Uncertainty quantification of combustion noise by generalized polynomial chaos and state-space models

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Many physical systems are subject to uncertainty in operating regimes, boundary conditions, and physical parameter values. The generalized polynomial chaos (gPC) framework offers methods to represent and propagate uncertainties through the governing equations by means of spectral expansions in random space. The present study combines intrusive gPC with a state-space thermoacoustic model to account for uncertainties in combustion noise prediction of confined flames. The acoustic waves, flame response and acoustic reflection coefficients are modeled as stochastic variables and projected onto a finite set of gPC basis functions. By solving the resulting set of equations once, it is possible to determine probability density functions of acoustic quantities at each node of the discretized domain. Results of the proposed method are satisfactorily validated against Monte Carlo simulation and compared with experiments. We show that the contribution of the flame response uncertainties (magnitude and phase) to the sound pressure level produced by combustion is particularly important within a frequency range, which is close to the frequency characterizing the intrinsic thermoacoustic feedback loop. Additionally, we demonstrate the simplicity of performing global sensitivity analysis once the gPC coefficients are available. Furthermore, non-intrusive gPC is applied to the deterministic state-space model of the system and computational costs are compared with those of the intrusive gPC counterpart.

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1. Introduction

Combustion noise is defined as the noise created by combustion and produced in a direct or indirect manner. On the one hand, indirect combustion noise originates as the result of acceleration of combustion-generated entropy fluctuations. This type of noise is typically created in zones exhibiting strong gradients in the mean flow. On the other hand, direct combustion noise is directly produced by turbulent combustion, and is mainly associated with fluctuations of heat release rate [1–4]. A recent study has shown that acceleration of entropy gradients occurring at the flame front should also be included in the source term formulation, in addition to fluctuations of heat release rate, if the phase of the acoustic waves produced is of interest [5]. Because in this work we do not study indirect combustion noise, direct combustion noise will be referred to in this article simply as combustion noise.

Numerical approaches to quantifying combustion noise are in general of the hybrid type [6]. In a first step, a high-fidelity numerical simulation (e.g., Large Eddy Simulation LES) is performed to evaluate the sources of noise produced by the turbulent flame. In a second step a set of acoustic equations is solved, where the sources of combustion noise calculated in the previous step are accounted for. This system of equations is generally linear and solved for the fluctuating acoustic quantity of interest (acoustic pressure, density and/or velocity). The aforementioned procedure presumes that the sources of combustion noise are not affected by the surrounding acoustic field [4,7]. Nevertheless, in [8] it was shown that this assumption neglects the role of the Intrinsic Thermo-Acoustic (ITA) feedback loop as a mechanism of combustion noise generation. The effect of the acoustic waves on the sources of noise can be recovered if the flame response to upstream velocity perturbations is included as input in the hybrid approach [8].

The reliability of hybrid approaches is limited by the presence of uncertainties in the sources of combustion noise, flame response model, mean flow field, acoustic boundary conditions and the acoustic model itself. Complete elimination of uncertainties is unrealistic, and reliable predictions of combustion noise should account for these uncertainties by means of stochastic models and Uncertainty Quantification (UQ).
The present study proposes an efficient method, based on intrusive generalized polynomial chaos (igPC) and State-Space (SS) models, to perform UQ in hybrid approaches devoted to the assessment of combustion noise. It consists of: i) describing the acoustic system as a SS system of equations; ii) expanding the SS system in terms of gPC to account for the uncertainties in the input variables; iii) solving the expanded linear system for the acoustic variables. The outputs are stochastic variables characterized by Probability Density Functions (PDF) that provide all statistical moments (expected value, variance, etc.) of the quantities of interest. It should also be noted that the UQ method proposed in this study is not a sampling method. Only one single solution of an extended system of equations is necessary to evaluate the quantities of interest. Results from the proposed approach can be postprocessed at low cost to perform detailed UQ studies by means of global sensitivity analysis [9]. Accordingly, it becomes computationally affordable to quantify the relative importance of all input uncertainties on the uncertainty in the sound pressure level. In addition to uncertainty propagation, we propose some strategies to quantify input uncertainties in the flame response model and in acoustic boundaries characterized by perforated plates.

