AA242B: MECHANICAL VIBRATIONS

Undamped Vibrations of n-DOF Systems

These slides are based on the recommended textbook: M. Géradin and D. Rixen, "Mechanical Vibrations: Theory and Applications to Structural Dynamics," Second Edition, Wiley, John & Sons, Incorporated, ISBN-13:9780471975465

Outline

- Equilibrium configuration

$$q_s(t) = q_s(0), \qquad \dot{q}_s(t) = 0, \qquad s = 1, \cdots, n \tag{1}$$

- Recall the Lagrange equations of motion

$$-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}}{\partial q_s} - \frac{\partial \mathcal{V}}{\partial q_s} - \frac{\partial D}{\partial \dot{q}_s} + Q_s(t) = 0$$

where $\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2$

- Recall the generalized gyroscopic forces

$$F_s = -\sum_{r=1}^{n} \frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r}\dot{q}_r + \frac{\partial \mathcal{T}_1}{\partial q_s} = \sum_{r=1}^{n}\left(\frac{\partial^2 \mathcal{T}_1}{\partial q_s \partial \dot{q}_r} - \frac{\partial^2 \mathcal{T}_1}{\partial q_r \partial \dot{q}_s}\right)\dot{q}_r, \quad s = 1, \cdots, n$$

- **<u>Definition</u>**: the effective potential energy is defined as

$$\boxed{\mathcal{V}^\star = \mathcal{V} - \mathcal{T}_0 = \mathcal{V}^\star(\mathbf{q}, t)}$$

- The Lagrange equations of motion can be re-written as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) - \frac{\partial \mathcal{T}_2}{\partial q_s} = Q_s(t) - \frac{\partial \mathcal{V}^\star}{\partial q_s} - \frac{\partial D}{\partial \dot{q}_s} + F_s - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}\right)$$

- Recall that

$$\begin{aligned} \mathcal{T}_0(\mathbf{q},t) &= \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{3} m_k \left(\frac{\partial U_{i_k}}{\partial t}(\mathbf{q},t)\right)^2 && \text{(transport kinetic energy)} \\ \mathcal{D} &= \sum_{k=1}^{N}\int_0^{v_k(\dot{\mathbf{q}})} C_k f_k(\gamma)d\gamma && \text{(dissipation function)} \end{aligned}$$

- From the Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) - \frac{\partial \mathcal{T}_2}{\partial q_s} = Q_s(t) - \frac{\partial \mathcal{V}^\star}{\partial q_s}(\mathbf{q},t) - \frac{\partial D}{\partial \dot{q}_s} + F_s - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}\right)$$

it follows that an equilibrium configuration exists if and only if

$$0 = Q_s(t) - \frac{\partial \mathcal{V}^\star}{\partial q_s}(\mathbf{q},t)\ !$$

- Hence, at equilibrium

$$\boxed{Q_s(t) = 0 \quad \text{and} \quad \frac{\partial \mathcal{V}^\star}{\partial q_s} = \frac{\partial(\mathcal{V}-\mathcal{T}_0)}{\partial q_s} = 0, \quad s = 1,\cdots,n}$$

- Consider *first* a system that does not undergo a transport or overall motion $\Rightarrow \mathcal{T} = \mathcal{T}_2(\dot{\mathbf{q}})$
- The equilibrium position is then given by

$$\boxed{Q_s(t) = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{V}}{\partial q_s} = 0, \quad s = 1, \cdots, n}$$

- Assume next that this system is conservative $\Rightarrow \mathcal{E} = \mathcal{T} + \mathcal{V} = cst$
- Shift the origin of the generalized coordinates so that at equilibrium, $q_s = 0, \quad s = 1, \cdots, n$ (in which case the q_s represent the deviation from equilibrium)
- Since the potential energy is defined only up to a constant, choose this constant so that $\mathcal{V}(q_s = 0) = 0$
- Now, suppose that a certain energy $\mathcal{E}(0)$ is initially given to the system in equilibrium

- **Definition**: the equilibrium position $(q_s = 0, s = 1, \cdots, n)$ is said to be stable if

$$\boxed{\exists\ \mathcal{E}^\star\ /\ \forall\ \mathcal{E}(0) < \mathcal{E}^\star, \quad \mathcal{T}(t) \leq \mathcal{E}(0)}$$

- Consequences
  - $\mathcal{T} + \mathcal{V} = \mathcal{E} = cst = \mathcal{E}(0) \Rightarrow \boxed{\mathcal{V}(t) = \mathcal{E}(0) - \mathcal{T}(t)) \geq 0}$
  - at a stable equilibrium position, the potential energy is at a relative minimum
  - if $\mathcal{E}(0)$ is small enough, $\mathcal{V}(t)$ will be small enough $\Rightarrow$ and therefore deviations from the equilibrium position will be small enough

- "Linearization" of $\mathcal{T}$ and $\mathcal{V}$ around an equilibrium position $(q_s = 0, \dfrac{\partial \mathcal{V}}{\partial q_s} = 0)$
  - actually, this means obtaining a quadratic form of $\mathcal{T}$ and $\mathcal{V}$ in $\mathbf{q}$ and $\dot{\mathbf{q}}$, respectively, so that the corresponding generalized forces are linear
  - since $q_s(t)$ represent deviations from equilibrium, $\mathcal{V}$ can be expanded as follows

$$\begin{aligned}\mathcal{V}(\mathbf{q}) &= \mathcal{V}(0) + \sum_{s=1}^{n} \frac{\partial \mathcal{V}}{\partial q_s}\,|_{\mathbf{q}=0}\; q_s + \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} \frac{\partial^2 \mathcal{V}}{\partial q_s \partial q_r}\,|_{\mathbf{q}=0}\; q_s q_r + \mathcal{O}(\mathbf{q}^3) \\ &= \mathcal{V}(0) + \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} \frac{\partial^2 \mathcal{V}}{\partial q_s \partial q_r}\,|_{\mathbf{q}=0}\; q_s q_r + \mathcal{O}(\mathbf{q}^3)\end{aligned}$$

  - since the potential energy is defined only up to a constant, if this constant is chosen so that $\mathcal{V}(0) = 0$, then a second-order approximation of $\mathcal{V}(\mathbf{q})$ is given by

$$\boxed{\mathcal{V}(\mathbf{q}) = \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} \frac{\partial^2 \mathcal{V}}{\partial q_s \partial q_r}\,|_{\mathbf{q}=0}\; q_s q_r, \qquad \text{for } \mathbf{q} \neq 0}$$

- Stiffness matrix
  - let $\mathbf{K} = [k_{sr}]$ where $k_{sr} = k_{rs} = \dfrac{\partial^2 \mathcal{V}}{\partial q_s \partial q_r} |_{\mathbf{q}=0}$

$$\Longrightarrow \boxed{\mathcal{V}(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} > 0, \quad \text{for } \mathbf{q} \neq 0 \Rightarrow \mathbf{K} \text{ is symmetric positive definite}}$$

  - in the absence of sufficient boundary conditions – that is, in the presence of rigid body modes

$$\frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} \geq 0, \quad \text{for } \mathbf{q} \neq 0 \Rightarrow \mathbf{K}\text{is symmetric positive semi-definite}$$

- Recall that

$$\mathcal{T}_2 = \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{k=1}^{N}\sum_{i=1}^{3} m_k \frac{\partial U_{i_k}}{\partial q_s}\frac{\partial U_{i_k}}{\partial q_r}\dot{q}_s\dot{q}_r \quad \text{(relative kinetic energy)}$$

$$\begin{aligned}
\mathcal{T}_2(\mathbf{q},\dot{\mathbf{q}}) &= \mathcal{T}_2(0,0) + \sum_{s=1}^{n}\frac{\partial \mathcal{T}_2}{\partial q_s}\,|_{\mathbf{q}=0,\dot{\mathbf{q}}=0}\; q_s + \sum_{s=1}^{n}\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\,|_{\mathbf{q}=0,\dot{\mathbf{q}}=0}\; \dot{q}_s \\
&+ \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 \mathcal{T}_2}{\partial q_s \partial q_r}\,|_{\mathbf{q}=0,\dot{\mathbf{q}}=0}\; q_s q_r + \sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 \mathcal{T}_2}{\partial q_s \partial \dot{q}_r}\,|_{\mathbf{q}=0,\dot{\mathbf{q}}=0}\; q_s \dot{q}_r \\
&+ \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 \mathcal{T}_2}{\partial \dot{q}_s \partial \dot{q}_r}\,|_{\mathbf{q}=0,\dot{\mathbf{q}}=0}\; \dot{q}_s \dot{q}_r + \mathcal{O}(\mathbf{q}^3,\dot{\mathbf{q}}^3) \\
&= \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 \mathcal{T}_2}{\partial \dot{q}_s \partial \dot{q}_r}\,|_{\mathbf{q}=0,\dot{\mathbf{q}}=0}\; \dot{q}_s \dot{q}_r + \mathcal{O}(\mathbf{q}^3,\dot{\mathbf{q}}^3)
\end{aligned}$$

