A numerical evaluation of the dynamical systems approach to wall layer turbulence

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This work attempts to test predictions based on the Dynamical Systems approach to Wall Layer Turbulence. We analyze the Dynamical Systems model for the non-linear interaction mechanism between the coherent structures and deduce qualitative behavior to be expected. We then test for this behavior in data sets from D.N.S. The agreement is good, given the sub optimal conditions for the test. We discuss implications of this test and work to be done to deepen the understanding of control of turbulent boundary layers.

1. Introduction

The Dynamical Systems approach to wall region turbulence is a methodology for deriving low dimensional Dynamical Systems (which are systems of O.D.E's) to describe the interaction of Coherent Structures in the near wall region. Coherent Structures are defined through the Proper Orthogonal Decomposition (to be described later). See Aubry et al. (1988) and Berkooz et al. (1990-1). The most significant achievement of the Dynamical Systems approach to date was suggesting a non-linear interaction mechanism that produces the so called "bursting" observed in experiments (Kline et al. 1967, Kim et al. 1970, Corino and Brodkey 1969). These events were also observed in numerical simulations of wall bounded flows (Moin and Kim, 1985). The reaction to the Dynamical Systems Approach in the Turbulence community ranged from enthusiasm to technical criticism to basic objections. Technical criticism, like Moffat’s observations (Moffat 1990), were instrumental in furthering our understanding of the results (Berkooz et al. 1990-2, Holmes et al. 1990). Basic objections stemmed from previous experience where interesting dynamical behavior of Dynamical Systems models was due to the process which led to the Dynamical System (abbreviated D.S. from now on), and not to physical content of the equation they were modeling. The celebrated Lorenz system is such an example, the original problem it started off as was the Benard convection problem.

This work attempts to establish a connection between the D.S. approach and the “real world” (i.e. computer D.N.S.) by showing that the D.S. models predict non trivial behavior of the wall layer region that may be observed in D.N.S.

There are technical and fundamental difficulties in establishing this connection. A technical difficulty facing a detailed comparison is that the spanwise periodic domain used in the D.S. is small compared to most full size wall bounded D.N.S. In addition

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the D.S. concentrate on a series of events in time, while spatially there is only one event. A full D.N.S. has several events occurring at different spatial locations, sometimes with overlap. This problem was overcome by using the "Minimal Flow Unit of Near Wall Turbulence" studied by Jimenez and Moin (1990). The minimal flow unit is the smallest computational domain, in a channel flow computation that would sustain a turbulent flow. This flow is not ideal for our purpose. Two point velocity correlation functions are not reproduced accurately, thus suggesting that the structures (in a P.O.D. sense) observed will not be the same as a real flow. The effects of the outer layer are not well produced above $y^+ = 40$, as will be outlined later; this effect is of importance to the D. S. models. The box sizes used in the minimal flow unit are smaller then the boxes used in previous studies of D.S. ($100^+$ for the minimal flow compared to $333^+$ for the D.S. The eigenfunctions we had were not computed for this flow. They were derived from a large flow box D.N.S. (see Moin and Moser 1989). This caused us to interpolate the eigenfunctions, thus getting a less then optimal basis. The question of interpolating eigenfunctions, or two point correlation functions, will be addressed in the sequel. The main advantage of the minimal flow unit is that it produces well defined events which is in the spirit of the events produced by the D.S. Overall these data sets were the most readily available and therefore used.

Determining what will constitute a connection is a fundamental question. We suggest that the D.S. model for the non linear interaction mechanism that produces the burst will be analyzed to give qualitative predictions. The predictions will be compared to behavior of the corresponding real world objects to see what parts of the model perform accurately and what parts need improvement. A more quantitative study (i.e. measuring the short time tracking capability of D.S.) is underway.

This work is organized as follows: In section 2. we describe the essential ingredients of the D.S. approach. We describe the non linear interaction mechanism suggested to be the mechanism for the production of bursts. Based on the description we deduce some qualitative predictions that will be tested in the sequel. In section 3. we describe the numerical procedures developed in this work to test the predictions outlined in section 2. and the results of the comparison. Section 4. contains a discussion, conclusions, describes the relevance of this work to control applications, and suggests further topics for work and their contribution. Section 5. contains acknowledgements.

