, (5); and the diffusion term, (6).

The dissipation and diffusion terms already exist in incompressible flows, while the pressure terms (4 and 5) are due to the flow compressibility. We shall, however, see in section 2 that flow compressibility also affects the form of the dissipation term.

### 1.2 Data bases for the modeling of the turbulent kinetic energy equation

Direct numerical simulations (DNS) of compressible, homogeneous turbulence are underway at CTR. For compressible flows, homogeneity requires that the mean pressure, temperature, and density be constant over space and that the mean velocity gradient satisfies the constraint

$$\frac{\partial}{\partial t} \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial \bar{u}_k}{\partial x_j} = 0 \quad (1.3)$$

Examples of allowed mean flows include constant flow, shear flow, and time-dependent irrotational strained flows.

DNS of isotropic decaying turbulence and of sheared flows have been performed by Blaisdell [1990]. Since turbulent statistics are point-independent for homogeneous flows, the diffusion term, (6), is null. As the mean pressure is constant over space, no direct information is available for the pressure gradient term, (5). We shall, however, see in section 3 that this term can be modelled together with the pressure-dilatation term.

### 1.3 Motivation

Homogeneous flows can be used to investigate models for the dissipation term, (3), and the pressure-dilatation term, (4). The turbulent kinetic energy balance of a sheared flow plotted in figure 1 shows that the pressure-dilatation term, often neglected, is about 10% of the dissipation and is thus important when compared with the time derivative of the turbulent kinetic energy.

Section 2 is devoted to the modeling of the dissipation term, while section 3 deals with the pressure-dilatation term modeling.

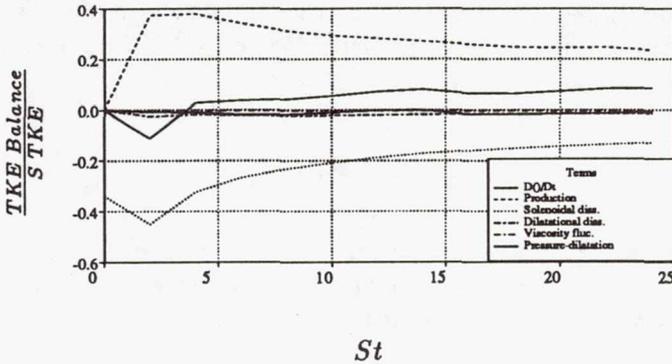


FIGURE 1. Turbulent kinetic energy balance for shear flow.

## 2. Modeling of the dissipation term

### 2.1 Decomposition of the dissipation term

As shown in figure 1, DNS confirms the *a priori* idea that the contribution of viscosity fluctuations to the dissipation is negligible. For homogeneous flows, the dissipation term can thus be written as

$$\overline{\rho \varepsilon} = \tau_{ij} \overline{\frac{\partial u_i''}{\partial x_j}} \simeq \underbrace{\overline{\mu \omega_i'' \omega_i''}}_{\overline{\rho \varepsilon_s}} + \underbrace{\frac{4}{3} \mu \overline{\frac{\partial u_i''}{\partial x_i} \frac{\partial u_j''}{\partial x_j}}}_{\overline{\rho \varepsilon_d}} \quad (1.4)$$

where  $\omega$  is the vorticity.

The solenoidal dissipation,  $\varepsilon_s$ , represents the contribution of the vorticity, i.e. the usual form of the dissipation term in incompressible homogeneous turbulence. The dilatational dissipation,  $\varepsilon_d$ , is due to the flow compressibility.

For homogeneous flows, it is possible to split the velocity field into a solenoidal part (normal to the wave number vector in Fourier space) and a dilatational part (parallel to the wave number vector). The above splitting of the dissipation term then reflects the splitting of the velocity field. It must be pointed out that this splitting of the velocity is not unique in inhomogeneous flows. However, for inhomogeneous flows, the dissipation term, (3), can still be split into solenoidal and dilatational parts *plus* a diffusion term.

The solenoidal and dilatational dissipations are plotted as part of the balance presented in figure 1. The dilatational contribution is small compared with the solenoidal one, about 10%, but is not negligible in the balance.