- Hence, a second-order approximation of $\mathcal{T}_2(\dot{\mathbf{q}})$ is given by

$$\boxed{\mathcal{T}_2(\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} > 0, \qquad \text{for } \dot{\mathbf{q}} \neq 0}$$

where

$$\mathbf{M} = \left[m_{sr} = m_{rs} = \frac{\partial^2 \mathcal{T}_2}{\partial \dot{q}_s \partial \dot{q}_r} \Big|_{\mathbf{q}=0} = \sum_{k=1}^{N} m_k \sum_{i=1}^{3} \frac{\partial U_{ik}}{\partial q_s} \Big|_{q=0} \frac{\partial U_{ik}}{\partial q_r} \Big|_{q=0} \right]$$

is the mass matrix and is symmetric positive definite

- Free-vibrations about a stable equilibrium position of a conservative system that does not undergo a transport or overall motion ($\mathcal{T}_0 = \mathcal{T}_1 = 0$)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) - \frac{\partial \mathcal{T}_2}{\partial q_s} = -\frac{\partial \mathcal{V}}{\partial q_s}$$

$$\Longrightarrow \frac{d}{dt}(\mathbf{M}\dot{\mathbf{q}}) - \mathbf{0} = -\mathbf{K}\mathbf{q}$$

$$\Longrightarrow \boxed{\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0}$$

- Consider *next* the more general case of a system in steady motion (a transported system) whose equilibrium configuration defined by

$$\frac{\partial \mathcal{V}^\star}{\partial q_s} = \frac{\partial(\mathcal{V} - \mathcal{T}_0)}{\partial q_s} = 0, \quad s = 1, \cdots, n$$

  corresponds to the balance of forces deriving from a potential $\left(\frac{\partial \mathcal{V}}{\partial q_s}\right)$ and centrifugal forces $\left(\frac{\partial \mathcal{T}_0}{\partial q_s}\right)$
- It is an equilibrium configuration in the sense that $\dot{q}_s$ — which represent here the generalized velocities relative to a steady motion — are zero but the system is not idle

- Linearizations
  - effective potential energy $\mathcal{V}^\star \Rightarrow$ effective stiffness matrix $\mathbf{K}^\star$

$$\boxed{\mathcal{V}^\star(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T\mathbf{K}^\star\mathbf{q} > 0, \qquad \text{for } \mathbf{q} \neq 0}$$

where $\mathbf{K}^\star = \left[k^\star_{sr} = k_{sr} - \left(\frac{\partial^2 \mathcal{T}_0}{\partial q_s \partial q_r}\right)|_{\mathbf{q}=0}\right]$

  - mutual kinetic energy

$$\begin{aligned}
\mathcal{T}_1 &= \sum_{s=1}^{n}\sum_{k=1}^{N}\sum_{i=1}^{3}\frac{\partial U_{i_k}}{\partial t} m_k \frac{\partial U_{i_k}}{\partial q_s}\dot{q}_s = \sum_{s=1}^{n}\dot{q}_s\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}(\mathbf{q}) \\
&= \sum_{s=1}^{n}\dot{q}_s\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}(0) + \sum_{r=1}^{n}\frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s q_r}|_{\mathbf{q}=0}\, q_r + \mathcal{O}(\mathbf{q}^2)\right) \\
&\approx \sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s}q_r|_{\mathbf{q}=0}\,\dot{q}_s q_r \Rightarrow \boxed{\mathcal{T}_1(\mathbf{q},\dot{\mathbf{q}}) = \dot{\mathbf{q}}^T\mathbf{F}\mathbf{q}}
\end{aligned}$$

where $\mathbf{F} = \left[f_{sr} = \frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r}|_{\mathbf{q}=0}\right]$

- Equations of free-vibration around an equilibrium configuration
  - the equilibrium configuration generated by a steady motion remains stable as long as $\mathcal{V}^\star = \mathcal{V} - \mathcal{T}_0 \geq 0$
  - this corresponds to the fact that $\mathbf{K}^\star$ remains positive definite
  - in the neighborhood of such a configuration, the equations of motion (for a conservative system undergoing transport or overall motion) are

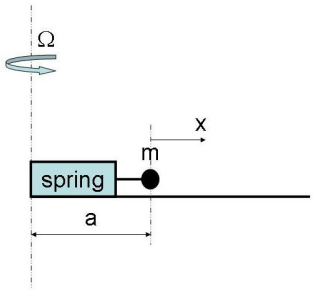
$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) - \underbrace{\frac{\partial \mathcal{T}_2}{\partial q_s}}_{0} = \underbrace{Q_s(t)}_{0} - \frac{\partial \mathcal{V}^\star}{\partial q_s} - \underbrace{\frac{\partial D}{\partial \dot{q}_s}}_{0} + F_s - \underbrace{\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}\right)}_{0}$$

where $F_s = \sum_{r=1}^{n}\left(\frac{\partial^2 \mathcal{T}_1}{\partial q_s \partial \dot{q}_r} - \frac{\partial^2 \mathcal{T}_1}{\partial q_r \partial \dot{q}_s}\right)\dot{q}_r = (\mathbf{F}^T - \mathbf{F})\dot{\mathbf{q}}$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{K}^\star\mathbf{q} = 0}$$

where $\mathbf{G} = \mathbf{F} - \mathbf{F}^T = -\mathbf{G}^T$ is the gyroscopic coupling matrix

- Example



$$q = x \qquad X = (a+x)\cos\Omega t \qquad Y = (a+x)\sin\Omega t \qquad v^2 = \dot{X}^2 + \dot{Y}^2 = (a+x)^2\Omega^2 + \dot{x}^2$$

$$\mathcal{V} = \frac{1}{2}kx^2 \qquad \mathcal{T}_0 = \frac{1}{2}\Omega^2 m(a+x)^2 \qquad \mathcal{T}_1 = 0 \qquad \mathcal{T}_2 = \frac{1}{2}m\dot{x}^2 \qquad \mathcal{V}^\star = \frac{1}{2}kx^2 - \frac{1}{2}\Omega^2 m(a+x)^2$$

- equilibrium configuration

$$\frac{\partial \mathcal{V}^\star}{\partial x} = 0 \Longrightarrow kx - \Omega^2 m(a+x) = 0 \Longrightarrow x_{eq} = \frac{\Omega^2 ma}{k - \Omega^2 m}$$

- the system becomes unstable for $\Omega^2 = \dfrac{k}{m}$
- $k^\star = \dfrac{\partial^2 \mathcal{V}^\star}{\partial x^2} = k - \Omega^2 m \Rightarrow$ system is unstable for $\Omega^2 \geq \dfrac{k}{m}$

- Free-vibration equations: $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0$
- $\mathbf{q}(t) = \mathbf{q}_a e^{i\omega t} \Rightarrow (\mathbf{K} - \omega^2\mathbf{M})\mathbf{q}_a = 0 \Rightarrow \det(\mathbf{K} - \omega^2\mathbf{M}) = 0$
- If the system has n degrees of freedom (dofs), $\mathbf{M}$ and $\mathbf{K}$ are $n \times n$ matrices $\Rightarrow n$ eigenpairs $(\omega_i^2, \mathbf{q}_{a_i})$
- Rigid body mode(s): $\omega_j^2 = 0 \Rightarrow \mathbf{K}\mathbf{q}_{a_j} = 0$
- For a rigid body mode, $\mathcal{V}(\mathbf{q}_{a_j}) = \dfrac{1}{2}\mathbf{q}_{a_j}^T\mathbf{K}\mathbf{q}_{a_j} = 0$