This article has the following structure. A brief overview of the current methods for UQ in thermoacoustics is carried out in the next section, whereas the gPC approach is introduced in Section 3. The SS framework is introduced in Section 4, and a SS model of combustion noise for confined flames is proposed in Section 5. This model is extended by means of igPC in Section 6. Section 7 focuses on the modeling of the input uncertainties (flame response and acoustic boundaries). Finally, the numerical results are presented in Section 8.

2. Uncertainty quantification in thermoacoustics

UQ concentrates on the study of systems that are characterized by random inputs and outputs, where randomness is caused by the presence of uncertainties that can be of epistemic nature (lack of knowledge) or aleatory, i.e., because of intrinsic variability [10]. Once the random characteristics of the inputs are inferred, forward UQ focuses on methodology to propagate these uncertainties to the outputs. This step is usually the most complex and computationally intensive for realistic engineering simulations.

In thermoacoustics UQ is commonly performed by means of the deterministic model resolution paradigm [11,12] and follows two steps:

1. Simplification of the equations: A system of acoustic equations (e.g. Linearized Navier–Stokes Equations, Linearized Euler Equations or Helmholtz equation) describes the acoustic states of a system that are produced by external sources. In absence of external sources, these equations are homogeneous and reduce to eigenvalue problems, where the outputs are the eigenfrequencies and eigenmodes. By making assumptions such as small variance of the input parameters, surrogate models are derived from the acoustic equations. As a result, the computational cost is significantly reduced, and sampling methods for UQ become computationally affordable.

2. Sampling of the probability space: Monte Carlo sampling is commonly used to perform multiple realizations of the surrogate model [13,14]. It is easy to implement and provides reliable results if a sufficient amount of data can be retrieved at a low computational cost to compute the statistics (expectation, variance, kurtosis, etc.).

To the best of the authors’ knowledge, there has been no study that focuses on UQ of combustion noise estimation. Instead, most UQ-studies in thermoacoustics have focused on combustion instabilities, where surrogate models replace the homogeneous system of acoustic equations. The main interest in these studies is to quantify the probability of a system to be thermoacoustically unstable under predetermined input uncertainties. Among the UQ techniques studied in thermoacoustics, the adjoint approach has proven to be computationally efficient and accurate when restricted to small input uncertainties [15]. In contrast, considering large input uncertainties requires higher-order correction terms of the adjoint formulation [16]. In addition to the adjoint method, the active subspace method has been successfully implemented in [17,18]. This method is beneficial for systems with a large number of input uncertainties that can be accurately approximated by a small number of linear combinations of variables (called active variables), as is the case in annular combustion chambers, where numerous burners are present. Methods making use of the Gaussian Process approach have been demonstrated successful for large numbers of input random variables [19–21]. Recently, non-intrusive generalized polynomial chaos (NigPC) was applied to a laminar flame configuration, accounting for uncertainties in the high-fidelity numerical simulations that provide the flame response to upstream velocity perturbations [22].

The performance of the deterministic model resolution paradigm depends on the quality of the surrogate model. Generally, this surrogate model is reliable only as long as it is close to a predetermined condition of operation, often equal to the expected values of the input parameters. In contrast, the igPC method does not depend on tuning around an operating condition and is not restricted to small input variances. The following section gives a brief overview of gPC.

3. Generalized polynomial chaos

For a large variety of problems, generalized Polynomial Chaos (gPC) provides an efficient means to represent and propagate uncertainty via spectral expansions in stochastic variables with known or estimated distributions [23,24]. For instance, Legendre polynomials admit an optimal representation of a stochastic process in uniform random variables. However, orthogonal polynomials are not limited to well-known distributions, as described in e.g. [25]. In this work we will estimate optimal orthogonal polynomials whenever data is available, and otherwise make assumptions on the distribution of parameters and rely on tabulated polynomials.

Let \( \xi = (\xi_1, \ldots, \xi_d) \) be a vector of independent random variables with finite moments. The joint PDF \( f_k(\xi) \) is given by the product of the marginal PDFs \( f_k(\xi_j) \), i.e.,

\[
f_k(\xi) = f_{\xi_1}(\xi_1) \cdots f_{\xi_d}(\xi_d).
\]

