Chapter 15

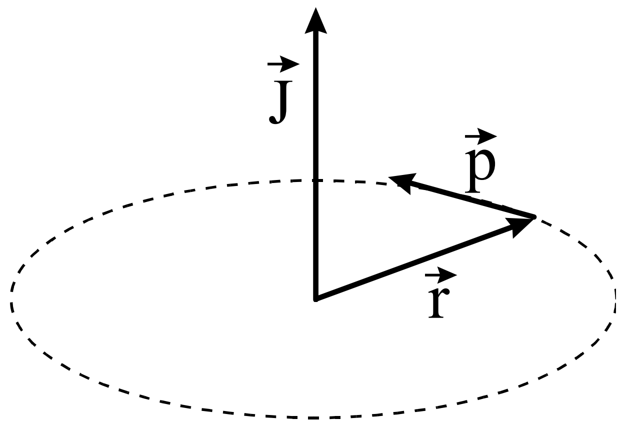
Angular Momentum

Classical

$$\vec{J} = \vec{r} \times \vec{p}$$

radius vector from origin

linear momentum



$$\vec{J} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

determinant form of cross product

$$\hat{i} \rightarrow \hat{x}$$
$$\hat{j} \rightarrow \hat{y}$$
$$\hat{k} \rightarrow \hat{z}$$

$$J_x = y\,p_z - z\,p_y$$

$$J_y = z\,p_x - x\,p_z$$

$$J_z = x\,p_y - y\,p_x$$

$$\vec{J} \cdot \vec{J} = J^2 = J_x^2 + J_y^2 + J_z^2$$

Q.M. Angular Momentum

In the Schrödinger Representation, use Q.M. operators for *x* and *p*, etc.

$$\underline{P}_x = -i\hbar\frac{\partial}{\partial x} \qquad \underline{x} = x$$

Substituting

$$\underline{\vec{J}} = -i\hbar\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \underline{x} & \underline{y} & \underline{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$\underline{J}_x = -i\hbar\left(\underline{y}\frac{\partial}{\partial z} - \underline{z}\frac{\partial}{\partial y}\right)$$

$$\underline{J}_z = -i\hbar\left(\underline{x}\frac{\partial}{\partial y} - \underline{y}\frac{\partial}{\partial x}\right)$$

$$\underline{J}_y = -i\hbar\left(\underline{z}\frac{\partial}{\partial x} - \underline{x}\frac{\partial}{\partial z}\right)$$

$$\underline{\vec{J}} \cdot \underline{\vec{J}} = \underline{J}_x^2 + \underline{J}_y^2 + \underline{J}_z^2$$

Commutators

Consider

$$\left[\underline{J}_x, \underline{J}_y\right] = \underline{J}_x \underline{J}_y - \underline{J}_y \underline{J}_x$$

substituting operators in units of $\hbar$

$$\underline{J}_x \underline{J}_y = -\left(\underline{y}\frac{\partial}{\partial z} - \underline{z}\frac{\partial}{\partial y}\right)\left(\underline{z}\frac{\partial}{\partial x} - \underline{x}\frac{\partial}{\partial z}\right)$$

Keep track of what commutes.

$$= -\left(\underline{y}\frac{\partial}{\partial z}\underline{z}\frac{\partial}{\partial x} - \underline{y}\frac{\partial}{\partial z}\underline{x}\frac{\partial}{\partial z} - \underline{z}\frac{\partial}{\partial y}\underline{z}\frac{\partial}{\partial x} + \underline{z}\frac{\partial}{\partial y}\underline{x}\frac{\partial}{\partial z}\right)$$

Similarly

$$\underline{J}_y \underline{J}_x = -\left(\underline{z}\frac{\partial}{\partial x}\underline{y}\frac{\partial}{\partial z} - \underline{z}\frac{\partial}{\partial x}\underline{z}\frac{\partial}{\partial y} - \underline{x}\frac{\partial}{\partial z}\underline{y}\frac{\partial}{\partial z} + \underline{x}\frac{\partial}{\partial z}\underline{z}\frac{\partial}{\partial y}\right)$$

