CME 345: Model Reduction (Spring 2017)

Homework #4
Due June 5, 2017

Problem 1: Cross-Gramian

In addition to the infinite reachability and controllability gramians, one can define a third one, called cross-gramian and defined as the solution $\mathcal{X}$ to the Sylvester equation

$$\mathbf{A}\mathcal{X} + \mathcal{X}\mathbf{A} + \mathbf{B}\mathbf{C} = \mathbf{0}$$

1. Show that
$$\mathcal{X} = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{C}e^{\mathbf{A}t}dt$$
is solution of the above equation.

2. **Definition:** An LTI system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is called symmetric if $\mathbf{D} = \mathbf{D}^T$ and there exists a matrix $\mathbf{\Psi} = \mathbf{\Psi}^T \in \mathbb{R}^{N\times N}$ such that
$$\begin{aligned} \mathbf{A}\mathbf{\Psi} &= \mathbf{\Psi}\mathbf{A}^T \\ \mathbf{B} &= \mathbf{\Psi}\mathbf{C}^T \end{aligned}$$
Show that for a symmetric system, $\mathbf{A}^k\mathbf{\Psi} = \mathbf{\Psi}(\mathbf{A}^T)^k$, $\forall k = 0, 1, \cdots$.

3. Show that $e^{\mathbf{A}t}\mathbf{\Psi} = \mathbf{\Psi}e^{\mathbf{A}^T t}$.

4. Give the relation between $\mathcal{P}$, $\mathcal{X}$ and $\mathbf{\Psi}$ for symmetric systems.

5. From now on, assume that $\mathbf{\Psi}$ is nonsingular. Show that $\mathbf{\Psi}^{-1}e^{\mathbf{A}t} = e^{\mathbf{A}^T t}\mathbf{\Psi}^{-1}$.

6. Use the result of the previous question to establish the relation between $\mathcal{Q}$, $\mathbf{\Psi}$ and $\mathcal{X}$ for symmetric systems.

   (Hint: start from the expression for $\mathcal{Q}$ and express it in terms of $\mathcal{X}$ and $\mathbf{\Psi}$.)

7. Conclude that $\mathcal{X}^2 = \mathcal{P}\mathcal{Q}$ in the case of symmetric systems.

8. What is $\mathcal{X}^2$ when the dynamical system is balanced?

Problem 2: Moment Matching

Consider an LTI system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ for which $\mathbf{D} = \mathbf{0}$. The goal of this problem is to prove the following theorem which was covered in chapter 7.

Theorem: Let $s_i \in \mathbb{C},\ i = 1, \cdots, 2l$, $\mathbf{V}$ be a right ROB satisfying

$$\text{range}(\mathbf{V}) = \bigcup_{i=1}^{l} \mathcal{K}_k\left((s_i\mathbf{I}_N - \mathbf{A})^{-1}, (s_i\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{b}\right)$$

and $\mathbf{W}$ be a left ROB satisfying

$$\text{range}(\mathbf{W}) = \bigcup_{i=l+1}^{2l} \mathcal{K}_k\left((s_i\mathbf{I}_N - \mathbf{A}^T)^{-1}, (s_i\mathbf{I}_N - \mathbf{A}^T)^{-1}\mathbf{c}^T\right)$$

and $\mathbf{W}^T\mathbf{V}$ is nonsingular. Then, the ROM obtained by Petrov-Galerkin projection of the HDM $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ using $\mathbf{W}$ and $\mathbf{V}$ satisfies

$$\eta_{r,j}(s_i) = \eta_j(s_i) \Leftrightarrow \mathbf{H}_r^{(j)}(s_i) = \mathbf{H}^{(j)}(s_i),\ \forall i = 1, \cdots, 2l,\ \forall j = 0, \cdots, k-1$$

1. Show that when $\mathbf{W}^T\mathbf{V} \neq \mathbf{I}_k$, the moments of the transfer function associated with the ROM
$$\mathbf{H}_r(s) = \mathbf{c}_r(s\mathbf{W}^T\mathbf{V} - \mathbf{A}_r)^{-1}\mathbf{b}_r = \mathbf{c}\mathbf{V}(s\mathbf{W}^T\mathbf{V} - \mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b}$$
are given by
$$\eta_{r,m}(s) = m!\mathbf{c}\mathbf{V}\left((s\mathbf{W}^T\mathbf{V} - \mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\left(\mathbf{W}^T\mathbf{V}\right)\right)^m (s\mathbf{W}^T\mathbf{V} - \mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b}$$
(Hint: exploit the identity $\dfrac{d\mathbf{M}^{-1}}{ds}(s) = -\mathbf{M}^{-1}(s)\dfrac{d\mathbf{M}}{ds}(s)\mathbf{M}^{-1}(s)$.)
2. Prove the above theorem.