CME 345: Model Reduction (Spring 2017)

Homework #3
Due May 22, 2017

Final Project: Proposal

Describe in a few sentences the project that you would like to conduct. In particular, specify the nature of the HDM (HDR (linear, nonlinear, parameterized, ...) and some of the model reduction techniques that you would like to apply to this model (if you already know which ones).

Problem: Method of Snapshots for Multiple Parametric Configurations

Consider the parameterized advection-diffusion problem of the first homework assignment. A Matlab implementation of this problem (advectionDiffusion.m) can be downloaded from

`https://web.stanford.edu/group/frg/course_work/CME345.html`

Recall that this problem is parameterized by a vector $\boldsymbol{\mu} \in \mathbb{R}^3$ constituted by the advection speed in the x-direction $\mathcal{U}_1$, the diffusivity coefficient κ, and the parameter $\overline{y}$ associated with the Dirichlet boundary condition. Hence, the parameter space is

$$\boldsymbol{\mu} = (\mathcal{U}_1, \kappa, \overline{y}) \in \mathcal{D} = [0, 0.5] \times [5 \times 10^{-4}, 0.025] \times [0.4, 0.6]$$

Here, the objective is to build a Galerkin projection-based ROM that is robust in the entire parameter space $\mathcal{D}$.

Part I: Computing the Candidate Parameter Points

1. To assess the accuracy of the constructed ROM in the parameter space $\mathcal{D}$, discretize uniformly $\mathcal{D}$ with 10 points in each of the three dimensions $\left(\text{resulting in } 10^3 \text{ candidate parameter points } \left\{\boldsymbol{\mu}_c^{(1)}, \ldots, \boldsymbol{\mu}_c^{(10^3)}\right\}\right)$, and compute the HDM solutions

$$\left\{\mathbf{w}\left(\boldsymbol{\mu}_r^{(1)}\right), \cdots, \mathbf{w}\left(\boldsymbol{\mu}_r^{(10^3)}\right)\right\}$$

associated with these parameters. Store the corresponding solutions.

Recall that these solutions will not be used to construct the ROM, but only to test its accuracy.

For given ROM solutions $\left\{\widetilde{\mathbf{w}}\left(\boldsymbol{\mu}_r^{(1)}\right), \cdots, \widetilde{\mathbf{w}}\left(\boldsymbol{\mu}_r^{(10^3)}\right)\right\} = \left\{\mathbf{V}\mathbf{q}\left(\boldsymbol{\mu}_r^{(1)}\right), \cdots, \mathbf{V}\mathbf{q}\left(\boldsymbol{\mu}_r^{(10^3)}\right)\right\}$, consider the two following accuracy measures:

- The maximum relative error:

$$E_{\max} = \max_{i=1,\cdots,10^3} \frac{\left\|\mathbf{w}\left(\boldsymbol{\mu}_r^{(i)}\right) - \widetilde{\mathbf{w}}\left(\boldsymbol{\mu}_r^{(i)}\right)\right\|_2}{\left\|\mathbf{w}\left(\boldsymbol{\mu}_r^{(i)}\right)\right\|_2}$$

- The average relative error:

$$E_{\text{avg}} = \frac{1}{10^3} \sum_{i=1}^{10^3} \frac{\left\|\mathbf{w}\left(\boldsymbol{\mu}_r^{(i)}\right) - \widetilde{\mathbf{w}}\left(\boldsymbol{\mu}_r^{(i)}\right)\right\|_2}{\left\|\mathbf{w}\left(\boldsymbol{\mu}_r^{(i)}\right)\right\|_2}$$

Part II: Building a ROM Using Random Sampling

2. Sample randomly 10 points $\{\boldsymbol{\mu}_r^{(1)}, \cdots, \boldsymbol{\mu}_r^{(10)}\}$ using a uniform distribution in the parameter domain $\mathcal{D}$ and compute the associated HDMs $\{\mathbf{w}(\boldsymbol{\mu}_r^{(1)}), \cdots, \mathbf{w}(\boldsymbol{\mu}_r^{(10)})\}$. (Hint: you can use the function `rand` in Matlab).