Subtracting

$$\left[\underline{J}_x, \underline{J}_y\right] = -\left[\underline{y}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\underline{z} - \underline{z}\frac{\partial}{\partial z}\right) + \underline{x}\frac{\partial}{\partial y}\left(\underline{z}\frac{\partial}{\partial z} - \frac{\partial}{\partial z}\underline{z}\right)\right]$$

$$\left[\underline{J}_x,\underline{J}_y\right]=-\left[\underline{y}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\underline{z}-\underline{z}\frac{\partial}{\partial z}\right)+\underline{x}\frac{\partial}{\partial y}\left(\underline{z}\frac{\partial}{\partial z}-\frac{\partial}{\partial z}\underline{z}\right)\right]$$

$$=-\left(\underline{y}\frac{\partial}{\partial x}-\underline{x}\frac{\partial}{\partial y}\right)\left[\frac{\partial}{\partial z},\underline{z}\right]$$

$$=\left(\underline{x}\frac{\partial}{\partial y}-\underline{y}\frac{\partial}{\partial x}\right)\left[\frac{\partial}{\partial z},\underline{z}\right]$$

$$=i\underline{J}_z\left[\frac{\partial}{\partial z},\underline{z}\right]$$

But $\left[\frac{\partial}{\partial z},\underline{z}\right]=1$ **because** $\frac{\partial}{\partial \underline{z}}=\frac{\underline{P}_z}{-i\hbar}$

$$-\frac{1}{i\hbar}\left[\underline{P}_z,\underline{z}\right]=-\frac{1}{i\hbar}(-1)\left[\underline{z},\underline{P}_z\right]$$ **Using** $\left[\underline{z},\underline{P}_z\right]=i\hbar$

$$=\frac{1}{i\hbar}(i\hbar)=1$$

Therefore,

$$\left[\underline{J}_x,\underline{J}_y\right]=i\underline{J}_z$$

$$\left[\underline{J}_x,\underline{J}_y\right]=i\hbar\underline{J}_z$$ **in conventional units**

The commutators in units of $\hbar$ are

$$\left[\underline{J}_x, \underline{J}_y\right] = i\,\underline{J}_z$$

$$\left[\underline{J}_y, \underline{J}_z\right] = i\,\underline{J}_x$$

$$\left[\underline{J}_z, \underline{J}_x\right] = i\,\underline{J}_y .$$

Using these it is found that

$$\left[\underline{J}^2, \underline{J}_z\right] = \left[\underline{J}^2, \underline{J}_x\right] = \left[\underline{J}^2, \underline{J}_y\right] = 0$$

Components of angular momentum do not commute.

$\underline{J}^2$ commutes with all components.

Therefore,

$\underline{J}^2$ and one component of angular momentum can be measured simultaneously.

Call this component $\underline{J}_z$.

Therefore,

$\underline{J}^2$ and $\underline{J}_z$ matrices can be simultaneously diagonalized by the same unitary transformation.

Furthermore,

$$\left[\underline{H},\underline{J}\right]=0$$ **($\underline{J}$ looks like rotation)**

Therefore,

$$\left[\underline{H},\underline{J}^2\right]=0$$

$\underline{H}$, $\underline{J}^2$, $\underline{J}_z$ are all simultaneous observables.

Diagonalization of $\underline{J}^2$ and $\underline{J}_z$

$\underline{J}^2$ and $\underline{J}_z$ commute.

Therefore, set of vectors

$$|\lambda\, m\rangle$$

Labeling kets with eigenvalues.

are eigenvectors of both operators.

$\underline{\underline{J}}^2$ and $\underline{\underline{J}}_z$ are simultaneously diagonal in the basis $|\lambda\, m\rangle$

$$\underline{J}^2|\lambda\, m\rangle = \lambda|\lambda\, m\rangle$$

$$\underline{J}_z|\lambda\, m\rangle = m|\lambda\, m\rangle$$

(in units of $\hbar$)

Form operators

$$\underline{J}_{+} = \underline{J}_{x} + i\,\underline{J}_{y} \qquad\qquad \underline{J}_{-} = \underline{J}_{x} - i\,\underline{J}_{y}$$

From the definitions of $\underline{J}_{+}$ and $\underline{J}_{-}$ and the angular momentum commutators, the following commutators and identities can be derived.

Commutators

$$\left[\underline{J}_{+}, \underline{J}_{z}\right] = -\underline{J}_{+}$$

$$\left[\underline{J}_{-}, \underline{J}_{z}\right] = \underline{J}_{-}$$

$$\left[\underline{J}_{+}, \underline{J}_{-}\right] = 2\underline{J}_{z}$$

Identities

$$\underline{J}_{+}\underline{J}_{-} = \underline{J}^{2} - \underline{J}_{z}^{2} + \underline{J}_{z}$$

$$\underline{J}_{-}\underline{J}_{+} = \underline{J}^{2} - \underline{J}_{z}^{2} - \underline{J}_{z}$$

Expectation value

$$\langle \lambda m | \underline{J}^2 | \lambda m \rangle \geq \langle \lambda m | \underline{J}_z^2 | \lambda m \rangle$$

Because

$$\langle \lambda m | \underline{J}^2 | \lambda m \rangle = \langle \lambda m | \underline{J}_z^2 | \lambda m \rangle + \langle \lambda m | \underline{J}_x^2 | \lambda m \rangle + \langle \lambda m | \underline{J}_y^2 | \lambda m \rangle$$

Positive numbers because $\underline{J}$'s are Hermitian – give real numbers. Square of real numbers – positive.