By the finite-moments assumption, for each random variable \( \xi_i \) (\( i = 1, \ldots, d \)) there exists a set of basis polynomials \( \{\psi^{(i)}(\xi_j)\}_{j=1}^{\infty} \) that are orthogonal w.r.t. \( f_{\xi_j} \), i.e.,

\[
\mathbb{E}(\psi^{(i)}(\xi_j)\psi^{(i)}(\xi_k)) = \int \psi^{(i)}(\xi_j)\psi^{(i)}(\xi_k)f_{\xi_j}d\xi_j = \delta_{jk},
\]

where \( \mathbb{E} \) denotes the expectation operator and \( \delta_{jk} \) is the Kronecker-delta function. The basis functions satisfy a three-term recurrence relation,

\[
\psi^{(i)}_{j+1}(\xi_j) = (\xi_j - \hat{\alpha}^{(i)}_{j})\psi^{(i)}_{j} - \hat{\beta}^{(i)}_{j+1}\psi^{(i)}_{j+1}, \quad j = 0, 1, \ldots, \psi^{(i)}_{0} = 0, \psi^{(i)}_{d} = 1.
\]

The recurrence coefficients \( \hat{\alpha}^{(i)}_{j}, \hat{\beta}^{(i)}_{j} \) define the basis polynomials (as well as Gauss-type quadrature rules) with respect to the PDF \( f_{\xi_j} \). The gPC basis functions of arbitrary order can be computed for any PDF using the Stieltjes procedure [26] if the PDF \( f_{\xi_j} \) is known, or up to order \( n \) using the modified Chebyshev algorithm if all moments up to \( 2n \) are known [25]. For the experimental cases where \( f_{\xi_j} \) is unknown, the analytical moments are
replaced by the sample moments, before applying the Chebyshev algorithm, or a PDF can be estimated through kernel density estimation. In the case of independent random variables $\xi$, multivariate basis functions can be generated as products of the univariate basis functions

$$
\Psi_\alpha(\xi) = \psi_\alpha^1(\xi_1) \cdots \psi_\alpha^d(\xi_d), \quad \text{for all multi-indices } \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d.
$$

(2)

By construction, the basis functions $\Psi_\alpha$ are orthogonal with respect to the PDF $f_\xi$. In practice, the basis is truncated to a finite number of terms by restricting the total polynomial order to some integer $q$, i.e. $\sum \alpha_j \leq q$, but other truncation schemes are possible. This results in an expansion in $P = (d + q)!/d!q!$ terms. For ease of notation, we map the multi-index $\alpha$ to a single index $i = 1, \ldots, P$.

Any finite-variance function $c(\xi)$ can be described as a weighted sum of the orthogonal stochastic basis functions $\{\Psi_i\}_{i=1}^P$, as given by

$$
c(\xi) = \sum_{k=1}^P c_k \Psi_k(\xi),
$$

(3)

where the truncation to $P$ terms should reflect the desired accuracy of the gPC expansion. When $P$ grows, (3) converges in the mean-squared sense. The coefficients $c_k$ may be functions of space and time, admitting a separation of stochastic and physical space, and are defined by the projections

$$
c_k = \mathbb{E}_\xi(c(\xi) \Psi_k(\xi)) = \int c(\xi) \Psi_k(\xi) f_\xi(\xi) d\xi.
$$

(4)

Two variants of solution methods are encountered to compute the stochastic modes: NigPC and igPC. In the former method, several deterministic computations are performed in order to create a collection of realizations of the output of interest. Interpolation or quadrature rules are applied to approximate the coefficients of each polynomial. In the igPC method, the system of equations under consideration is projected onto the space spanned by the basis functions. Subsequently, the resulting enlarged and modified system of equations is solved once for the gPC coefficients, as illustrated in Fig. 1. Significant theoretical progress on gPC has been achieved during the last decade in the field of computational fluid dynamics [11, 12, 27, 28].

### 4. The state-space model

The dynamics of complex systems is well described by SS models if the input-output map can be considered linear. The system's complexity, understood as the number of subsystems with different space and time scales associated to very diverse physical mechanisms, can efficiently be broken down into many simple SS subsystems that are connected by predefined input and outputs [29]. A generic SS model reads:

$$
\dot{x} = \hat{A}x + \hat{B}u
$$

(5)

$$
y = \hat{C}x + \hat{D}u,
$$

(6)

$$
\hat{u} = \hat{F}y + u
$$

(7)