- Consider two distinct eigenpairs $(\omega_i^2, \mathbf{q}_{a_i})$ and $(\omega_j^2, \mathbf{q}_{a_j})$

$$\mathbf{q}_{a_j}^T \mathbf{K} \mathbf{q}_{a_i} = \mathbf{q}_{a_j}^T \omega_i^2 \mathbf{M} \mathbf{q}_{a_i} \tag{2}$$

$$\mathbf{q}_{a_i}^T \mathbf{K} \mathbf{q}_{a_j} = \mathbf{q}_{a_i}^T \omega_j^2 \mathbf{M} \mathbf{q}_{a_j} \tag{3}$$

- Because $\mathbf{M}$ and $\mathbf{K}$ are symmetric

$$(2) - (3)^T \Rightarrow 0 = (\omega_i^2 - \omega_j^2) \mathbf{q}_{a_j}^T \mathbf{M} \mathbf{q}_{a_i}$$

since $\omega_i^2 \neq \omega_j^2 \Rightarrow \boxed{\mathbf{q}_{a_j}^T \mathbf{M} \mathbf{q}_{a_i} = 0 \quad \text{and} \quad \mathbf{q}_{a_j}^T \mathbf{K} \mathbf{q}_{a_i} = 0}$

- Physical interpretation of the orthogonality conditions

$$\mathbf{q}_{a_j}^T \mathbf{M} \mathbf{q}_{a_i} = 0 \Rightarrow \mathbf{q}_{a_j}^T \left(\omega_i^2 \mathbf{M} \mathbf{q}_{a_i}\right) = \left(\omega_i^2 \mathbf{M} \mathbf{q}_{a_i}\right)^T \mathbf{q}_{a_j} = 0$$

which implies that the virtual work produced by the inertia forces of mode i during a virtual displacement prescribed by mode j is zero

$$\mathbf{q}_{a_j}^T \mathbf{K} \mathbf{q}_{a_i} = 0 \Rightarrow \left(\mathbf{K} \mathbf{q}_{a_i}\right)^T \mathbf{q}_{a_j} = 0$$

which implies that the virtual work produced by the elastic forces of mode i during a virtual displacement prescribed by mode j is zero

- Rayleigh quotient

$$\mathbf{K}\mathbf{q}_{a_i} = \omega_i^2 \mathbf{M}\mathbf{q}_{a_i} \Rightarrow \mathbf{q}_{a_i}^T \mathbf{K}\mathbf{q}_{a_i} = \omega_i^2 \mathbf{q}_{a_i}^T \mathbf{M}\mathbf{q}_{a_i} \Rightarrow \omega_i^2 = \frac{\mathbf{q}_{a_i}^T \mathbf{K}\mathbf{q}_{a_i}}{\mathbf{q}_{a_i}^T \mathbf{M}\mathbf{q}_{a_i}} = \frac{\gamma_i}{\mu_i}$$

- γ_i = generalized stiffness coefficient of mode i (measures the contribution of mode i to the elastic deformation energy)
- μ_i = generalized mass coefficient of mode i (measures the contribution of mode i to the kinetic energy)
- Since the amplitude of $\mathbf{q}_{a_i}$ is determined up to a factor only $\Rightarrow$ γ_i and μ_i are determined up to a constant factor only
- Mass normalization

$$\begin{aligned} \mathbf{q}_{a_j}^T \mathbf{M}\mathbf{q}_{a_i} &= \delta_{ij} \\ \mathbf{q}_{a_j}^T \mathbf{K}\mathbf{q}_{a_i} &= \omega_i^2 \delta_{ij} \end{aligned}$$

- What happens if a multiple circular frequency is encountered?
- *Theorem: to a multiple root ω_p^2 of the system*

$$\mathbf{K}\mathbf{q}_a = \omega^2 \mathbf{M}\mathbf{q}_a$$

corresponds a number of linearly independent eigenvectors $\{\mathbf{q}_{a_i}\}$ equal to the root multiplicity

- n-dof system: $\mathbf{M} \in \mathbb{R}^{n \times n}$, $\mathbf{K} \in \mathbb{R}^{n}$, and $\mathbf{q} \in \mathbb{R}^{n}$
- Coupled system of ordinary differential equations

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} &= \mathbf{0} \\ \mathbf{q}(0) &= \mathbf{q}_0 \\ \dot{\mathbf{q}}(0) &= \dot{\mathbf{q}}_0 \end{cases}$$

- Natural vibration modes (eigenmodes)

$$\mathbf{K}\mathbf{q}_{a_i} = \omega_i^2 \mathbf{M}\mathbf{q}_{a_i}, \quad i = 1, \cdots, n$$

$$\mathbf{Q} = [\mathbf{q}_{a_1} \quad \mathbf{q}_{a_2} \quad \cdots \quad \mathbf{q}_{a_n}] \Rightarrow \left\{ \begin{array}{ccc} \mathbf{Q}^T\mathbf{K}\mathbf{Q} & = & \mathbf{\Omega}^2 \\ \mathbf{Q}^T\mathbf{M}\mathbf{Q} & = & \mathbf{I} \end{array} \right.$$

$$\mathbf{\Omega}^2 = \begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{pmatrix}$$

- Truncated eigenbasis

$$\mathbf{Q}_r = [\mathbf{q}_{a_1} \quad \mathbf{q}_{a_2} \quad \cdots \quad \mathbf{q}_{a_r}], \qquad r \ll n \Rightarrow \left\{ \begin{array}{ccc} \mathbf{Q}_r^T\mathbf{K}\mathbf{Q}_r & = & \mathbf{\Omega}_r^2 \\ \mathbf{Q}_r^T\mathbf{M}\mathbf{Q}_r & = & \mathbf{I}_r \end{array} \right. \begin{array}{l} \text{(reduced stiffness matrix)} \\ \text{(reduced mass matrix)} \end{array}$$

$$\mathbf{\Omega}_r^2 = \begin{pmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_r^2 \end{pmatrix}$$

- Modal superposition: $\mathbf{q} = \mathbf{Q}_r\mathbf{y} = \sum_{i=1}^{r} y_i \mathbf{q}_{a_i}$ where $\mathbf{y} = [y_1 \quad y_2 \quad \cdots \quad y_r]^T$ and y_i is called the modal displacement
- Substitute in $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$

$$\Longrightarrow \mathbf{M}\mathbf{Q}_r\ddot{\mathbf{y}} + \mathbf{K}\mathbf{Q}_r\mathbf{y} = \mathbf{0}$$

$$\Longrightarrow \mathbf{Q}_r^T\mathbf{M}\mathbf{Q}_r\ddot{\mathbf{y}} + \mathbf{Q}_r^T\mathbf{K}\mathbf{Q}_r\mathbf{y} = \mathbf{0}$$

$$\Longrightarrow \boxed{\mathbf{I}_r\ddot{\mathbf{y}} + \mathbf{\Omega}_r^2\mathbf{y} = \mathbf{0}}$$

- Uncoupled differential equations (modal equations)

$$\boxed{\ddot{y}_i + \omega_i^2 y_i = 0, \qquad i = 1, \cdots r}$$

$$\ddot{y}_i + \omega_i^2 y_i = 0, \qquad i = 1, \cdots r$$

- Case 1: $\omega_i^2 = 0$ (rigid body mode)

$$y_i = a_i t + b_i$$

- Case 2: $\omega_j^2 \neq 0$

$$y_j = c_j \cos \omega_j t + d_j \sin \omega_j t$$

- General case: r_b rigid body modes

$$\mathbf{q} = \sum_{i=1}^{r_b} (a_i t + b_i) \mathbf{q}_{a_i} + \sum_{j=1}^{r-r_b} (c_j \cos \omega_j t + d_j \sin \omega_j t) \mathbf{q}_{a_j}$$