Therefore,
the sum of three positive numbers is greater than or equal to one of them.

Now

$$\langle \lambda m | \underline{J}^2 | \lambda m \rangle = \lambda$$

$$\langle \lambda m | \underline{J}_z^2 | \lambda m \rangle = m^2$$

Therefore,

$$\lambda \geq m^2$$

Eigenvalues of $\underline{J}^2$ are greater than or equal to square of eigenvalues of $\underline{J}_z$.

Using

$$\left[\underline{J}_+,\underline{J}_z\right]=-\underline{J}_+$$

$$\Longrightarrow\quad \underline{J}_z\underline{J}_+=\underline{J}_+\underline{J}_z+\underline{J}_+$$

Consider

$$\underline{J}_z\left[\underline{J}_+|\lambda m\rangle\right]=\underline{J}_+\underline{J}_z|\lambda m\rangle+\underline{J}_+|\lambda m\rangle$$

$$=\underline{J}_+m|\lambda m\rangle+\underline{J}_+|\lambda m\rangle$$

$$=(m+1)\left[\underline{J}_+|\lambda m\rangle\right]$$

eigenvalue **eigenvector**

Furthermore,

$$\left[\underline{J}^2,\underline{J}_+\right]=0$$

$\underline{J}^2$ commutes with $\underline{J}_+$ because it commutes with $\underline{J}_x$ and $\underline{J}_y$.

Then

$$\underline{J}^2\left[\underline{J}_+|\lambda m\rangle\right]=\underline{J}_+\underline{J}^2|\lambda m\rangle$$

$$=\lambda\left[\underline{J}_+|\lambda m\rangle\right]$$

eigenvalue **eigenvector**

$$\underline{J}_z\left[\underline{J}_+\left|\lambda\, m\right\rangle\right]=\left(m+1\right)\left[\underline{J}_+\left|\lambda\, m\right\rangle\right]$$

eigenvalue **eigenvector**

$$\underline{J}^2\left[\underline{J}_+\left|\lambda\, m\right\rangle\right]=\lambda\left[\underline{J}_+\left|\lambda\, m\right\rangle\right]$$

eigenvalue **eigenvector**

Thus,

$\underline{J}_+\left|\lambda m\right\rangle$ **is eigenvector of $\underline{J}_z$ with eigenvalue $m + 1$**

and of $\underline{J}^2$ with eigenvalue λ.

$\underline{J}_+$ is a raising operator.

It increases m by 1

and leaves λ unchanged.

Repeated applications of

$$\underline{J}_+ \quad \text{to} \quad |\lambda\, m\rangle$$

gives new eigenvectors of $\underline{J}_z$ (and $\underline{J}^2$) with larger and larger values of *m*.

But,

this must stop at a largest value of *m*, m_{max}

because

$$\lambda \geq m^2. \qquad \text{(}m\text{ increases, }\lambda\text{ doesn't change)}$$

Call largest value of *m* (m_{max}) *j*.

$$m_{max} = j$$

For this value of *m*, that is, $m = j$

$$\underline{J}_+ |\lambda\, j\rangle = 0 \qquad \text{with} \qquad |\lambda\, j\rangle \neq 0$$

Can't raise past max value.

In similar manner can prove

$$\underline{J}_- \left| \lambda\, m \right\rangle$$

is an eigenvector of $\underline{J}_z$ with eigenvalues $m - 1$
and of $\underline{J}^2$ with eigenvalues λ.

Therefore,

$\underline{J}_-$ **is a lowering operator.**

It reduces the value of m by 1 and leaves λ unchanged.

Operating $\underline{J}_-$ repeatedly on $\left| \lambda\, j \right\rangle$

$$\underline{J}_- \left| \lambda\, j \right\rangle$$

largest value of m

gives eigenvectors with sequence of m eigenvalues

$$m = j,\ j-1,\ j-2,\ \cdots$$

But,

$$\lambda \geq m^2$$

Therefore, can't lower indefinitely.

Must be some

$$\left|\lambda\, j'\right\rangle$$

such that

$$\underline{J}_-\left|\lambda\, j'\right\rangle = 0 \qquad \text{with} \qquad \left|\lambda\, j'\right\rangle \neq 0$$

Smallest value of m.
Can't lower below smallest value.