where the vectors $x$, $u$ and $y$ stand for the system states, inputs and outputs, respectively. The state matrix $\hat{A}$ characterizes the system states, and the input matrix $\hat{B}$ connects the inputs to the states. The output matrix $\hat{C}$ relates the outputs to the states, the feedthrough matrix $\hat{D}$ directly relates inputs to outputs, and the connectivity matrix $F$ relates the inputs and outputs of all subsystems. A representation of the SS matrices is displayed in Fig. 2 with an example of two interconnected subsystems. The symbol ‘$\cdot$’ denotes a block-diagonal matrix composed by blocks (sub-matrices) describing each subsystem, and implies that the connectivity between sub-systems (relation of input and outputs) is disregarded. To account for the connectivity between sub-systems, Eqs. (6) and (7) are combined to obtain $\hat{u}(I - \hat{F}\hat{D}) = \hat{F}\hat{C}x + u$, and introduced in Eq. (5) resulting in

$$
\dot{x} = \hat{A}x + \hat{B}(I - \hat{F}\hat{D})^{-1}\hat{F}\hat{C}x + \hat{B}(I - \hat{F}\hat{D})^{-1}u.
$$

(8)

By applying the Laplace transform to Eq. (8), written as $\mathcal{L}[a(t)] = \hat{a}$, we obtain the frequency description of the system,

$$
A\hat{x} - s\hat{x} = \hat{b},
$$

(9)

where $\mathcal{L}[x(t)] = \hat{x}$, and $s = \sigma + ko$ is the Laplace complex variable, whose real part $\sigma$ denotes the growth rate of the thermacoustic mode $\hat{x}$ and the imaginary part represents the oscillation.

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**Fig. 1.** Schematic representation of deterministic and stochastic approaches for UQ.
frequency. Eq. (9) presents a linear system, which is used in this study to quantify the noise produced by a confined, turbulent premixed flame. In the following the symbol () is dropped for readability purposes.

5. SS model of combustion noise produced by a confined flame

We investigate a turbulent premixed combustor that contains a plenum, a flame holder, a swirled premixed flame and a combustion chamber, as illustrated in Fig. 3. This system, known as ‘BRS’ combustor (German acronym), has been widely used to study combustion noise and combustion instabilities [8,30,31]. Each element (plenum, flame holder, flame, combustion chamber) is represented by a SS system of equations [32–34]. The main advantage of the SS representation is that each element can be properly described by a separate system (i.e., acoustic propagation in the duct element, finite impulse response in the flame element, etc) and easily connected with each other by e.g. incoming and outgoing acoustic waves. The geometric and thermodynamic parameters of the turbulent combustor are displayed in Table 1. The combustor subsystems and corresponding equations are described in the following.

![Fig. 2. State-space representation of two interconnected subsystems.](image)

**Fig. 2.** State-space representation of two interconnected subsystems.

![Fig. 3. Combustor representation.](image)

**Fig. 3.** Combustor representation.

**Table 1** Geometrical and thermodynamic parameters used in the acoustic network model. Indices denote P: Plenum, M: Mid-section, C: Combustion chamber (see Fig. 3).

| Length (m) | \( l_0 = 0.17 \), \( l_1 = 0.18 \), \( l_2 = 0.3 + l \) |
| Mean velocity at Mid-section (m.s\(^{-1}\)) | \( \bar{u}_M = \bar{u}_{ref} = 11.3 \) |
| Cross section area (m\(^2\)) | \( A_P = \frac{\pi}{2} 0.21 \), \( A_M = \frac{\pi}{4} (0.04^2 - 0.016^2) \), \( A_C = 0.09^2 \) |
| Temperatures (K) | \( T_0 = 293 \), \( T_n = 1600 \) |
| Microphone located at \( l_{mic} = 0.12 \) m |

5.1. Ducts (plenum, mid-section and combustion chamber)

The duct element describes the acoustics of the plenum, mid-section and combustion chamber under the plane wave approximation. The upstream \( g \) and downstream \( f \) traveling acoustic waves are defined by:

\[
f = \frac{1}{2} \left( \frac{\partial p'}{\partial t} + u' \right), \quad g = \frac{1}{2} \left( \frac{\partial p'}{\partial t} - u' \right),
\]

where \([\cdot]' = [\cdot] - \bar{[\cdot]}\) denotes a fluctuation and \([\cdot]\) denotes a temporal average. The propagation of these waves is characterized by the transport equations