- Initial conditions

$$\mathbf{q}(0) = \mathbf{q}_0 = \mathbf{Q}\mathbf{y}(0) \qquad \text{and} \qquad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 = \mathbf{Q}\dot{\mathbf{y}}(0)$$

$$\mathbf{q}_0 = \mathbf{Q}_r \mathbf{y}(0) \qquad \text{and} \qquad \dot{\mathbf{q}}_0 = \mathbf{Q}_r \dot{\mathbf{y}}(0)$$

$$\Longrightarrow \mathbf{q}_{a_i}^T \mathbf{M} \mathbf{q}_0 = \mathbf{q}_{a_i}^T \mathbf{M} \mathbf{Q}_r \mathbf{y}(0)$$

From the orthogonality properties of the natural mode shapes (eigenvectors) it follows that

$$\mathbf{q}_{a_i}^T \mathbf{M} \mathbf{q}_0 = \mathbf{q}_{a_i}^T \mathbf{M} \mathbf{Q}_r \mathbf{y}(0) = \mathbf{q}_{a_i}^T \mathbf{M} \mathbf{q}_{a_i} y_i(0) = y_i(0) \Rightarrow y_i(0) = \mathbf{q}_{a_i}^T \mathbf{M} \mathbf{q}(0)$$

- Case 1: $\omega_i^2 = 0$ (rigid body mode) $\Rightarrow a_i \times 0 + b_i = \mathbf{q}_{a_i}^T \mathbf{M} \mathbf{q}(0) \Rightarrow b_i = \mathbf{q}_{a_i}^T \mathbf{M} \mathbf{q}(0)$
- Case 2: $\omega_j^2 \neq 0 \Rightarrow c_j \times 1 + d_j \times 0 = \mathbf{q}_{a_j}^T \mathbf{M} \mathbf{q}(0) \Rightarrow c_j = \mathbf{q}_{a_j}^T \mathbf{M} \mathbf{q}(0)$
- Case 1: $\omega_i^2 = 0$ (rigid body mode) $\Rightarrow a_i = \mathbf{q}_{a_i}^T \mathbf{M} \dot{\mathbf{q}}(0)$
- Case 2: $\omega_j^2 \neq 0 \Rightarrow d_j = \dfrac{1}{\omega_j} \mathbf{q}_{a_j}^T \mathbf{M} \dot{\mathbf{q}}(0)$
- Thus, the general solution is

$$\boxed{\mathbf{q}(t) = \sum_{i=1}^{r_b} \left(\left[\mathbf{q}_{a_i}^T \mathbf{M} \dot{\mathbf{q}}(0) \right] t + \mathbf{q}_{a_i}^T \mathbf{M} \mathbf{q}(0) \right) \mathbf{q}_{a_i} + \sum_{j=1}^{r-r_b} \left(\mathbf{q}_{a_j}^T \mathbf{M} \mathbf{q}(0) \cos \omega_j t + \mathbf{q}_{a_j}^T \mathbf{M} \dot{\mathbf{q}}(0) \frac{\sin \omega_j t}{\omega_j} \right) \mathbf{q}_{a_j}}$$

$$\forall \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{x} = \sum_{s=1}^{n} \alpha_s \mathbf{q}_{a_s} \Rightarrow \mathbf{q}_{a_j}^T \mathbf{M}\mathbf{x} = \sum_{s=1}^{n} \alpha_s \mathbf{q}_{a_j}^T \mathbf{M}\mathbf{q}_{a_s} = \alpha_j$$

$$\Longrightarrow \forall \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{x} = \sum_{s=1}^{n} \left(\mathbf{q}_{a_s}^T \mathbf{M}\mathbf{x}\right) \mathbf{q}_{a_s} = \sum_{s=1}^{n} \mathbf{q}_{a_s} \left(\mathbf{q}_{a_s}^T \mathbf{M}\mathbf{x}\right) = \sum_{s=1}^{n} \left(\mathbf{q}_{a_s} \mathbf{q}_{a_s}^T\right) \mathbf{M}\mathbf{x} = \left(\sum_{s=1}^{n} (\mathbf{q}_{a_s} \mathbf{q}_{a_s}^T)\mathbf{M}\right) \mathbf{x}$$

$$\Rightarrow \boxed{\sum_{s=1}^{n} \left(\mathbf{q}_{a_s} \mathbf{q}_{a_s}^T\right) \mathbf{M} = \mathbf{I}}$$

- This is the same result as $\mathbf{Q}^T\mathbf{M}\mathbf{Q} = \mathbf{I}$
- A given load $\mathbf{p}$ can be expanded in terms of the inertia forces generated by the eigenmodes, $\mathbf{M}\mathbf{q}_{a_j}$ as follows

$$\mathbf{p} = \sum_{j=1}^{n} \beta_j \mathbf{M}\mathbf{q}_{a_j} \Rightarrow \mathbf{q}_{a_i}^T \mathbf{p} = \sum_{j=1}^{n} \beta_j \mathbf{q}_{a_i}^T \mathbf{M}\mathbf{q}_{a_j} = \beta_i$$

$$\Longrightarrow \beta_i = \mathbf{q}_{a_i}^T \mathbf{p} = \text{modal participation factor} \rightarrow \mathbf{p} = \sum_{j=1}^{n} \left(\mathbf{q}_{a_j}^T \mathbf{p}\right) \mathbf{M}\mathbf{q}_{a_j}$$

Spectral Expansions

- Recall that $\sum_{s=1}^{n} \left(\mathbf{q}_{a_s} \mathbf{q}_{a_s}^T \right) \mathbf{M} = \mathbf{I}$
- Hence, $\forall \mathbf{A} \in \mathbb{R}^n, \quad \mathbf{A} = \sum_{s=1}^{n} \mathbf{A}\mathbf{q}_{a_s} \mathbf{q}_{a_s}^T \mathbf{M}$ and $\mathbf{A} = \sum_{s=1}^{n} \mathbf{q}_{a_s} \mathbf{q}_{a_s}^T \mathbf{M}\mathbf{A}$
- $\mathbf{A} = \mathbf{M} \Rightarrow \mathbf{M} = \sum_{s=1}^{n} \mathbf{M}\mathbf{q}_{a_s} \mathbf{q}_{a_s}^T \mathbf{M} = \sum_{s=1}^{n} \mathbf{M}\mathbf{q}_{a_s} \left(\mathbf{M}\mathbf{q}_{a_s}\right)^T$ (because $\mathbf{M}$ is symmetric)
- $\mathbf{A} = \mathbf{K} \Rightarrow \mathbf{K} = \sum_{s=1}^{n} \mathbf{K}\mathbf{q}_{a_s} \mathbf{q}_{a_s}^T \mathbf{M} = \sum_{s=1}^{n} \omega_s^2 \mathbf{M}\mathbf{q}_{a_s} \mathbf{q}_{a_s}^T \mathbf{M} = \sum_{s=1}^{n} \omega_s^2 \mathbf{M}\mathbf{q}_{a_s} \left(\mathbf{M}\mathbf{q}_{a_s}\right)^T$
- $\mathbf{A} = \mathbf{M}^{-1} \Rightarrow \mathbf{M}^{-1} = \sum_{s=1}^{n} \mathbf{q}_{a_s} \mathbf{q}_{a_s}^T \mathbf{M}\mathbf{M}^{-1} \Rightarrow \boxed{\mathbf{M}^{-1} = \sum_{s=1}^{n} \mathbf{q}_{a_s} \mathbf{q}_{a_s}^T}$
- $\mathbf{A} = \mathbf{K}^{-1} \Rightarrow \mathbf{K}^{-1} = \sum_{s=1}^{n} \mathbf{q}_{a_s} \mathbf{q}_{a_s}^T \mathbf{M}\mathbf{K}^{-1} = \sum_{s=1}^{n} \mathbf{q}_{a_s} \left(\mathbf{M}\mathbf{q}_{a_s}\right)^T \mathbf{K}^{-1} \Rightarrow \boxed{\mathbf{K}^{-1} = \sum_{s=1}^{n} \frac{\mathbf{q}_{a_s} \mathbf{q}_{a_s}^T}{\omega_s^2}}$

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} &= \mathbf{s}_a \cos\omega t \\ \mathbf{q}(0) &= \mathbf{q}_0 \\ \dot{\mathbf{q}}(0) &= \dot{\mathbf{q}}_0 \end{cases}$$