Thus,

$$j = j' + \text{an integer.}$$

largest value of m

smallest value of m

Went from largest value to smallest value in unit steps.

We have

largest value of m

$$\underline{J}_+|\lambda\, j\rangle = 0$$

$$\underline{J}_-|\lambda\, j'\rangle = 0$$

smallest value of m

Left multiplying top equation by $\underline{J}_-$ and bottom equation by $\underline{J}_+$

$$\underline{J}_-\underline{J}_+|\lambda\, j\rangle = 0$$

$$\underline{J}_+\underline{J}_-|\lambda\, j'\rangle = 0$$

identities

$$\underline{J}_-\underline{J}_+ = \underline{J}^2 - \underline{J}_z^2 - \underline{J}_z$$

$$\underline{J}_+\underline{J}_- = \underline{J}^2 - \underline{J}_z^2 + \underline{J}_z$$

Then

$$\underline{J}_-\underline{J}_+|\lambda\, j\rangle = 0 = \left(\underline{J}^2 - \underline{J}_z^2 - \underline{J}_z\right)|\lambda\, j\rangle$$

$$\underline{J}_+\underline{J}_-|\lambda\, j'\rangle = 0 = \left(\underline{J}^2 - \underline{J}_z^2 + \underline{J}_z\right)|\lambda\, j'\rangle$$

and operating

$$\underline{J}_-\underline{J}_+|\lambda\, j\rangle = 0 = \left(\lambda - j^2 - j\right)|\lambda\, j\rangle$$

$$\underline{J}_+\underline{J}_-|\lambda\, j'\rangle = 0 = \left(\lambda - j'^2 + j'\right)|\lambda\, j'\rangle$$

$$\underline{J}_-\underline{J}_+\left|\lambda\, j\right\rangle = 0 = \left(\lambda - j^2 - j\right)\left|\lambda\, j\right\rangle \qquad \underline{J}_+\underline{J}_-\left|\lambda\, j'\right\rangle = 0 = \left(\lambda - j'^2 + j'\right)\left|\lambda\, j'\right\rangle$$

Because $\left|\lambda\, j\right\rangle \neq 0$ **and** $\left|\lambda\, j'\right\rangle \neq 0$

the coefficients of the kets must equal 0.

Therefore,

$$\lambda = j(j+1) \qquad \text{and} \qquad \lambda = (-j')(-j'+1)$$

Because $j > j'$

$$j' = -j$$

and

2*j* = an integer $\qquad j = \text{integer}/2;$ ***j* can have integer or half integer values.**

because we go from *j* to *j'* = -*j* in unit steps with lowering operator $\underline{J}_-$.

Thus, the eigenvalues of $\underline{J}^2$ are

$$\lambda = j(j+1) \qquad \text{and} \qquad j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2}, \cdots \quad \text{(largest } m \text{ for a } \lambda\text{)}$$

The eigenvalues of $\underline{J}_z$ are $m = j,\ j-1, \cdots, -j+1,\ -j$

largest *m* $\qquad$ **change by unit steps** $\qquad$ **smallest value of *m***

Final results

$$\underline{J}^2 \left| jm \right\rangle = j(j+1) \left| jm \right\rangle$$

$$\underline{J}_z \left| jm \right\rangle = m \left| jm \right\rangle$$

There are $(2j + 1)$ m-states for a given j, going from j to $-j$ in integer steps.

Can derive

$$\underline{J}_+ \left| jm \right\rangle = \sqrt{\left(j-m\right)\left(j+m+1\right)} \left| jm+1 \right\rangle$$

$$\underline{J}_- \left| jm \right\rangle = \sqrt{\left(j+m\right)\left(j-m+1\right)} \left| jm-1 \right\rangle$$

Angular momentum states can be grouped by the value of j.
Eigenvalues of $\underline{J}^2$, $\lambda = j(j+1)$.

$$j = 0,\ 1/2,\ 1,\ 3/2,\ 2, \cdots$$

$$j = 0 \qquad m = 0 \qquad\qquad |0\,0\rangle$$

$$j = 1/2 \qquad m = 1/2,\ -1/2 \qquad\qquad \left|\frac{1}{2}\ \frac{1}{2}\right\rangle \ \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$$

$$j = 1 \qquad m = 1,\ 0,\ -1 \qquad\qquad |1\ 1\rangle\ |1\ 0\rangle\ |1\ -1\rangle$$

$$j = 3/2 \qquad m = 3/2,\ 1/2,\ -1/2,\ -3/2 \qquad\qquad \left|\frac{3}{2}\ \frac{3}{2}\right\rangle \left|\frac{3}{2}\ \frac{1}{2}\right\rangle \left|\frac{3}{2}\ -\frac{1}{2}\right\rangle \left|\frac{3}{2}\ -\frac{3}{2}\right\rangle$$

$$j = 2 \qquad m = 2,\ 1,\ 0,\ -1,\ -2 \qquad\qquad |2\ 2\rangle\ |2\ 1\rangle\ |2\ 0\rangle\ |2\ -1\rangle\ |2\ -2\rangle$$