### 2.2 Existing models

Sarkar *et al.* [1989] confirmed, from DNS of isotropic decaying turbulence, the results of their asymptotic analysis which predicts the existence of an equipartition of energy between the variance of the pressure associated with the dilatational

velocity field and the dilatational contribution to the kinetic energy. Then, assuming that the dilatational pressure variance scales with the square of the turbulent Mach number,  $M_t = \sqrt{q^2}/a$ , and that the solenoidal and dilatation velocity fields have similar Taylor microscales, they obtain

$$\varepsilon_d = M_t^2 \varepsilon_s \quad (2.1)$$

Zeman [1990] assumes that the dilatation is mainly due to shock-like structures embedded within the energetic turbulent eddies. From shock relations and an assumed Gram-Charlier pdf for the velocity fluctuations, he states

$$\varepsilon_d = c_D F(M_t, K) \varepsilon_s \quad c_D \sim 0.75 \quad (2.2)$$

where  $K$  is the kurtosis of the velocity fluctuations.

Both Sarkar and Zeman assume that the evolution of the solenoidal dissipation is given by the same transport equation as for incompressible flows.

### 2.3 Comparison with DNS results

In the simulations performed by Blaisdell, the ratio of the dilatational dissipation to the total dissipation is insensitive to initial conditions in sheared flows but is very sensitive to initial conditions in isotropic decay. This is shown in figure 2.

It seems that the flow keeps a memory of the initial dissipation partition for isotropic decay but not for sheared flows. This is at variance with the above analysis, at least for isotropic decay, but Erlebacher *et al.* [1990] have shown, from linear acoustics, that the compressible part of the flow, i.e. the acoustic part, can decouple from the solenoidal part for isotropic decay and reach various asymptotic levels according to the initial conditions. This analysis holds only for isotropic flows and low turbulent Mach numbers. From DNS of low Reynolds number ( $Re_\lambda \sim 20$ ), two-dimensional isotropic turbulence, Erlebacher [1990] evidenced a sensitivity to initial flow conditions.

Another explanation for this behavior is that turbulence remembers its initial conditions in the final period of decay. As shown in figure 3, the Reynolds number,  $Re_\lambda$ , based upon the Taylor microscale  $\lambda_{11}$  and  $q$ , becomes very small for the isotropic decay case.

To check the influence of the low Reynolds numbers, two new isotropic decaying turbulence runs were performed. To avoid low Reynolds numbers, the initial Reynolds number was held constant at  $Re_\lambda \simeq 50$  during the turbulence development period by decreasing the viscosity, after which the viscosity was held constant. The initial turbulent Mach number is 0.5. The sensitivity to initial conditions is still observed. Consequently, it seems important to keep information about initial conditions in isotropic decay while a unique equilibrium behavior seems to be achieved in the presence of shear.

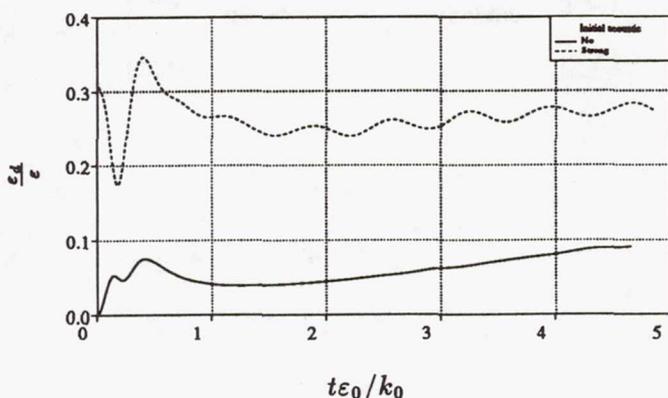
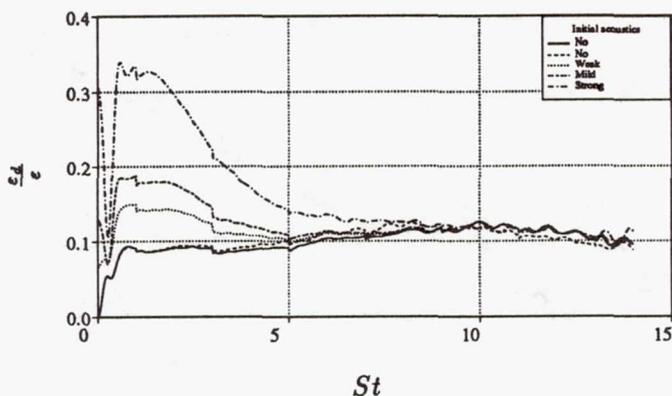
a. Isotropic decaying turbulence  $M_{t_0} = 0.7$ b. Sheared turbulence  $M_{t_0} = 0.5$ 