2. The D.S. approach and the Non Linear Interaction Mechanism.

The D.S. approach to the wall layer region of the turbulent boundary layer relies on four distinct elements. These are: The Proper Orthogonal Decomposition, Galerkin projection or truncation, a model to describe the feedback of the evolving coherent structures on what is effectively the local mean velocity profile, and a Smagorinsky type subgrid scale model to model the loss of energy to the unresolved modes. For a more thorough exposition the reader is referred to Aubry et al. (1988), Berkooz et al. (1990-1) and Berkooz et al. (1990-3).
The P.O.D. has been described in several references. See Lumley (1970) for the most comprehensive discussion. For our purpose it will suffice to give the following necessary and sufficient condition which characterizes the P.O.D. Given a random signal (assume one dimensionality for simplicity) \( u(x, t) : [a, b] \to \mathbb{R} \) such that \( u(x, t) \in L^2([a, b]) \) for every \( t \). The P.O.D gives an optimal basis for a representation of the signal in the following sense. Let \( \{ \phi_i(x) \}_{i=1}^{\infty} \) be the P.O.D basis and \( \{ \psi_i(x) \}_{i=1}^{\infty} \) is any orthonormal basis s.t (equality in \( L^2 \) sense)

\[
 u(x, t) = \sum_{i=1}^{\infty} a_i(t) \phi_i(x) = \sum_{i=1}^{\infty} b_i(t) \psi_i(x)
\]

where \( a_i(t) \) and \( b_i(t) \) are random coefficients. If \( \langle ., . \rangle \) denotes the time average then

\[
 \sum_{i=1}^{n} \langle a_i a_i^* \rangle \geq \sum_{i=1}^{n} \langle b_i b_i^* \rangle
\]

for every \( n \). Recall that in such a representation \( \langle a_i a_i^* \rangle \) represents the average energy content of the i-th mode. Hence for a given number of modes the P.O.D basis captures the most energy. This is due to the fact that the sum of the \( n \) largest eigenvalues of an operator are greater than the sum of any \( n \) elements on the diagonal w.r.t any basis of the functional space. See Holmes et al. (1990). For an application of the P.O.D to the wall region in a pipe flow see Herzog (1986). For an application in a D.N.S see Moin and Moser (1989). The last two sources discuss the application of the P.O.D in three dimensions, where two of the directions are homogenous, which is the case we are interested in.

Galerkin projection, or truncation, is a tool in common use in C.F.D. Basically one picks a finite number of basis functions and gets dynamical equations for their amplitudes from the equations of motion. It is the core of a spectral method code and in a different sense, for a finite difference code. In our context we perform the same operation, the difference being that we retain a very small number of modes, and instead of picking a numerically convenient basis we use a physically relevant one.

The feedback of the evolving coherent structures on the “local” mean velocity profile is given by the formula:

\[
 U(x_2) = \frac{1}{\nu} \int_0^{x_2} \langle u_1 u_2 \rangle \, dx_2 + \frac{u_2^2}{\nu} \left( x_2 - \frac{x_2^2}{2H} \right) \tag{1}
\]

where \( H \) is a channel half width. See Tennekes and Lumley pg 150. It is derived using time stationarity of the flow. It is then used when in place of \( u_1 \) and \( u_2 \) one puts the dynamical values of these quantities. This proves to be an important element since it provides a feedback that prevents the dynamical coefficients from growing indefinitely as would be the case if one was to put the average value of the mean velocity profile.

The loss to higher modes is modelled by a Smagorinsky type subgrid scale model. The value of the Smagorinsky constant for a given truncation may be computed...
in terms of the eigenfunctions and eigenvalues to within an order 1 number. See Berkooz et al. (1990-2). The order 1 number gives rise to a parameter in the problem called the bifurcation parameter \( \alpha \). This parameter is used to tune the D.S. to obtain physically relevant behavior, corresponding to proper modeling of loss to higher modes.

At this point we should remark that most models studied by our group at Cornell contained no streamwise variation in the representation of the velocity field. It has been shown (Berkooz et al. 1990-2) that this functional subspace has some nice and useful properties. Recently, however, N. Aubry and her student S. Sanghi, have studied higher dimensional models with streamwise variation. See Aubry and Sanghi (1989). Their results show that the essentials of the dynamics described in previous reports persist in these higher dimensional models. A point of nomenclature, we will be discussing dynamics in the functional space, which is of 10 real dimensions if 5 structures are resolved, or the regular physical, 3 dimensional space where the fluid flows. The reader can distinguish between them from the context.