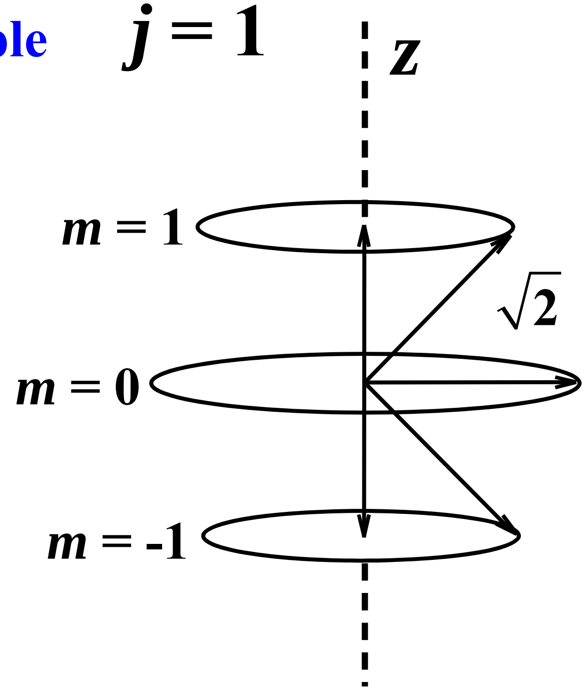
etc.

Eigenvalues of $\underline{J}^2$ are the square of the total angular momentum.

The length of the angular momentum vector is

$\sqrt{j(j+1)}$ **or in conventional units** $\hbar\sqrt{j(j+1)}$

Example $j = 1$



Eigenvalues of $\underline{J}_z$ are the projections of the angular momentum on the *z* axis.

The matrix elements of $\underline{J}^2$ $\underline{J}_z$ $\underline{J}_+$ $\underline{J}_-$ are

$$\langle j'm'|\underline{J}^2|jm\rangle = j(j+1) \quad \delta_{j'j}\delta_{m',m}$$

$$\langle j'm'|\underline{J}_z|jm\rangle = m \quad \delta_{j'j}\delta_{m',m}$$

$$\langle j'm'|\underline{J}_+|jm\rangle = \sqrt{(j-m)(j+m+1)} \quad \delta_{j'j}\delta_{m',m+1}$$

$$\langle j'm'|\underline{J}_-|jm\rangle = \sqrt{(j+m)(j-m+1)} \quad \delta_{j'j}\delta_{m',m-1}$$

The matrices for the first few values of j are (in units of $\hbar$)

$j = 0$

$$\underline{\underline{J}}_+ = (0) \qquad \underline{\underline{J}}_- = (0)$$

$$\underline{\underline{J}}_z = (0) \qquad \underline{\underline{J}}^2 = (0)$$

$j = 1/2$

$$\underline{\underline{J}}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \underline{\underline{J}}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\underline{\underline{J}}_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \quad \underline{\underline{J}}^2 = \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}$$

$j = 1$

$$\underline{\underline{J}}_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{\underline{J}}_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\underline{\underline{J}}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \underline{\underline{J}}^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The $|j\,m\rangle$ are eigenkets of the $\underline{J}^2$ and $\underline{J}_z$ operators – diagonal matrices.

The raising and lowering operators $\underline{J}_+$ and $\underline{J}_-$ have matrix elements one step above and one step below the principal diagonal, respectively.

Particles such as atoms

$$|\psi\rangle = R(r)Y_\ell^m(\theta,\varphi)$$

spherical harmonics from solution of H atom

The $Y_\ell^m(\theta,\varphi)$ are the eigenvectors of the operators

$\underline{L}^2$ and $\underline{L}_z$.

The

$$Y_\ell^m(\theta,\varphi) = |\, j\, m\rangle = |\,\ell\, m\rangle$$

$$\underline{L}^2\, Y_\ell^m(\theta,\varphi) = \ell(\ell+1)Y_\ell^m(\theta,\varphi)$$

$$\underline{L}_z\, Y_\ell^m(\theta,\varphi) = m\, Y_\ell^m(\theta,\varphi)$$

Addition of Angular Momentum

Examples

Orbital and spin angular momentum - ℓ and s.
These are really coupled – spin-orbit coupling.