FIGURE 2. Dissipation partition time evolution.

### 3. Modeling the pressure-dilatation term

#### 3.1 Existing models

Sarkar *et al.* [1989] proposed to include the pressure-dilatation term in the expression for the dilatational dissipation as, from their isotropic decay simulations, this term is small and its fluctuations are larger than its average.

The same approach has been used by Zeman [1990] who later proposed to relate this term to the pressure-variance evolution. He uses a return law in which the pressure variance is assumed to return to an equilibrium value on an acoustic time scale. Zeman's model [1990a] reads

$$p \frac{\partial u_i''}{\partial x_i} = -\frac{1}{2} \frac{1}{\bar{\rho} a^2} \frac{D p'^2}{Dt} \quad (3.1)$$

$$\frac{D p'^2}{Dt} = \frac{p_e^2 - p'^2}{\tau_a} \quad (3.2)$$

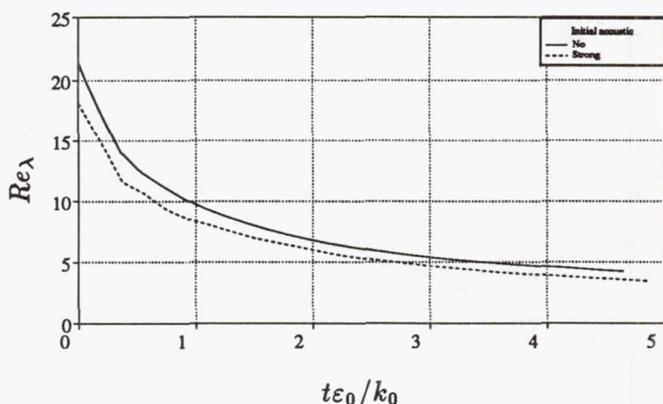
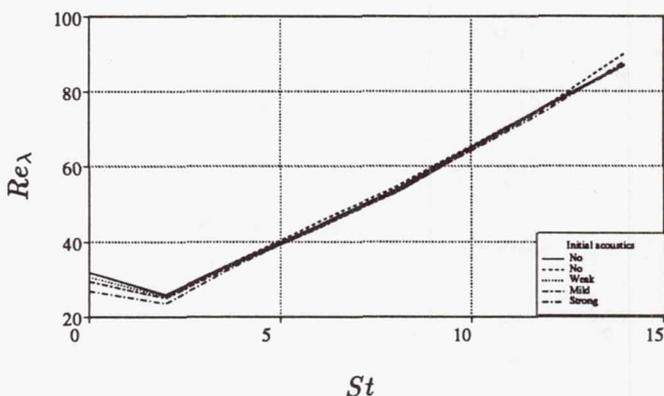
a. Isotropic decaying turbulence  $M_{t_0} = 0.7$ b. Sheared turbulence  $M_{t_0} = 0.5$ 

FIGURE 3. Taylor microscale Reynolds number time evolution.

$$\overline{p_e^2} = \overline{\rho}^2 q^2 a^2 \frac{\alpha M_t^4}{1 + \alpha M_t^4} \quad (3.3)$$

where equation (3.1), which is given by linear acoustics, has been shown to have a wider range of validity. Zeman [1990b] also proposed a revised version of equation (3.3) in which the right hand side (RHS) varies as  $M_t^2$  instead of  $M_t^4$ .