- Solution can be decomposed as
  $\mathbf{q} = \mathbf{q}_H(\text{homogeneous}) \quad + \quad \mathbf{q}_P(\text{particular})$
- $\mathbf{q}_p = \mathbf{q}_a \cos\omega t \Rightarrow \mathbf{q}_a = (\mathbf{K} - \omega^2\mathbf{M})^{-1}\mathbf{s}_a$ where $(\mathbf{K} - \omega^2\mathbf{M})^{-1}$ is called the **admittance** or **dynamic influence** matrix
- The forced response is the part of the response that is synchronous to the excitation — that is, $\mathbf{q}_p$

- Rigid body modes: $\{\mathbf{u}_{a_i}\}_{i=1}^{r_b}$
- $\forall \mathbf{q}_a \in \mathbb{R}^n, \quad \mathbf{q}_a = \sum_{i=1}^{r_b} \alpha_i \mathbf{u}_{a_i} + \sum_{j=1}^{n-r_b} \beta_j \mathbf{q}_{a_j}$

$$\Longrightarrow \mathbf{s}_a = (\mathbf{K} - \omega^2 \mathbf{M})\mathbf{q}_a = \sum_{i=1}^{r_b} \alpha_i (\mathbf{K} - \omega^2 \mathbf{M})\mathbf{u}_{a_i} + \sum_{j=1}^{n-r_b} \beta_j (\mathbf{K} - \omega^2 \mathbf{M})\mathbf{q}_{a_j}$$

$$\Longrightarrow \mathbf{s}_a = -\sum_{i=1}^{r_b} \alpha_i \omega^2 \mathbf{M}\mathbf{u}_{a_i} + \sum_{j=1}^{n-r_b} \beta_j (\omega_j^2 - \omega^2)\mathbf{M}\mathbf{q}_{a_j}$$

- Premultiply by $\mathbf{u}_{a_j}^T \Rightarrow \alpha_j = -\dfrac{\mathbf{u}_{a_j}^T \mathbf{s}_a}{\omega^2}$ and premultiply by $\mathbf{q}_{a_i}^T \Rightarrow \beta_i = \dfrac{\mathbf{q}_{a_i}^T \mathbf{s}_a}{(\omega_i^2 - \omega^2)}$

$$\Longrightarrow \mathbf{q}_a = -\sum_{i=1}^{r_b} \frac{\mathbf{u}_{a_i}^T \mathbf{s}_a}{\omega^2} \mathbf{u}_{a_i} + \sum_{j=1}^{n-r_b} \frac{\mathbf{q}_{a_j}^T \mathbf{s}_a}{(\omega_j^2 - \omega^2)} \mathbf{q}_{a_j} = \left(-\sum_{i=1}^{r_b} \frac{\mathbf{u}_{a_i}\mathbf{u}_{a_i}^T}{\omega^2} + \sum_{j=1}^{n-r_b} \frac{\mathbf{q}_{a_j}\mathbf{q}_{a_j}^T}{(\omega_j^2 - \omega^2)} \right) \mathbf{s}_a$$

- Since $\mathbf{q}_a = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{s}_a$

$$\Longrightarrow \boxed{(\mathbf{K} - \omega^2 \mathbf{M})^{-1} = -\frac{1}{\omega^2} \sum_{i=1}^{r_b} \mathbf{u}_{a_i}\mathbf{u}_{a_i}^T + \sum_{j=1}^{n-r_b} \frac{\mathbf{q}_{a_j}\mathbf{q}_{a_j}^T}{\omega_j^2 - \omega^2}} \quad (4)$$

- Which excitation $\mathbf{s}_{a_m}$ will generate a harmonic response with an amplitude corresponding to $\mathbf{q}_{a_m}$?

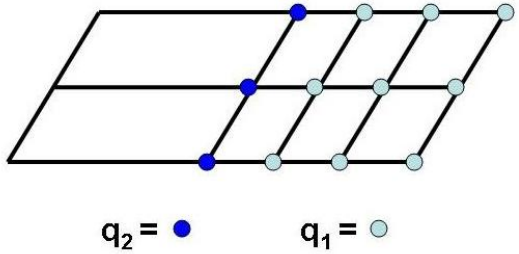
$$\begin{aligned} \mathbf{q}_{a_m} &= (\mathbf{K} - \omega^2\mathbf{M})^{-1}\mathbf{s}_{a_m} \Rightarrow \mathbf{s}_{a_m} = (\mathbf{K} - \omega^2\mathbf{M})\mathbf{q}_{a_m} \\ \Longrightarrow \mathbf{s}_{a_m} &= \mathbf{K}\mathbf{q}_{a_m} - \omega^2\mathbf{M}\mathbf{q}_{a_m} = (\omega_m^2 - \omega^2)\mathbf{M}\mathbf{q}_{a_m} \end{aligned}$$

- At resonance $(\omega^2 = \omega_m^2)$ $\mathbf{s}_{a_m} = 0 \Rightarrow$ no force is needed to maintain $\mathbf{q}_{a_m}$ once it is reached

- The inverse of an admittance is an impedance

$$\mathbf{Z}(\omega^2) = (\mathbf{K} - \omega^2 \mathbf{M})$$

- Application: substructuring (or domain decomposition)



  - the dynamical behavior of a substructure is described by its harmonic response when forces are applied onto its interface boundaries
  - a subsystem is typically described by $\mathbf{K}$ and $\mathbf{M}$, has n_1 free dofs $\mathbf{q}_1$, and is connected to the rest of the system by $n_2 = n - n_1$ boundary dofs $\mathbf{q}_2$ where the reaction forces are denoted here by $\mathbf{g}_2$

$$\begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{g}_2 \end{pmatrix}$$

$$\Longrightarrow \mathbf{Z}_{11}\mathbf{q}_1 + \mathbf{Z}_{12}\mathbf{q}_2 = 0 \Rightarrow \mathbf{q}_1 = -\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}\mathbf{q}_2$$

$$\Longrightarrow (\mathbf{Z}_{22} - \mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12})\mathbf{q}_2 = \mathbf{Z}_{22}^{\star}\mathbf{q}_2 = \mathbf{g}_2$$

  - $Z_{22}^{\star}$ is the "reduced" impedance (reduced to the boundary)

- Spectral expansion of $\mathbf{Z}_{22}^{\star}$
  - look at $\mathbf{Z}_{11}^{-1}$ and let $(\tilde{\omega}_i^2, \tilde{\mathbf{q}}_{a_i})$ denote the n_1 eigenpairs of the associated dynamical subsystem: from (4), it follows that $Z_{11}^{-1} = \sum_{j=1}^{n_1} \frac{\tilde{\mathbf{q}}_{a_j}\tilde{\mathbf{q}}_{a_j}^T}{\tilde{\omega}_j^2 - \omega^2}$
  - apply twice the relation

$$\frac{1}{\tilde{\omega}_j^2 - \omega^2} = \frac{1}{\tilde{\omega}_j^2} + \frac{\omega^2}{\tilde{\omega}_j^2(\tilde{\omega}_j^2 - \omega^2)}$$

$$\Longrightarrow \mathbf{Z}_{11}^{-1} = \mathbf{K}_{11}^{-1} + \omega^2 \sum_{j=1}^{n_1} \frac{\tilde{\mathbf{q}}_{a_j}\tilde{\mathbf{q}}_{a_j}^T}{\tilde{\omega}_j^2(\tilde{\omega}_j^2 - \omega^2)} = \mathbf{K}_{11}^{-1} + \omega^2 \sum_{j=1}^{n_1} \frac{\tilde{\mathbf{q}}_{a_j}\tilde{\mathbf{q}}_{a_j}^T}{\tilde{\omega}_j^4} + \omega^4 \sum_{j=1}^{n_1} \frac{\tilde{\mathbf{q}}_{a_j}\tilde{\mathbf{q}}_{a_j}^T}{\tilde{\omega}_j^4(\tilde{\omega}_j^2 - \omega^2)}$$