\[
\dot{f} + \hat{c} \frac{df}{dx} = 0; \quad \dot{g} - \hat{c} \frac{dg}{dx} = 0.
\]
The density $\bar{\rho}$ and speed of sound $\bar{c}$ are defined either at cold conditions (plenum and mid-section) or at hot conditions (combustion chamber). The SS subsystem describing the duct element, and derived from Eq. (11), is described in A.1. Once discretized in space, the acoustic waves along the ducts are vectors $\mathbf{f}$ and $\mathbf{g}$, where the $j_{th}$ element ($f_j$ and $g_j$) is associated to a position $x_i$. Similarly, the discretized acoustic pressure along the combustor is written as a vector $p = (f + g)\hat{c}^2$.

Eq. (11) has analytical solutions $f$ and $g$. Instead of making use of those solutions, in this work we discretize Eq. (11) because: i) We want to relax the assumption of constant propagation speed along the duct; ii) The proposed discretization allows the derivation of an equation in the frequency domain, which is linear in the Laplace variable $s$. Such linearity can be exploited in future works, when aiming at solving the eigenvalue problem resulting from Eq. (9) when $\bar{b} = 0$.

5.2. Flame element

Most of the time in combustion noise studies, the flame is assumed to be insensitive to the surrounding acoustic field [1]. Accordingly, only one-way coupling is considered, where the flame produces acoustic waves but does not react to them. Recently, [8] showed that accounting for the flame response to acoustic fluctuations is essential to capture peaks in the combustion noise spectrum associated to the intrinsic thermoacoustic (ITA) feedback loop. The ITA loop is described as follows. Fluctuations of unsteady heat release rate produce downstream and upstream traveling acoustic waves. The latter modulates the velocity upstream of the flame, which in turn promote perturbations of the heat release rate, thus closing the loop. The ITA feedback loop is independent of the surrounding acoustic conditions and can exist even in anechoic combustors [35].

In this work the flame is described by an element associated to three subsystems: one defining the reference velocity, and the other two characterizing the flame response and the flame acoustic scattering.

5.2.1. Reference velocity

Premixed flames respond to acoustic waves by the acoustic modulated reference velocity, defined by $\bar{\nu}^{\text{ref}} = f_{\text{ref}} - g_{\text{ref}}$, where $\bar{f}_{\text{ref}}$ indicates a reference position immediately upstream of the flame. The corresponding SS system is described in A.2.

5.2.2. The flame response

The flame response $F(\omega)$ links upstream velocity perturbations at $\bar{\nu}^{\text{ref}}$ to global fluctuations of the unsteady heat release rate $\dot{Q}$. In time domain, the flame response can be described as a distributed time lag filter, which in turn is defined by an associated impulse response composed of coefficients $h_k$. Input and output are then related by the convolution

$$\frac{\dot{Q}}{Q} = \frac{1}{\bar{Q}} \sum_{k=0}^{L} h_k \nu_{\text{ref}}^{(\text{ref})},$$

In frequency domain, the input and output relationship can be written as

$$\frac{\dot{Q}}{Q} = F(\omega) \frac{\bar{\nu}}{\nu_{\text{ref}}},$$

where the flame response $F(\omega)$ is obtained from the Fourier transform of the impulse response

$$F(\omega) = n(\omega) e^{i\phi(\omega)} = \sum_{k=0}^{N_u} h_k e^{-i\omega k \Delta t}.$$
5.4. Assembly of the system

The matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, F$ and vectors $x, \tilde{u}, y$ associated with the interconnected system are shown in A.5 and A.6. The gPC-SS framework introduced in this work is not limited to the SS model proposed, as it applies to any system of equations that can be expressed as a SS system of equations, such as LNSE or LEE [40,41].

6. The stochastic state-space model

We account for uncertainties in the SS system by substituting the gPC expansion (3) into all elements of Eq. (8),

$$x = \sum_{k=1}^{P} \psi_k x_k, \quad u = \sum_{k=1}^{P} \psi_k u_k, \quad y = \sum_{k=1}^{P} \psi_k y_k, \quad \tilde{A} = \sum_{k=1}^{P} \psi_k \tilde{A}_k,$$

and similarly for $\tilde{B}, \tilde{C}, \tilde{D}$ and $F$. The gPC coefficients can be scalars, vectors or matrices. In the following example, it is shown how a stochastic system of equations is built from a generic model.