### 3.2 Modeling of the pressure-dilatation term

DNS of sheared flows have shown that the mean value of the pressure-dilatation term is comparable to the dilatational dissipation and thus gives a significant contribution to the kinetic energy budget. Moreover, this term exhibits oscillations which seem to scale on an acoustic time scale and are difficult to model. It appears, however, from comparisons of two runs with identical initial spectra but different seeds in the initial random field generation, that these oscillations are due to the noise of the biggest eddies for which only a small statistical sample is obtained in a single

simulation. Consequently, we only try to reproduce a smoothed pressure-dilatation evolution, not its oscillations.

### 3.2.1 Transport equation for the velocity-pressure gradient term

Instead of using linear acoustics and scaling relations, we first investigated the transport equation for the pressure-dilatation term. The three pressure terms which appear in the turbulent kinetic energy equation, (1.1), i.e. the pressure-dilatation term, (4); the mean pressure gradient term, (5); and the contribution of pressure to the diffusion term, (6), are an expansion of the initial term  $-u_i'' \frac{\partial p}{\partial x_i}$ .

It is easier to derive an equation for  $-u_i'' \frac{\partial p}{\partial x_i}$  than for the pressure-dilatation term. Moreover, *only* for homogeneous flows can the pressure-dilatation term be interpreted as an energy exchange term between turbulent kinetic energy and potential pressure fluctuation energy  $\overline{p'^2}/2\gamma\bar{p}$ . However, the term  $-u_i'' \frac{\partial p}{\partial x_i}$  always represents the reversible energy exchange between the turbulent kinetic energy and the internal energy.

Assuming that the fluid is a perfect gas, the pressure can be linked to the internal energy per unit mass,  $e$ , as  $p = (\gamma - 1)\rho e$  so that the energy equation can be written as a pressure evolution equation. From this equation, together with the momentum and continuity equations, the transport equation for  $u_i'' \frac{\partial p}{\partial x_i}$  can be derived as

$$\begin{aligned}
 \underbrace{\frac{\partial}{\partial t} \overline{u_i'' \frac{\partial p}{\partial x_i}} + \bar{u}_l \frac{\partial}{\partial x_l} \overline{u_i'' \frac{\partial p}{\partial x_i}}}_{(a)} &= \underbrace{-2\overline{u_l'' \frac{\partial p}{\partial x_l} \frac{\partial \tilde{u}_i}{\partial x_l}}}_{(b)} - \underbrace{\gamma \overline{u_i'' \frac{\partial p}{\partial x_i} \frac{\partial \tilde{u}_l}{\partial x_l}}}_{(c)} - \underbrace{\gamma \overline{p u_i'' \frac{\partial^2 \tilde{u}_l}{\partial x_i \partial x_l}}}_{(c)} \\
 &+ \underbrace{p \frac{\partial u_i''}{\partial x_l} \frac{\partial u_l''}{\partial x_i} + (\gamma - 1) p \frac{\partial u_i''}{\partial x_i} \frac{\partial u_l''}{\partial x_l} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \frac{\partial p}{\partial x_i}}_{(d)} \\
 &+ \underbrace{\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} \frac{\partial \bar{p}}{\partial x_i} - \frac{1}{\rho} \frac{\partial}{\partial x_l} \left[ \tau_{il} - \bar{\rho} u_i'' u_l'' \right] \frac{\partial \bar{p}}{\partial x_i}}_{(e)} \\
 &+ \underbrace{\frac{1}{\rho} \frac{\partial \tau_{il}}{\partial x_l} \frac{\partial p}{\partial x_i} - (\gamma - 1) \left( \tau_{lk} \frac{\partial u_l''}{\partial x_k} - \frac{\partial q_l}{\partial x_l} \right) \frac{\partial u_i''}{\partial x_i}}_{(f)} \\
 &= \underbrace{\frac{\partial}{\partial x_l} \left[ \overline{u_i'' \frac{\partial p u_l''}{\partial x_i}} - (\gamma - 1) \overline{p u_l'' \frac{\partial u_i''}{\partial x_i}} + (\gamma - 1) u_l'' \left( \tau_{ij} \frac{\partial u_i''}{\partial x_j} - \frac{\partial q_i}{\partial x_i} \right) \right]}_{(g)}
 \end{aligned} \tag{3.4}$$

where (a) represents the advection by the mean flow, (b) the mean velocity gradient effects, (c) the mean dilatation and mean dilatation gradient effects, (d) third order effects, and (e) the mean pressure gradient effect, while (f) are the viscous terms and (g) the diffusion terms.