ESR – electron spins coupled to nuclear spins

Inorganic spectroscopy – unpaired d electrons

Molecular excited triplet states – two unpaired electrons

Could consider separate angular momentum vectors
j_1 and j_2.
These are distinct.
But will see, that when they are coupled, want to combine the angular momentum vectors into one resultant vector.

Specific Case

$$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$$

$$m_1 = \pm\frac{1}{2} \qquad m_2 = \pm\frac{1}{2}$$

Four product states

$$\begin{array}{lcl} j_1\ m_1\ j_2\ m_2 & & m_1 m_2 \\ \left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle & = & \left|\frac{1}{2}\ \frac{1}{2}\right\rangle \end{array}$$

j_1 and j_2 omitted because they are always the same.

$$\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$$

Called the $m_1 m_2$ representation

$$\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = \left|-\frac{1}{2}\ \frac{1}{2}\right\rangle$$

The two angular momenta are considered separately.

$$\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\ -\frac{1}{2}\right\rangle$$

$|m_1 m_2\rangle \rightarrow |j_1 j_2 m_1 m_2\rangle$ **$m_1 m_2$ representation**

Want different representation $\rightarrow$ **Unitary Transformation to coupled rep.**
Angular momentum vectors added.

New States labeled $|jm\rangle$

$$|j_1 j_2 jm\rangle = |jm\rangle$$

jm representation

$\left|jm\right\rangle$ ➡ Eigenkets of operators in *jm* representation.

$\underline{J}^2$ and $\underline{J}_z$

where

$$\underline{J} = \underline{J}_1 + \underline{J}_2$$

$$\underline{J}_z = \underline{J}_{1z} + \underline{J}_{2z}$$

$$\underline{J}^2 \left|jm\right\rangle = j(j+1)\left|jm\right\rangle$$

vector sum of j_1 and j_2

$$\underline{J}_z \left|jm\right\rangle = m\left|jm\right\rangle$$

Want unitary transformation from the $m_1 m_2$ representation to the *jm* representation.

Want

$$|jm\rangle = \sum_{m_1 m_2} C_{m_1 m_2} |m_1 m_2\rangle$$

$$C_{m_1 m_2} = \langle m_1 m_2 | jm\rangle$$

$C_{m_1 m_2}$ **are the Clebsch-Gordan coefficients; Wigner coefficients; vector coupling coefficients**

$|m_1 m_2\rangle$ **are the basis vectors**

N **states in the** $m_1 m_2$ **representation** $\Longrightarrow$ ***N*** **states in the** jm **representation.**

$$N = (2j_1 + 1)(2j_2 + 1)$$

$\underline{J}^2$ **and** $\underline{J}_z$ **obey the normal commutator relations.**

Prove by using $\underline{J} = \underline{J}_1 + \underline{J}_2$
and cranking through commutator relations
using the fact that $\underline{J}_1$ **and** $\underline{J}_2$ **and their components commute.**
Operators operating on different state spaces commute.

Finding the transformation

$$\underline{J}_z = \underline{J}_{1z} + \underline{J}_{2z} \quad \longrightarrow \quad m = m_1 + m_2$$ **or coupling coefficient vanishes.**

To see this consider

$$\left| jm \right\rangle = \sum_{m_1 m_2} C_{m_1 m_2} \left| m_1 m_2 \right\rangle$$

Operate with $\underline{J}_z$

equal

$$\underline{J}_z \left| jm \right\rangle = m \left| jm \right\rangle = \left(\underline{J}_{1z} + \underline{J}_{2z} \right) \sum_{m_1 m_2} C_{m_1 m_2} \left| m_1 m_2 \right\rangle$$

$$= \sum_{m_1 m_2} \left(m_1 + m_2 \right) C_{m_1 m_2} \left| m_1 m_2 \right\rangle$$

These must be equal.

Other terms

$C_{m_1 m_2} = 0$

if

$m_1 + m_2 \neq m$

Largest value of m

$$m = j_1 + j_2 = m_1^{\max} + m_2^{\max}$$

since largest

$$m_1 = j_1 \qquad \text{and} \qquad m_2 = j_2$$

Then the largest value of j is

$$j = j_1 + j_2$$

because the largest value of j equals the largest value of m.

There is only one state with the largest j and m.

There are a total of $(2j + 1)$ m states associated with the largest $j = j_1 + j_2$.

Next largest *m* (*m* – 1)

$$m = j_1 + j_2 - 1$$

But $m = m_1 + m_2$

Two ways to get *m* - 1

$$m_1 = j_1 \text{ and } m_2 = j_2 - 1$$

$$m_1 = j_1 - 1 \text{ and } m_2 = j_2$$

Can form two orthogonal and normalized combinations.