After performing modeling described above and analyzing the resultant D.S. using techniques from bifurcation theory and dynamical systems theory, the following non linear interaction mechanism is suggested. The interaction of the coherent structures is dominated by a spherical type heteroclinic attractor. One can visualize this mechanism as follows, imagine a sphere in the real invariant subspace of phase space. On it there are distinct patches that are "pseudo-attractors", i.e. sets which are capable of attracting the dynamics for a while but points eventually escape them with a small perturbation present. The simplest examples of such systems were studied for the case of the "pseudo attractor" being a hyperbolic fixed point. See Holmes and Stone (1990). In the D.S. of the wall region it is the coupling with the outer layer that provides the perturbation term. The behavior of the wall region may be envisioned as a point travelling close to the sphere, lingering for a while in a well defined area, this corresponds to periods of quiescent behavior of the fluid. The point then jumps toward a new attracting patch, this corresponds to the burst and sweep, until the point settles for a while around a new point, only to start the cycle again - see figure 4.

The attractors described in Aubry et al. (1988) and Berkooz et al. (1990-1) are specific cases of this more general kind of attractor. They consisted of a circle of fixed points, where bursts occurred as jumps from one side of the circle to the other. However, if one wants to describe the general type of attractor that one will find one has to talk about a sphere. In higher dimensions the jumps may occur between different parts of the sphere as different heteroclinic structures may coexist and may be intertwined, thus allowing more freedom in the destination of the jump. This formulation has two advantages. It is independent on the number of modes chosen, and does not produce behavior that might be considered over simplified.

The pressure term that represents the coupling to the outer layer serves to perturb the points and thus prevent them from being attracted to a quiescent state indefinitely. The dynamics of the bursting are thus determined by the dynamics of the large scale structure where the outer layer is a constant source of instability.
A remarkable observation of Armbruster Guckenheimer and Holmes is that these types of attractors are structurally stable (in the D.S sense) in the context of systems with symmetry like that of the system derived from a boundary layer. This implies that the it is possible to meet such attractors for a non negligible set in "parameter space", and that the occurrence of such dynamical features is robust. The original result was proven for 2 dimensional systems. The principle for generalization to higher dimensions can be deduced from the original proof. Recently Holmes and Campbell (1990) and Berkooz (1990-4) have shown generalizations to higher dimensions.

Based on the model described above we can make the following predictions:

1. The p.d.f of the radius of in functional space of the dynamics should have a well defined peak (for a given flow). This is due to the basic conjecture regarding the shape of the attractor.
2. The distribution of points in phase space should be such that intervals of of time corresponding to the quiescent periods of the flow should be clustered together. This is from the conjecture that the quiescent periods correspond to motion around the "pseudo-attractors" around the sphere.
3. One should be able to correlate between streaky motion in phase space and bursting periods.

3. Numerical Methods and Results

3.1 The Data Set Used.

As mentioned earlier the data sets used were those studied by Jimenez and Moin. The simulation was run so as to keep a constant mass flux through the channel by varying the pressure drop. A Reynolds number was compiled based on channel half width and the center line velocity of a parabolic velocity profile which the same mass flux. Specifically we had a Reynolds number of 2000. The friction velocity was approximately 1/22 of the centerline velocity. The computational domain was \( \pi \times 2 \times 0.35\pi \), streamwise, wall normal and spanwise respectively. Our realizations were spaced 0.625 external time units apart, or 3 wall time units. We had 352 realizations totalling 220 external units, or 1056 wall units. The characteristic time between bursts reported by Jimenez and Moin is approximately 100 external time units, thus the flow is expected to contain two bursts. See fig. 1, which contains the time history of the wall shear stress. An increase in the wall shear is associated with the bursting process where low momentum fluid from the near wall region is exchanged with high momentum fluid from higher up, thus increasing the wall shear stress.