For homogeneous flows, terms (e) and (g) are zero. Moreover, for isotropic decay or sheared flows, term (c) is zero. It must also be pointed out that the three terms

$$-2\overline{u_i'' \frac{\partial p}{\partial x_i} \frac{\partial \tilde{u}_i}{\partial x_l}} + \overline{p \frac{\partial u_i''}{\partial x_l} \frac{\partial u_l''}{\partial x_i}} - \overline{\frac{1}{\rho} \frac{\partial p}{\partial x_i} \frac{\partial p}{\partial x_i}}$$

exactly cancel in incompressible flows thanks to the Poisson equation for the pressure.

### 3.2.2 Closure of the transport equation

The first two terms in (d) can be approximated as

$$\overline{p \frac{\partial u_i''}{\partial x_l} \frac{\partial u_l''}{\partial x_i}} \simeq \overline{\bar{p} \frac{\partial u_i''}{\partial x_l} \frac{\partial u_l''}{\partial x_i}} \simeq \overline{\bar{p} \frac{\partial u_i''}{\partial x_i} \frac{\partial u_l''}{\partial x_l}} = \frac{\bar{p}}{\mu} \varepsilon_d$$

i.e. these terms blow up as the Reynolds number tends towards infinity. Consequently, the pressure-dilatation budget is the balance of terms which tend towards infinity while their difference remains finite. A term by term modeling approach cannot be applied. Using an equation for  $\overline{p \frac{\partial u_i''}{\partial x_i}}$  leads to the same behavior at high Reynolds numbers.

As with the dissipation equation, a heuristic approach is used to include the physics in a modelled equation. However, equation (3.4) does provide information about the rôle of the compressibility terms. The modeling of the equation will be discussed only for homogeneous flows so that we turn back to the pressure-dilatation term. This term should tend towards zero for incompressible flows, i.e. it must be damped on an acoustic time scale. All terms on the RHS of (3.4) cancel in the incompressible limit. On the other hand, DNS are in good agreement with (3.1) and show that the pressure-dilatation term tends, after some transients, to be positive for decaying turbulence and negative for sheared flows, i.e. to scale upon the turbulent kinetic energy temporal evolution. At last, assuming that the pressure level scales upon  $M_t^2$  and that the production term scales upon an acoustic time scale leads to the form

$$\frac{d}{dt} \overline{p \frac{\partial u_i''}{\partial x_i}} = -C_1 \rho \frac{1}{\tau_a} M_t^2 \frac{dk}{dt} - C_2 \frac{1}{\tau_a} \overline{p \frac{\partial u_i''}{\partial x_i}} \quad (3.5)$$

$$\text{with } \tau_a = \frac{k^{3/2}}{\varepsilon a} \quad C_1 = 0.25 \quad C_2 = 0.2$$

It must be mentioned that Zeman's model can be rewritten as a transport equation for the pressure-dilatation term. A somewhat different form, including turbulent kinetic energy and dissipation time derivatives and a similar damping term is thus obtained.

### 3.2.3 Comparison with DNS and Zeman's model

The evolution of the pressure-dilatation term given by DNS is compared with predictions from the above model, (3.5), and from Zeman's model, assuming either an  $M_t^4$  or  $M_t^2$  behavior in (3.3). Only the evolution of the pressure-dilatation term is predicted, using the DNS values of kinetic energy, dissipation, and other quantities. The comparison starts at a time when the turbulent flow is developed. The aim is to reproduce the mean evolution of the pressure-dilatation term, not its oscillations which are the signature of the biggest eddies of the DNS.