   Build a ROB $\mathbf{V}$ of dimension $k = 10$ using POD and the 10 HDM-based solutions of the considered advection-diffusion problem (note that no truncation is performed here as the size of the ROB is equal to the number of computed snapshots, assuming that these are linearly independent).

   Assess the accuracy of the corresponding ROM by computing $E_{\max}^{\text{random}}$ and $E_{\text{avg}}^{\text{random}}$.

   (Hint: since there is some randomness in the choice of the samples, repeat the above procedure 20 times and compute $E_{\max}^{\text{random}}$ and $E_{\text{avg}}^{\text{random}}$ by averaging respectively the quantities $E_{\max}^{(i)}$ and $E_{\text{avg}}^{(i)}$ obtained for $i = 1, \cdots, 20$).

3. Sample randomly 10 points $\{\boldsymbol{\mu}_{lhs}^{(1)}, \cdots, \boldsymbol{\mu}_{lhs}^{(10)}\}$ using a latin hypercube sampling in the parameter domain $\mathcal{D}$ and compute the associated HDM-based solutions of the considered advection-diffusion problem $\{\mathbf{w}(\boldsymbol{\mu}_{lhs}^{(1)}), \cdots, \mathbf{w}(\boldsymbol{\mu}_{lhs}^{(10)})\}$. (Hint: you can use the function `lhsdesign` in Matlab).

   Build a ROB $\mathbf{V}$ of dimension $k = 10$ using POD and the 10 HDM-based solutions of the considered advection-diffusion problem.

   Assess the accuracy of the corresponding ROM by computing $E_{\max}^{\text{lhs}}$ and $E_{\text{avg}}^{\text{lhs}}$.

   (Hint: as in question 2, repeat the above procedure 20 times and average the resulting errors).

Part III: Building a ROM Using Greedy Sampling

4. The parameterized linear HDM system can be written as

$$\mathbf{A}(\boldsymbol{\mu})\mathbf{w}(\boldsymbol{\mu}) = \mathbf{b}(\boldsymbol{\mu})$$

For a given ROM solution $\mathbf{q}(\boldsymbol{\mu})$, define the residual as $\mathbf{r}(\mathbf{q}(\boldsymbol{\mu}), \boldsymbol{\mu}) = \mathbf{A}(\boldsymbol{\mu})\mathbf{V}\mathbf{q}(\boldsymbol{\mu}) - \mathbf{b}(\boldsymbol{\mu})$, and write the error indicator $\mathcal{I}(\mathbf{q}(\boldsymbol{\mu}), \boldsymbol{\mu}) = \|\mathbf{r}((\mathbf{q}(\boldsymbol{\mu}), \boldsymbol{\mu})\|^2$ as a quadratic expression in $\mathbf{q}(\boldsymbol{\mu})$.

5. Assuming that $\mathbf{A}(\boldsymbol{\mu}) \in \mathbb{R}^{N\times N}$, $\mathbf{b}(\boldsymbol{\mu}) \in \mathbb{R}^N$, $\mathbf{V} \in \mathbb{R}^{N\times k}$ and $\mathbf{q}(\boldsymbol{\mu}) \in \mathbb{R}^k$ are given, does the number of operations associated with computing the error indicator $\mathcal{I}(\mathbf{q}(\boldsymbol{\mu}), \boldsymbol{\mu})$ scale with N? Why?

6. The greedy sampling approach proceeds by evaluating the error indicator for a large number of candidate parameters and selecting the parameter point where the largest ROM error is indicated.