6.1. Example: projection of linear equation

We consider an algebraic equation $ax = b$, where $b$ and $a$ are given stochastic parameters with known gPC expansions (Eq. (3)). The solution reads $x = b/a$ if $a \neq 0$. The algebraic model

$$\sum_{k=1}^{P} \psi_k \tilde{a}_k x_k = \sum_{k=1}^{P} \psi_k b_k$$

is projected on the set of basis functions $\psi_j$ where $j = 1, \ldots, P$ to obtain

$$x_j = \sum_{k=1}^{P} \psi_k \tilde{a}_k x_k = \sum_{k=1}^{P} \psi_k b_k$$

where the orthonormality between polynomial functions (Eq. (11)) has been exploited. The resulting system of equations can be written as

$$\begin{bmatrix} G(a)_{1,1} & \cdots & G(a)_{1,P} \\ \vdots & \ddots & \vdots \\ G(a)_{P,1} & \cdots & G(a)_{P,P} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_P \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_P \end{bmatrix},$$

where $a = (a_1, \ldots, a_P)^T$, and the stochastic Galerkin matrix $G(a)$ is given by

$$G(a)_{jk} = \sum_{l=1}^{P} \tilde{a}_l \psi_j \psi_k \quad j, k = 1, \ldots, P.$$

Finally, Eq. (20) is solved for the $P$ unknowns $x$. The gPC coefficients are not stochastic, and therefore uncertainty quantification of the system can be performed without sampling.

6.2. Obtaining the gPC coefficients for the Stochastic-SS model

It is convenient to write Eq. (8) as

$$x = (\tilde{A} + Y)x + Z\tilde{u},$$

where $Y \equiv \tilde{B}W\tilde{C}$ and $Z \equiv \tilde{B}V$. Additionally, we define $W \equiv VF$, and $V \equiv (I - TD)^{-1}$.

Following the same procedure as in the Example, we obtain the expanded system of equations related to Eq. (8):

$$\begin{bmatrix} M_{1,1} & \cdots & M_{1,P} \\ \vdots & \ddots & \vdots \\ M_{P,1} & \cdots & M_{P,P} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_P \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_P \end{bmatrix},$$

where

$$M_{jk} = \sum_{k=1}^{P} G(\tilde{A} + Y)_{jk} - I\delta_{jk} \quad \text{and} \quad b_j = \sum_{k=1}^{P} G(Z)_{jk}u_k.$$

The coefficients $Y_j$ and $Z_j$ are readily obtained by solving

$$Y_j = \sum_{k=1}^{P} \tilde{B}_k H(W, \tilde{C})_{jk} \quad \text{and} \quad Z_j = \sum_{k=1}^{P} \tilde{C}_k G(V)_{jk},$$

where

$$H(W, \tilde{C})_{jk} \equiv \sum_{l=1}^{P} \sum_{m=1}^{P} W_l \tilde{C}_m \psi_j \psi_l \psi_m.$$  

6.3. Obtaining the gPC coefficients of inverse functions

Before obtaining the coefficients $Y_j$ and $Z_j$, it is necessary to evaluate the coefficients $V_{kl}$ and $W_{kl}$. Recalling Eq. (23), we expand the relations $(I - TD)V = I$ and $(I - TD)W = F$ and obtain

$$\begin{bmatrix} I - H_{1,1} & \cdots & H_{1,P} \\ \vdots & \ddots & \vdots \\ H_{P,1} & \cdots & I - H_{P,P} \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_P \end{bmatrix} = \begin{bmatrix} I \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I - H_{1,1} & \cdots & H_{1,P} \\ \vdots & \ddots & \vdots \\ H_{P,1} & \cdots & I - H_{P,P} \end{bmatrix} \begin{bmatrix} W_1 \\ \vdots \\ W_P \end{bmatrix} = \begin{bmatrix} F_1 \\ \vdots \\ F_P \end{bmatrix}.$$

Note that the arguments of the function $H$ (written as $W$ and $\tilde{C}$ in Eq. (26)) are $F$ and $D$ in Eq. (27), to be solved for the coefficients $V_j$ and $W_j$.

The Stochastic-SS system (Eq. (24)) is linear. Nevertheless, the propagation of input uncertainties to the output random variable $x$ is nonlinear because: (a) the inverse problem $x = M^{-1}b$ is nonlinear, (b) some input random variables (like the time delay $\tau$, see Appendix A) is non-linearly related to the matrices $M_{jk}$, (c) The matrices $V$ and $W$ are nonlinear functions of the matrices $F$ and $D$.