The comparisons for some sheared flows are plotted in figure 4. Case (4.a), which is the run with the most gridpoints and the longest development, has been used to calibrate the coefficients  $C_1$  and  $C_2$ . Particular attention has been paid to the initial evolution of the pressure-dilatation term to avoid the increase predicted by Zeman's model. As the turbulent Mach number varies weakly, there are several possible sets of coefficients. However, they eventually lead to similar results for the other flows. Zeman's models give a good prediction of the pressure-dilatation level but not of its time derivative; the  $M_t^4$  assumption here seems to be the best.

For case (4.b), the model seems to slightly overestimate the pressure-dilatation term, but the agreement is fair. Zeman's best model is now the  $M_t^2$  assumption.

For case (4.c), all models predict an unrealistic initial increase of the pressure-dilatation term. Both the present model and Zeman's  $M_t^2$  model give good levels; the present model seems to better estimate the time derivative.

The comparison for isotropic decaying turbulence is plotted in figure 5. No model is able to give good predictions of the isotropic decay case. Zeman's  $M_t^2$  model gives good prediction of the strong acoustics case while the  $M_t^4$  model gives good predictions of the low acoustics cases. It is possible to tune the coefficients of our model to have a fair prediction of case (5.b), but then prediction of the other flows is poor. This seems again to be due to the sensitivity of isotropic decaying flows to the initial conditions.

## 4. Proposals for model improvement

A simple way to improve the modeling of isotropic decaying flows is to account for the initial partition between solenoidal and compressible modes by using transport equations for both the pressure-dilatation term and the dilatational dissipation. A way to extend the above model could be

$$\begin{aligned} \frac{d}{dt} p \frac{\partial u_i''}{\partial x_i} &= -C_1 \frac{\varepsilon_d}{k} P + C_2 \frac{\varepsilon_s}{\tau_a} - C_3 \frac{1}{\tau_a} p \frac{\partial u_i}{\partial x_i} \\ \frac{d\varepsilon_d}{dt} &= W_1 \frac{\varepsilon_d}{k} P + W_2 \frac{\varepsilon_s}{\tau_a} - W_3 \frac{\varepsilon_d}{\tau_a} \end{aligned}$$

where information about the constants and their Mach number dependency could be deduced from the equilibrium solutions for isotropic decay and sheared flows.

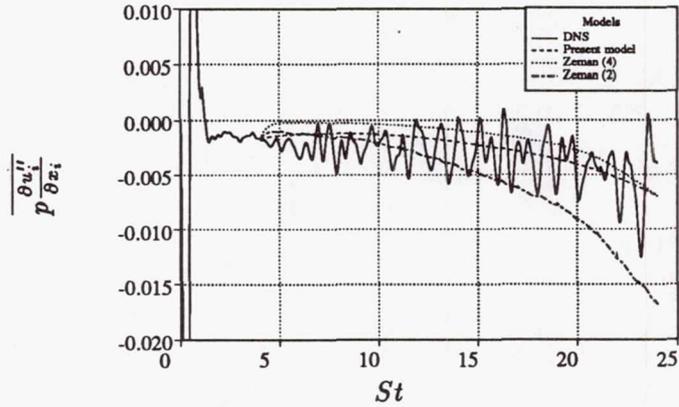
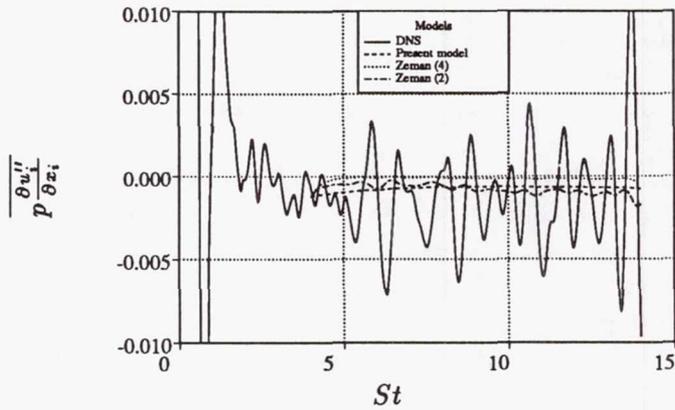
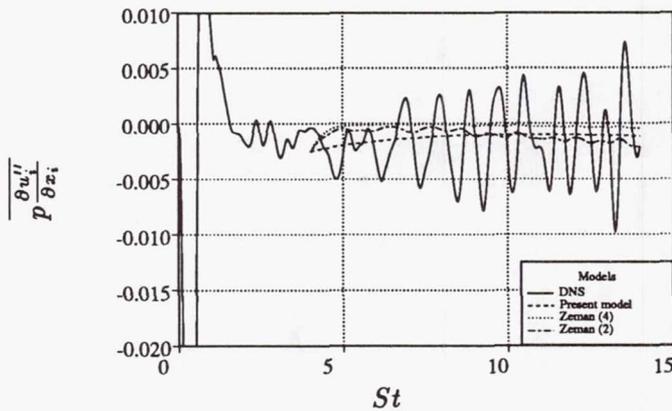
a.  $M_{t_0} = 0.4$  No initial acousticsb.  $M_{t_0} = 0.5$  Strong initial acousticsc.  $M_{t_0} = 0.5$  No initial acoustics