3.2 The Eigenfunctions and the Projection.

The eigenfunctions used in this study are based on eigenfunctions from a full channel D.N.S. This posed a requirement to fabricate eigenfunctions for the specific box used. The procedure applied was to interpolate the eigenfunctions, where the interpolation is performed according to wave lengths in wall variables. This procedure seemed to produce reasonable results for this case. This is due to the fact that
the eigenfunctions have a distinct shape, fairly robust as a function of wave number. As R. Moser suggested, it is preferable to interpolate the 2-point velocity correlation tensor and recompute the eigenfunctions. This will also solve the problem that interpolating is not valid for the second eigenfunction, due to loss of orthogonality. The question of interpolating eigenfunctions, or the two point velocity correlation tensor, will be addressed in the sequel.

We decided to choose $y^+ = 39$ as the upper bound for the modeled domain, that gave us 39 grid points. The eigenfunctions were then normalized. The normalized eigenfunctions were used in the study. See fig 5.a, 5.b, 5.c for the streamwise, normal, cross stream components of the eigenfunctions.

Once the eigenfunctions were computed we had to determine which of them correspond to large scale structures. The criterion chosen was that they should have such a shape as to be able to extract energy from interaction with the mean flow (i.e. a positive linear coefficient in the D.S.) and they should also have such a shape that their energy cannot grow without bound, (negative cubic). This selection ruled out eigenfunctions 5,6,7. One can also see by observation that they have fine structure that would generate scales which would not be considered large scale. We also chose not to include the first eigenfunction in the analysis since its average did not go to zero during the period simulated. As Jimenez and Moin point out a long averaging period is needed for the statistics to stabilize. We thus are left with three complex or 6 real variables. See figures 6.a, 6.b, 6.c for the traces used in this study.

3.3 The p.d.f of the Radius in Phase Space.

The p.d.f of the radius in phase space was computed. See figure 2 for the result. The total amount of points was divided into 25 bins. Note that the p.d.f has a very well defined peak. More over it seems to strongly suggest an exponential tail. This is quite remarkable as Holmes and Stone (1990) predicted an exponential tail for the distribution of passage time between bursts. It is conceivable that a similar analysis for the distance from the invariant heteroclinic attractor will also give an exponential tail since the passage time and the distance from the attractor are very intimately linked. (see Holmes and Stone 1990)

3.4 Distribution in Phase Space.

Determining whether points in phase space are clustered together of form a streaky structure is a tricky question. The way we decided to compare “clusters” versus “streaks” is as follows. Given a set of $n$ points in phase space we create a list of distances between all possible pairs $d_1 \ldots d_f$ where $f = n(n - 1)/2$. We would then compute the flatness factor of this list. A higher flatness would indicate more of a streaky group of points. It is also a dimensionless quantity. The following table summarizes our findings.
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We see that there is a correlation between the flatness indicator and the wall shear stress, for the few points we have. Another indicator of dynamical activity is the velocity in phase space. Based on the interaction mechanism described previously one would expect the velocity in phase space to be higher at bursting periods. Figure 3, which shows the velocity in phase space as a function of time shows such a correlation.

4. Summary and Discussion

This study was intended to evaluate numerically qualitative predictions of the Dynamical Systems approach to wall layer turbulence. The predictions were reasonably confirmed, considering the amount of data available and the basis used.

The main achievement of the D.S. approach as it comes to light in this study, is that starting with time average quantities (2-point velocity correlations) one is able to make, through the proposed model for non linear interaction of coherent structures, predictions about dynamical behavior of large scale structures and validate them numerically.

This study suggests a follow up in several areas. First it would be desirable to perform this study on a sample size significantly larger, like the ones used by Jimenez and Moin. Better eigenfunctions could also contribute. Second, it would be important to study the variation of the eigenfunctions and the two point velocity correlation as the geometry changes. One could start with different box sizes for the same geometry and go on to a curved channel etc. This is an important question since as seen by the P.O.D the two point correlation contains important structural information.

This study encourages the pursuit of a practical control scheme for drag reduction or increase of mixing. It confirms our basic assumptions about the relation of the dynamical systems objects we have dealing with and real world objects (like wall shear stress). If the control scheme suggested, i.e. holding the dynamics as close as possible to the hyperbolic points, or the pseudo attractors, succeeds, one will indeed reduce drag. We still are pursuing the measurement of the short term tracking. The analysis performed here can be applied to existing drag reduction techniques, i.e. riblets or the control scheme studied by Moin, Kim, and Choi (1989). It would be interesting and important to gauge the effects of those controls on the dynamics and see whether from a dynamical point of view they share the same principals.
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