  - owing to the $\mathbf{M}_{11}$-orthonormality of the modes and the spectral expansion of $\mathbf{K}_{11}^{-1}$, the above expression can further be written as

$$\begin{aligned}
\mathbf{Z}_{11}^{-1} &= \mathbf{K}_{11}^{-1} + \omega^2 \mathbf{K}_{11}^{-1}\mathbf{M}_{11}\mathbf{K}_{11}^{-1} + \omega^4 \sum_{j=1}^{n_1} \frac{\tilde{\mathbf{q}}_{a_j}\tilde{\mathbf{q}}_{a_j}^T}{\tilde{\omega}_j^4(\tilde{\omega}_j^2 - \omega^2)} \\
\Longrightarrow Z_{22}^{\star} &= \mathbf{K}_{22} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12} \\
&- \omega^2[\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{M}_{12} + \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{M}_{11}\mathbf{K}_{11}^{-1}\mathbf{K}_{12}] \\
&- \omega^4 \sum_{i=1}^{n_1} \frac{[(\mathbf{K}_{21} - \tilde{\omega}_i^2\mathbf{M}_{21})\tilde{\mathbf{q}}_{a_i}][(\mathbf{K}_{21} - \tilde{\omega}_i^2\mathbf{M}_{21})\tilde{\mathbf{q}}_{a_i})]^T}{\tilde{\omega}_i^4(\tilde{\omega}_i^2 - \omega^2)}
\end{aligned}$$

$$
\begin{aligned}
Z_{22}^{\star} &= \mathbf{K}_{22} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12} \\
&\quad -\omega^2[\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{M}_{12} + \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{M}_{11}\mathbf{K}_{11}^{-1}\mathbf{K}_{12}] \\
&\quad -\omega^4 \sum_{i=1}^{n_1} \frac{[(\mathbf{K}_{21} - \tilde{\omega}_i^2\mathbf{M}_{21})\tilde{\mathbf{q}}_{a_i}][(\mathbf{K}_{21} - \tilde{\omega}_i^2\mathbf{M}_{21})\tilde{\mathbf{q}}_{a_i})]^T}{\tilde{\omega}_i^4(\tilde{\omega}_i^2 - \omega^2)}
\end{aligned}
$$

- The first term $\mathbf{K}_{22} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12}$ represents the stiffness of the statically condensed system
- The second term $\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{M}_{12} + \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{M}_{11}\mathbf{K}_{11}^{-1}\mathbf{K}_{12}$ represents the mass of the subsystem statically condensed on the boundary
- The last term represents the contribution of the subsystem eigenmodes since it is generated by $\tilde{\mathbf{q}}_{a_i}\tilde{\mathbf{q}}_{a_i}^T$
- $(\mathbf{K}_{21} - \tilde{\omega}_i^2\mathbf{M}_{21})\tilde{\mathbf{q}}_{a_i}$ is the dynamic reaction on the boundary

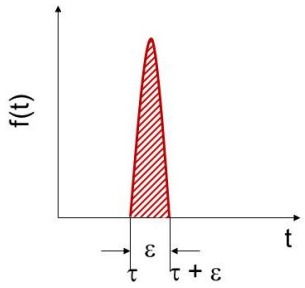
$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} &= \mathbf{p}(t) \\ \mathbf{q}(0) &= \mathbf{q}_0 \\ \dot{\mathbf{q}}(0) &= \dot{\mathbf{q}}_0 \end{cases}$$

- General approach
  - consider the simpler case where there is no rigid body mode $\Rightarrow$ eigenmodes $(\mathbf{q}_{a_i}, \omega_i^2), \omega_i^2 \neq 0, i = 1, \cdots, n$
  - modal superposition: $\mathbf{q} = \mathbf{Q}\mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{q}_{a_i}$
  - substitute in equations of dynamic equilibrium

$$\Longrightarrow \mathbf{M}\mathbf{Q}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{Q}\mathbf{y} = \mathbf{p}(t) \Rightarrow \mathbf{Q}^T\mathbf{M}\mathbf{Q}\ddot{\mathbf{y}} + \mathbf{Q}^T\mathbf{K}\mathbf{Q}\mathbf{y} = \mathbf{Q}^T\mathbf{p}(t)$$

  - modal equations $\boxed{\ddot{y}_i + \omega_i^2 y_i = \mathbf{q}_{a_i}^T \mathbf{p}(t), \quad i = 1, \cdots, n}$
  - $y_i(t)$ depend on two constants that can be obtained from the initial conditions
  - $\mathbf{q}(0) = \mathbf{Q}\mathbf{y}(0), \dot{\mathbf{q}}(0) = \mathbf{Q}\dot{\mathbf{y}}(0) \underset{orthogonality}{\Rightarrow} \underset{conditions}{} y_i(0), \dot{y}_i(0)$

- Response to an impulsive force



  - spring-mass system: m, k, $\omega^2 = \dfrac{k}{m}$
  - impulsive force $f(t)$: force whose amplitude could be infinitely large but which acts for a very short duration of time
  - magnitude of impulse: $I = \displaystyle\int_{\tau}^{\tau+\epsilon} f(t)dt$
  - impulsive force $= I\delta(t)$ where δ is the "delta" function **<u>centered at $t = 0$</u>** and satisfying $\displaystyle\int_0^{\epsilon} \delta(t)dt = 1$

- Response to an impulsive force (continue)
  - dynamic equilibrium

$$m\int_{\tau}^{\tau+\epsilon}\frac{d\dot{u}}{dt}dt + k\int_{\tau}^{\tau+\epsilon} u dt = \int_{\tau}^{\tau+\epsilon} f(t)dt = I$$

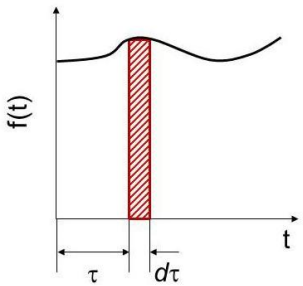
  - assume that at $t=\tau$, the system is at rest ($u(\tau)=0$ and $\dot{u}(\tau)=0$)
  - focus on the short (infinitesimal) interval of time $d\tau$
  - $\ddot{u}\approx A/\epsilon \Rightarrow \int_{\tau}^{\tau+\epsilon}\ddot{u}dt \approx A$ and $\int_{\tau}^{\tau+\epsilon} u dt \approx A\epsilon^2/6$
  - hence

$$\begin{aligned} m\int_{\tau}^{\tau+\epsilon}\frac{d\dot{u}}{dt}dt &\approx I \Rightarrow m\Delta\dot{u}\approx I \Rightarrow \Delta\dot{u}=\dot{u}(\tau+\epsilon)-\dot{u}(\tau)\approx\frac{I}{m}\Rightarrow \dot{u}(\tau+\epsilon)\approx\frac{I}{m} \\ \int_{\tau}^{\tau+\epsilon}\dot{u}dt &\approx 0 \Rightarrow \Delta u\approx 0 \Rightarrow u(\tau+\epsilon)-u(\tau)\approx 0 \Rightarrow u(\tau+\epsilon)\approx 0 \end{aligned}$$

  - the above equations provide initial conditions for the free-vibrations that start at the end of the impulsive load

$$\Longrightarrow u(t)\approx\frac{I}{m\omega}\sin\omega t \quad \text{for impulses of finite duration}$$

- Response to an impulsive force (continue)



  - for the differential time interval $d\tau$ ($d\tau \to 0$), the response analysis of the previous page becomes *exact*

$$\Longrightarrow du = \frac{f(\tau)d\tau}{m\omega}\sin\omega(t-\tau)$$

  - linear system $\Rightarrow$ superposition principle

$$du = \frac{f(\tau)d\tau}{m\omega}\sin\omega(t-\tau) \Rightarrow \boxed{u(t) = \frac{1}{m\omega}\int_0^t f(\tau)\sin\omega(t-\tau)d\tau}$$

- Time-integration of the normal equations
  - let $p_i(t) = \mathbf{q}_{a_i}^T \mathbf{p}(t) = i$-th modal participation factor
  - modal or **normal** equation: $\ddot{y}_i + \omega_i^2 y_i = p_i(t)$
  - $y_i(t) = y_i^H(t) + y_i^P(t)$, $y_i^H = A_i \cos \omega_i t + B_i \sin \omega_i t$
  - the particular solution $y_i^P(t)$ depends on the form of $p_i(t)$
  - however, the general form of a particular solution that satisfies the **rest** initial conditions is given by the **Duhamel's** integral

$$y_i^P(t) = \frac{1}{\omega_i} \int_0^t p_i(\tau) \sin \omega_i (t - \tau) d\tau$$