FIGURE 4. Evolution of the pressure-dilatation term for sheared turbulence.

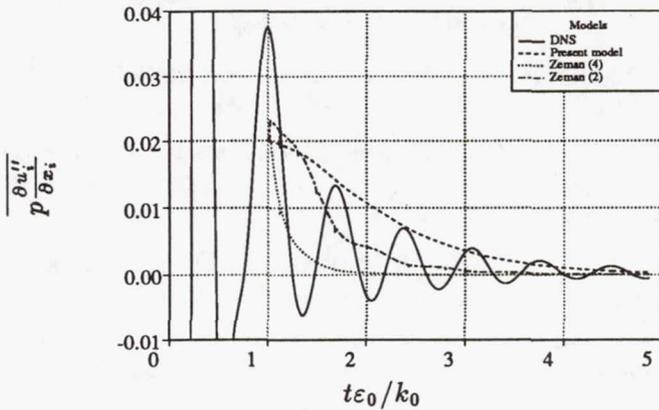
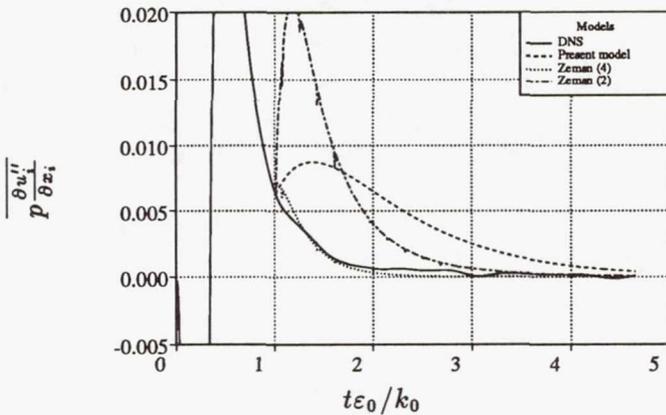
a.  $M_{t_0} = 0.7$  Strong initial acousticsb.  $M_{t_0} = 0.7$  No initial acoustics

FIGURE 5. Evolution of the pressure-dilatation term for isotropic decaying turbulence.

## 5. Conclusions

The dilatational dissipation and the pressure-dilatation term play important rôles in the kinetic energy budget and should not be neglected. The dilatational dissipation scales with the solenoidal dissipation and the turbulent Mach number, but this scaling also depends upon the initial conditions for isotropic decaying turbulence. A model to predict the evolution of the pressure-dilatation term has been proposed. It seems to work well for sheared flows but still requires validation and needs improvement to account for initial condition sensitivity.

Moreover, other important points such as turbulent stress, turbulent heat flux, and turbulent scalar flux modeling should be investigated. Homogeneous flows can provide helpful information, e.g. about the role of mean flow dilatation, but inhomogeneous flows are to be looked at to investigate the role of density or temperature gradients.

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