   In order to efficiently evaluate $\mathcal{I}(\mathbf{q}(\boldsymbol{\mu}), \boldsymbol{\mu})$ for a large number of candidate parameters $\boldsymbol{\mu} \in \{\boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_c\}$, the special form for this problem of the parametric dependency of the large-scale quantities $\mathbf{A}(\boldsymbol{\mu})$ and $\mathbf{b}(\boldsymbol{\mu})$ needs to be exploited. To this end:

   (a) Show that $\mathbf{A}(\boldsymbol{\mu})$ can be written under the form

   $$\mathbf{A}(\boldsymbol{\mu}) = f_1(\boldsymbol{\mu})\mathbf{A}_1 + f_2(\boldsymbol{\mu})\mathbf{A}_2$$

   where $f_1(\cdot)$ and $f_2(\cdot)$ are two scalar functions of $\boldsymbol{\mu}$ and $\mathbf{A}_1$ and $\mathbf{A}_2$ do not depend on $\boldsymbol{\mu}$.

   (b) Show that the right-hand side $\mathbf{b}(\boldsymbol{\mu})$ can be decomposed as

   $$\mathbf{b}(\boldsymbol{\mu}) = \begin{bmatrix} \mathbf{b}_1(\boldsymbol{\mu}) \\ \mathbf{0} \end{bmatrix}$$

   where the dimension of the vector $\mathbf{b}_1(\boldsymbol{\mu})$ is much smaller than that of $\mathbf{b}(\boldsymbol{\mu})$.

   (c) Show that $\mathbf{b}(\boldsymbol{\mu})^T\mathbf{b}(\boldsymbol{\mu})$ can be computed with a number of operations that does not depend on N.

   (d) Show that by pre-computing a few small-dimensional quantities once for all, the product $\left(\mathbf{V}^T\mathbf{A}(\boldsymbol{\mu})^T\mathbf{A}(\boldsymbol{\mu})\mathbf{V}\right)$ can then be computed for any $\boldsymbol{\mu}$ with a number of operations that does not depend on N.

   (e) Show that by pre-computing a few small-dimensional quantities once for all, the product $\left(\mathbf{b}(\boldsymbol{\mu})^T\mathbf{A}(\boldsymbol{\mu})\mathbf{V}\right)$ can then be computed for any $\boldsymbol{\mu}$ with a number of operations that does not depend on N.

   (f) Conclude by providing an expression for $\mathcal{I}(\mathbf{q}(\boldsymbol{\mu}), \boldsymbol{\mu})$ whose evaluation does not require an operation count that scales with N.

7. Show that by exploiting pre-computations based on the special forms of $\mathbf{A}(\boldsymbol{\mu})$ and $\mathbf{b}(\boldsymbol{\mu})$ for this problem, the computation of the reduced operators $\mathbf{V}^T\mathbf{A}(\boldsymbol{\mu})\mathbf{V}$ and $\mathbf{V}^T\mathbf{b}(\boldsymbol{\mu})$ (obtained from Galerkin projection) can be done for any $\boldsymbol{\mu}$ with an operation count that does not scale with N.

8. Implement in the provided Matlab code the pre-computations necessary for efficiently computing the reduced-order operators $\mathbf{V}^T\mathbf{A}(\boldsymbol{\mu})\mathbf{V}$ and $\mathbf{V}^T\mathbf{b}(\boldsymbol{\mu})$, and the error indicator $\mathcal{I}(\mathbf{q}(\boldsymbol{\mu}), \boldsymbol{\mu})$.

9. Perform a latin hypercube random sampling of the parameter domain $\mathcal{D}$ in $1,000$ points $\{\boldsymbol{\mu}_{cand}^{(1)}, \cdots, \boldsymbol{\mu}_{cand}^{(1000)}\}$.

   Implement the greedy sampling algorithm of Chapter 4 and run it for 9 iterations. You should obtain 10 HDM solutions (including the initial one). At each iteration, use the sampled parameter points to determine the parameter point of maximum error. (Note: do not compute the HDM-based solutions at these points – Hint: you can choose the first sampled point to be at the center of the parameter domain $\mathcal{D}$).

   Build a ROB $\mathbf{V}$ of dimension $k = 10$ using POD and the aforementioned 10 HDM-based solutions.

   Assess the accuracy of the associated ROM by computing $E_{\max}^{\text{greedy}}$ and $E_{\text{avg}}^{\text{greedy}}$. (Hint: as in questions 2 and 3, repeat the procedure 20 times and average the resulting errors).