  - complete solution

$$y_i(t) = A_i \cos \omega_i t + B_i \sin \omega_i t + \frac{1}{\omega_i} \int_0^t p_i(t) \sin \omega_i (t - \tau) d\tau$$

  - $A_i = y_i(0), B_i = \dfrac{\dot{y}_i(0)}{\omega_i}$ and $y_i(0)$ and $\dot{y}_i(0)$ can be determined from the initial conditions $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$ and the orthogonality conditions

- Response truncation and mode displacement method
  - computational efficiency $\Rightarrow \mathbf{q} = \mathbf{Q}_r \mathbf{y} = \sum_{i=1}^{r} y_i \mathbf{q}_{a_i}, \qquad r \ll n$
  - what is the effect of modal truncation?
  - consider the case where $\mathbf{p}(t) = \underbrace{\mathbf{g}}_{\substack{\text{static load}\\ \text{distribution}}} \times \underbrace{\phi(t)}_{\substack{\text{amplification}\\ \text{factor}}}$
  - for a system initially at rest ($\mathbf{q}(0) = 0$ and $\dot{\mathbf{q}}(0) = 0$)
    $y_i(0) = \mathbf{q}_{a_i}^T \mathbf{M}\mathbf{q}(0) = 0$ and $\dot{y}_i(0) = \mathbf{q}_{a_i}^T \mathbf{M}\dot{\mathbf{q}}(0) = 0 \Rightarrow A_i = B_i = 0$

$$\Longrightarrow y_i(t) = \frac{1}{\omega_i}\int_0^t p_i(\tau)\sin\omega_i(t-\tau)d\tau = \frac{\mathbf{q}_{a_i}^T\mathbf{g}}{\omega_i}\int_0^t \phi(\tau)\sin\omega_i(t-\tau)d\tau$$

$$\Longrightarrow \boxed{\mathbf{q}(t) = \sum_{i=1}^{r}\underbrace{\left[\mathbf{q}_{a_i}\mathbf{q}_{a_i}^T\mathbf{g}\right]}_{\text{spatial factor}}\underbrace{\left[\frac{1}{\omega_i}\int_0^t \phi(\tau)\sin\omega_i(t-\tau)d\tau\right]}_{\text{temporal factor}}}$$

- Response truncation and mode displacement method (continue)
  - general solution for restrained structure initially at rest

$$\Longrightarrow \mathbf{q}(t) = \sum_{i=1}^{r} \underbrace{\left[\mathbf{q}_{a_i}\mathbf{q}_{a_i}^T\mathbf{g}\right]}_{spatial\ \ factor} \left[\underbrace{\frac{1}{\omega_i}\int_0^t \phi(\tau)\sin\omega_i(t-\tau)d\tau}_{temporal\ \ factor\ \ \theta_i(t)}\right]$$

  - truncated response is accurate if neglected terms are small, which is true if:
    - $\mathbf{q}_{a_i}^T\mathbf{g}$ is small for $i = r+1, \cdots, n \Rightarrow \mathbf{g}$ is well approximated in the range of $\mathbf{Q}_r$
    - $\frac{1}{\omega_j}\int_0^t \phi(\tau)\sin\omega_j(t-\tau)d\tau$ is small for $j > r$, which depends on the frequency content of $\phi(t)$

$$\phi(t) = 1 \quad \Rightarrow \quad \theta_i(t) = \frac{1-\cos\omega_i t}{\omega_i} \rightarrow 0 \text{ for large circular frequencies}$$

$$\phi(t) = \sin\omega t \quad \Rightarrow \quad \theta_i(t) = \frac{\omega_i \sin\omega t - \omega\sin\omega_i t}{\omega_i(\omega_i^2 - \omega^2)}$$

- Mode acceleration method

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t) \Rightarrow \mathbf{K}\mathbf{q} = \mathbf{p}(t) - \mathbf{M}\ddot{\mathbf{q}}$$

  - apply truncated modal representation to the acceleration

$$\mathbf{q}(t) = \mathbf{Q}_r\mathbf{y}(t) \Rightarrow \ddot{\mathbf{q}}(t) = \mathbf{Q}_r\ddot{\mathbf{y}}(t)$$

$$\Longrightarrow \mathbf{K}\mathbf{q} = \mathbf{p}(t) - \sum_{i=1}^{r} \mathbf{M}\mathbf{q}_{a_i}\ddot{y}_i \Longrightarrow \boxed{\mathbf{q} = \mathbf{K}^{-1}\mathbf{p}(t) - \sum_{i=1}^{r} \frac{\mathbf{q}_{a_i}}{\omega_i^2}\ddot{y}_i}$$

  - recall that

$$\ddot{y}_i + \omega_i^2 y_i = \mathbf{q}_{a_i}^T\mathbf{p}(t) \tag{5}$$

  - and that for a system initially at rest

$$y_i(t) = \frac{1}{\omega_i}\int_0^t \mathbf{q}_{a_i}^T\mathbf{p}(\tau)\sin\omega_i(t-\tau)d\tau \tag{6}$$

  - (5) and (6) $\Rightarrow \ddot{y}_i(t) = \mathbf{q}_{a_i}^T\left(\mathbf{p}(t) - \omega_i\int_0^t p(\tau)\sin\omega_i(t-\tau)d\tau\right)$

- Mode acceleration method (continue)
  - substitute in $\mathbf{q}(t) = \mathbf{K}^{-1}\mathbf{p}(t) - \sum_{i=1}^{r} \frac{\mathbf{q}_{a_i}}{\omega_i^2}\ddot{y}_i$

$$\Longrightarrow \mathbf{q}(t) = \sum_{i=1}^{r} \frac{\mathbf{q}_{a_i}\mathbf{q}_{a_i}^T}{\omega_i} \int_0^t \mathbf{p}(\tau)\sin\omega_i(t-\tau)d\tau + \left(\mathbf{K}^{-1} - \sum_{i=1}^{r} \frac{\mathbf{q}_{a_i}\mathbf{q}_{a_i}^T}{\omega_i^2}\right)\mathbf{p}(t)$$

  - recall the spectral expansion $\mathbf{K}^{-1} = \sum_{i=1}^{n} \frac{\mathbf{q}_{a_i}\mathbf{q}_{a_i}^T}{\omega_i^2}$

$$\Longrightarrow \mathbf{q}(t) = \sum_{i=1}^{r} \frac{\mathbf{q}_{a_i}\mathbf{q}_{a_i}^T}{\omega_i} \int_0^t \mathbf{p}(\tau)\sin\omega_i(t-\tau)d\tau + \left(\sum_{i=r+1}^{n} \frac{\mathbf{q}_{a_i}\mathbf{q}_{a_i}^T}{\omega_i^2}\right)\mathbf{p}(t)$$

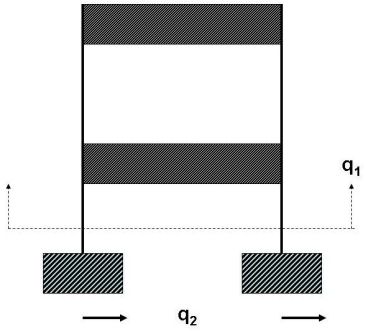
  - which shows that the mode acceleration method complements the truncated mode displacement solution with the missing terms using the modal expansion of the static response
  - how to deal with the computational cost issue?