One of the combinations belongs to

$$j = j_1 + j_2$$

Because this value of j has *m* values

$$m = (j_1 + j_2), (j_1 + j_2 - 1), \cdots, (-j_1 - j_2)$$

Other combination with $m = j_1 + j_2 - 1$

$\Rightarrow$ $j' = j_1 + j_2 - 1$

with $m = \underbrace{(j_1 + j_2 - 1)}_{\text{largest}}, (j_1 + j_2 - 2), \cdots, \underbrace{(-j_1 - j_2 + 1)}_{\text{smallest}}$

Doing this repeatedly

$\longrightarrow$ j **values from**

$$j = j_1 + j_2 \ \text{to} \ \left| j_1 - j_2 \right|$$ **in unit steps**

Each j has associated with it,
its $2j + 1$ m values.

Example

$$j_1 = \frac{1}{2}, \qquad j_2 = \frac{1}{2}$$

j values $\longrightarrow$ $j = j_1 + j_2$ to $|j_1 - j_2|$ in unit steps.

$$j = \frac{1}{2} + \frac{1}{2} = 1$$

$$j = \frac{1}{2} - \frac{1}{2} = 0$$

$$j = 1 \quad m = 1,\ 0,\ -1$$

$$j = 0 \quad m = 0$$

jm rep. kets $\qquad |1\,1\rangle, |1\,0\rangle, |1\,-1\rangle, |0\,0\rangle$

$m_1 m_2$ rep. kets $\qquad \left|\frac{1}{2}\,\frac{1}{2}\right\rangle, \left|\frac{1}{2}\,-\frac{1}{2}\right\rangle, \left|-\frac{1}{2}\,\frac{1}{2}\right\rangle, \left|-\frac{1}{2}\,-\frac{1}{2}\right\rangle$

Know jm kets $\Longrightarrow$ still need correct combo's of $m_1 m_2$ rep. kets

Generating procedure

Start with the *jm* ket with the largest value of *j* and the largest value of *m*.

$$|11\rangle$$

$$\underline{J}_z|1\,1\rangle = 1|1\,1\rangle \longrightarrow m = 1$$

But $m = m_1 + m_2$

Therefore,

$$m_1 = \frac{1}{2} \quad m_2 = \frac{1}{2}$$

because this is the only way to get

$$m_1 + m_2 = 1$$

Then $\underbrace{|11\rangle}_{jm} = \underbrace{\left|\frac{1}{2}\,\frac{1}{2}\right\rangle}_{m_1 m_2}$

Clebsch-Gordan coefficient = 1

Use lowering operators

$$\underline{J}_{-} = \underline{J}_{1-} + \underline{J}_{2-}$$

jm *$m_1 m_2$*

$$|11\rangle = \left|\frac{1}{2}\ \frac{1}{2}\right\rangle$$

jm *$m_1 m_2$*

$$\underline{J}_{-}|11\rangle = \sqrt{2}\,|10\rangle$$

from lowering op. expression

$$= \left(\underline{J}_{1-} + \underline{J}_{2-}\right)\left|\frac{1}{2}\frac{1}{2}\right\rangle = \underline{J}_{1-}\left|\frac{1}{2}\frac{1}{2}\right\rangle + \underline{J}_{2-}\left|\frac{1}{2}\frac{1}{2}\right\rangle$$

$$= 1\left|-\frac{1}{2}\ \frac{1}{2}\right\rangle + 1\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$$

from lowering op. expression (Use correct j_i and m_i values.)

Then

$$|10\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|-\frac{1}{2}\ \frac{1}{2}\right\rangle$$

Clebsch-Gordan Coefficients

Plug into raising and lowering op. formulas correctly.

$$\underline{J}_{+}\left|j\,m\right\rangle=\sqrt{(j-m)(j+m+1)}\left|j\,m+1\right\rangle$$

$$\underline{J}_{-}\left|j\,m\right\rangle=\sqrt{(j+m)(j-m+1)}\left|j\,m-1\right\rangle$$

For *jm* rep. $\longrightarrow \left|jm\right\rangle$
plug in *j* and *m*.