- Direct time-integration methods for solving

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} &= \mathbf{p}(t) \\ \mathbf{q}(0) &= \mathbf{q}_0 \\ \dot{\mathbf{q}}(0) &= \dot{\mathbf{q}}_0 \end{cases}$$

  - will be covered towards the end of this course

- The general case



$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} \quad \text{where } \mathbf{q}_2 \text{ is prescribed}$$

$$\begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{r}_2(t) \end{pmatrix}$$

- The first equation gives

$$\mathbf{M}_{11}\ddot{\mathbf{q}}_1 + \mathbf{K}_{11}\mathbf{q}_1 = -\mathbf{K}_{12}\mathbf{q}_2 - \mathbf{M}_{12}\ddot{\mathbf{q}}_2 \Rightarrow \mathbf{q}_1(t)$$

- Substitute in second equation

$$\Longrightarrow \mathbf{r}_2(t) = \mathbf{K}_{21}\mathbf{q}_1 + \mathbf{M}_{21}\ddot{\mathbf{q}}_1 + \mathbf{K}_{22}\mathbf{q}_2 + \mathbf{M}_{22}\ddot{\mathbf{q}}_2$$

- Quasi-static response of $\mathbf{q}_1$

$$\mathbf{0} + \mathbf{K}_{11}\mathbf{q}_1 = -\mathbf{K}_{12}\mathbf{q}_2 - \mathbf{0}$$

$$\Longrightarrow \mathbf{q}_1^{qs} = -\mathbf{K}_{11}^{-1}\mathbf{K}_{12}\mathbf{q}_2 = \mathbf{S}\mathbf{q}_2$$

- Decompose $\mathbf{q}_1 = \mathbf{q}_1^{qs} + \mathbf{z}_1 = \mathbf{S}\mathbf{q}_2 + \mathbf{z}_1$

$$\Longrightarrow \mathbf{q}(t) = \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{S} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{q}_2 \end{pmatrix}$$

where $\mathbf{z}_1$ represents the sole dynamics part of the response

- Substitute in the first dynamic equation and exploit above results

$$\Longrightarrow \mathbf{M}_{11}\mathbf{S}\ddot{\mathbf{q}}_2 + \mathbf{M}_{11}\ddot{\mathbf{z}}_1 + \mathbf{K}_{11}\mathbf{q}_1^{qs} + \mathbf{K}_{11}\mathbf{z}_1 = -\mathbf{K}_{12}\mathbf{q}_2 - \mathbf{M}_{12}\ddot{\mathbf{q}}_2$$

$$\Longrightarrow \begin{cases} \mathbf{M}_{11}\ddot{\mathbf{z}}_1 + \mathbf{K}_{11}\mathbf{z}_1 &= \mathbf{g}_1(t) \\ \mathbf{g}_1(t) &= -\mathbf{M}_{11}\ddot{\mathbf{q}}_1^{qs} - \mathbf{M}_{12}\ddot{\mathbf{q}}_2 = -(\mathbf{M}_{11}\mathbf{S} + \mathbf{M}_{12})\ddot{\mathbf{q}}_2 \end{cases}$$

- Consider next the system fixed to the ground and solve the corresponding EVP

$$\mathbf{K}_{11}\mathbf{x} = \omega^2 \mathbf{M}_{11}\mathbf{x} \Rightarrow \widetilde{\mathbf{Q}} = [\cdots \tilde{\mathbf{q}}_{a_i} \cdots]$$

$$\Longrightarrow \mathbf{z}_1(t) = \widetilde{\mathbf{Q}}\eta(t)$$

- Solve $\mathbf{M}_{11}\ddot{\mathbf{z}}_1 + \mathbf{K}_{11}\mathbf{z}_1 = \mathbf{g}_1(t)$ for $\mathbf{z}_1(t)$ using the modal superposition technique

- Case of a global support acceleration $\ddot{\mathbf{q}}_2(t) = \mathbf{u}_2\phi(t)$, where $\mathbf{u} = [\mathbf{u}_1 \quad \mathbf{u}_2]^T$ denotes a rigid body mode
  - decomposition of the solution into a rigid body motion and a relative displacement $\mathbf{y}$

$$\ddot{\mathbf{q}} = \ddot{\mathbf{q}}^{rb} + \ddot{\mathbf{y}} \Rightarrow \begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \phi(t) + \begin{bmatrix} \ddot{\mathbf{y}}_1 \\ 0 \end{bmatrix}$$

  - $\mathbf{u}_1 = \mathbf{S}\mathbf{u}_2 \Rightarrow \mathbf{g}_1(t) = -(\mathbf{M}_{11}\mathbf{S}\mathbf{u}_2 + \mathbf{M}_{12}\mathbf{u}_2)\phi(t)$

$$\Longrightarrow \boxed{\mathbf{g}_1(t) = -(\mathbf{M}_{11}\mathbf{u}_1 + \mathbf{M}_{12}\mathbf{u}_2)\phi(t)}$$

- Method of additional masses (approximate method)
  - suppose the system is subjected not to a specified $\mathbf{q}_2(t)$ but to an imposed

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{f}(t) \end{pmatrix}$$

  - suppose that masses associated with $\mathbf{q}_2$ are increased to $\mathbf{M}_{22} + \mathbf{M}_{22}^{\star}$
  - then

$$\begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} + \mathbf{M}_{22}^{\star} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{f}(t) \end{pmatrix}$$

  - by elimination one obtains
    $\ddot{\mathbf{q}}_2 = (\mathbf{M}_{22} + \mathbf{M}_{22}^{\star})^{-1} (\mathbf{f}(t) - \mathbf{K}_{22}\mathbf{q}_2 - \mathbf{K}_{21}\mathbf{q}_1 - \mathbf{M}_{21}\ddot{\mathbf{q}}_1)$ and

$$\mathbf{M}_{11}\ddot{\mathbf{q}}_1 + \mathbf{K}_{11}\mathbf{q}_1 = -\mathbf{K}_{12}\mathbf{q}_2 - \mathbf{M}_{12}(\mathbf{M}_{22} + \mathbf{M}_{22}^{\star})^{-1} (\mathbf{f}(t) - \mathbf{K}_{22}\mathbf{q}_2 - \mathbf{K}_{21}\mathbf{q}_1 - \mathbf{M}_{21}\ddot{\mathbf{q}}_1)$$

- Method of additional masses (continue)
  - therefore

$$\begin{aligned}\{\mathbf{M}_{11} \quad - \quad & \mathbf{M}_{12}(\mathbf{M}_{22}+\mathbf{M}_{22}^{\star})^{-1}\mathbf{M}_{21}\ \}\,\ddot{\mathbf{q}}_1 + \left\{\mathbf{K}_{11} - \mathbf{M}_{12}(\mathbf{M}_{22}+\mathbf{M}_{22}^{\star})^{-1}\mathbf{K}_{21}\right\}\mathbf{q}_1 \\ = \quad & -\mathbf{K}_{12}\mathbf{q}_2 - \mathbf{M}_{12}(\mathbf{M}_{22}+\mathbf{M}_{22}^{\star})^{-1}\left(\mathbf{f}(t) - \mathbf{K}_{22}\mathbf{q}_2\right)\end{aligned}$$

  - now if $(\mathbf{M}_{22}+\mathbf{M}_{22}^{\star})^{-1} \to 0$ and if $\mathbf{f}(t) = (\mathbf{M}_{22}+\mathbf{M}_{22}^{\star})\ddot{\mathbf{q}}_2$, one obtains

$$\mathbf{M}_{11}\ddot{\mathbf{q}}_1 + \mathbf{K}_{11}\mathbf{q}_1 = -\mathbf{K}_{12}\mathbf{q}_2 - \mathbf{M}_{12}\ddot{\mathbf{q}}_2 \Longrightarrow \mathbf{M}_{11}\ddot{\mathbf{q}}_1 + \mathbf{M}_{12}\ddot{\mathbf{q}}_2 + \mathbf{K}_{11}\mathbf{q}_1 + \mathbf{K}_{12}\mathbf{q}_2 = 0$$

    which is the same equation as in the general case of excitation through support motion
  - $\Rightarrow$ for $\mathbf{M}_{22}^{\star}$ very large, the response of the system with ground motion is the same as the response of the system with added lumped mass $\mathbf{M}_{22}^{\star}$ and the forcing function

$$\begin{pmatrix} \mathbf{0} \\ (\mathbf{M}_{22}+\mathbf{M}_{22}^{\star})\ddot{\mathbf{q}}_2 \end{pmatrix} \approx \begin{pmatrix} \mathbf{0} \\ \mathbf{M}_{22}^{\star}\ddot{\mathbf{q}}_2 \end{pmatrix}$$

    where $\ddot{\mathbf{q}}_2$ is given $\Rightarrow$ apply same solution method as for problems of response to external loading