For m_1m_2 rep. $\longrightarrow \left|m_1m_2\right\rangle$

$$\left|m_1m_2\right\rangle \;\; \textbf{means} \;\; \left|j_1 j_2 m_1 m_2\right\rangle$$

For $\underline{J}_{1-}$ and $\underline{J}_{2-}$ must put in

j_1 and m_1 when operating with $\underline{J}_{1-}$

and

j_2 and m_2 when operating with $\underline{J}_{2-}$

Lowering again

$$\underline{J}_- \left|1\,0\right\rangle = \sqrt{2}\left|1\,-1\right\rangle$$

$$\begin{array}{cc} m_1\ m_2 & m_1\ m_2 \end{array}$$

$$= \left(\underline{J}_{1-} + \underline{J}_{2-}\right)\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle + \left|-\frac{1}{2}\ \frac{1}{2}\right\rangle\right)$$

$$= \left[\frac{1}{\sqrt{2}}\left|-\frac{1}{2}\ -\frac{1}{2}\right\rangle + 0 + 0 + \frac{1}{\sqrt{2}}\left|-\frac{1}{2}\ -\frac{1}{2}\right\rangle\right]$$

Therefore,

$$\underset{jm}{\left|1\,-1\right\rangle} = \underset{m_1 m_2}{\left|-\frac{1}{2}\ -\frac{1}{2}\right\rangle}$$

Have found the three m states for $j = 1$ in terms of the $m_1 m_2$ states.

Still need $\left|00\right\rangle$

$$m = 0 = m_1 + m_2$$

Need jm $|0\,0\rangle$

$$m = 0$$

$$\therefore \quad m_1 + m_2 = 0$$

Two $m_1 m_2$ kets with $m_1 + m_2 = 0$

$$\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle, \left|-\frac{1}{2}\ \frac{1}{2}\right\rangle$$

The $|0\,0\rangle$ is a superposition of these.

Have already used one superposition of these to form $|1\,0\rangle$

$$|10\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|-\frac{1}{2}\ \frac{1}{2}\right\rangle$$

$|0\,0\rangle$ orthogonal to $|1\,0\rangle$ and normalized. Find combination of $\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle, \left|-\frac{1}{2}\ \frac{1}{2}\right\rangle$ normalized and orthogonal to $|1\,0\rangle$.

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left|-\frac{1}{2}\ \frac{1}{2}\right\rangle$$

Clebsch-Gordan Coefficients

Table of Clebsch-Gordan Coefficients

$j_1 = 1/2$

$j_2 = 1/2$

m_1	m_2	$j=1$, $m=1$	$j=1$, $m=0$	$j=0$, $m=0$	$j=1$, $m=-1$
$\frac{1}{2}$	$\frac{1}{2}$	1			
$\frac{1}{2}$	$-\frac{1}{2}$		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	
$-\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	
$-\frac{1}{2}$	$-\frac{1}{2}$				1

Next largest system

$$j_1 = 1 \qquad\qquad j_2 = \frac{1}{2}$$

$$m_1 = 1, 0, -1 \qquad\qquad m_2 = \frac{1}{2}, -\frac{1}{2}$$

$m_1 m_2$ **kets**

$$\left|1\,\frac{1}{2}\right\rangle \quad \left|1 - \frac{1}{2}\right\rangle \quad \left|0\,\frac{1}{2}\right\rangle \quad \left|0 - \frac{1}{2}\right\rangle \quad \left|-1\,\frac{1}{2}\right\rangle \quad \left|-1 - \frac{1}{2}\right\rangle$$

jm **states**

$$j = j_1 + j_2 = \frac{3}{2} \qquad\qquad m = \frac{3}{2}, \quad \frac{1}{2}, \quad -\frac{1}{2}, \quad -\frac{3}{2}$$

$$j = j_1 - j_2 = \frac{1}{2} \qquad\qquad m = \frac{1}{2}, \quad -\frac{1}{2}$$

jm **kets**

$$\left|\frac{3}{2}\frac{3}{2}\right\rangle \quad \left|\frac{3}{2}\frac{1}{2}\right\rangle \quad \left|\frac{3}{2}\;-\frac{1}{2}\right\rangle \quad \left|\frac{3}{2}\;-\frac{3}{2}\right\rangle \quad \left|\frac{1}{2}\frac{1}{2}\right\rangle \left|\frac{1}{2}\;-\frac{1}{2}\right\rangle$$

Table of Clebsch-Gordan Coefficients

$j_1 = 1$
$j_2 = 1/2$

	j	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
	m	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
1	$\frac{1}{2}$	1					
1	$-\frac{1}{2}$		$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$			
0	$\frac{1}{2}$		$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$			
0	$-\frac{1}{2}$				$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	
-1	$\frac{1}{2}$				$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	
-1	$-\frac{1}{2}$						1
m_1	m_2						

Example

$$\overset{j\,m}{\left|\frac{1}{2}\,\frac{1}{2}\right\rangle} = \sqrt{\frac{2}{3}}\overset{m_1\,m_2}{\left|1\;-\frac{1}{2}\right\rangle} - \sqrt{\frac{1}{3}}\overset{m_1\,m_2}{\left|0\,\frac{1}{2}\right\rangle}$$