7. Inferring input uncertainties in combustion noise

One of the most important and critical steps in UQ, aside from uncertainty propagation, which is the main subject of the present study, is inferring the distribution and correlation structure of the input uncertainties. We identify three main input uncertainties:

i) The reflecting acoustic properties of the surrounding environment: We assume that the uncertainties in the acoustic reflection at the outlet dominate those of the inlet, as the latter can be well represented by a hard wall. Only a few inaccurate experimental measurements of the outlet acoustic reflection are available. This makes the inference of the uncertain outlet acoustic reflection very challenging.

ii) The response of the flame to impinging acoustic waves: The flame response model can be considered reliable, as it is obtained by high-fidelity numerical simulation [35] combined with system identification techniques. We assume that uncertainties in the flame response are exclusively associated with the system identification procedure.
The source(s) of combustion noise: We do not consider uncertainties in the sources of combustion noise, but their inclusion would be straightforward as they are proportional to the noise produced. As mentioned later in Section 8.2, this proportionality implies that the PDF of the radiated noise is scaled by the noise sources: they do not influence the distribution type.

The present study focuses on two independent setups. In case A we consider a simplified model, where all uncertainties are assumed Gaussian and, consequently, Hermite polynomials are used. In case B we consider a more realistic model of uncertainties, where probability distributions are directly given by the input data. We employ Data-Driven (DD) polynomials for the corresponding gPC expansion, using the methods proposed in [25].

7.1. Uncertainties in the outlet acoustic reflection

The combustor outlet is equipped with a perforated plate open to the atmosphere, which promotes high acoustic dissipation along the mid-frequency band. Some few experimental measurements of \( r^{(\text{out})} \) are available. The uncertainties associated with \( r^{(\text{out})} \) are of epistemic nature: there is no documentation about the kind of perforated plate used (thickness, porosity, etc) and measurements may not be fully reliable. We follow two different strategies to handle uncertainties in \( r^{(\text{out})} \).

7.1.1. Case A

The first case consists of a basic model that barely relies on the measurements. The phase \( \varphi \) of \( r^{(\text{out})} \) is included by the addition of a virtual length \( l_c = 0.2 \text{ m} \) (see Table 1). This model describes the phase of the reflection coefficient as being approximately linear in the frequency \( \omega \). Figure 4(b) shows that there is good agreement in the low frequency region between the end-correction model and the experimental results. Additionally, the magnitude of \( r^{(\text{out})} \) is considered constant along \( \omega \), and a value of \( |r^{(\text{out})}| = 0.65 \) is assigned to it (see Fig. 4(a)). This model has successfully been used in [8].

We assume that both magnitude and phase of \( r^{(\text{out})} \) follow a Gaussian distribution with expected value as in the deterministic case and standard deviation equal to 0.1 for both \( |r^{(\text{out})}| \) and \( \varphi^{(\text{out})} \) (see Fig. 4(a, b)). We choose Hermite polynomials for the corresponding gPC expansion. As expected, the model fit is quite poor as there is no empirical justification for assuming a Gaussian distribution for the phase and gain of \( r^{(\text{out})} \).

7.1.2. Case B

Due to lack of reliability of the experimental measurements, we use the data to estimate a stochastic model. The idea consists of deriving a physical model that qualitatively describes the behavior of \( r^{(\text{out})} \) over the entire frequency range. This model should be linear in some few parameters \( k_j \) to allow the use of linear regression on the experimental data.

We settle for a simple acoustic model that describes a multi-perforated plate (with back cavity) as a spring-damping system, frequently used in the design of acoustic resonators [42]. The impedance reads \( Z_M = Z_{\text{M}} + iZ_{\text{C}} \), where the resistance is given by \( Z_{\text{R}}(\omega) = k_1/\omega + k_2 + k_3 \), and the reactance by \( Z_{\text{X}}(\omega) = k_1/\omega + k_2 + k_3 \). This model accounts for acoustic losses induced by flow inertia, viscosity and flow separation effects. The reflection coefficient \( r^{(\text{out})} \) is thus given by

\[
r^{(\text{out})} = \frac{Z_M - 1}{Z_M + 1} e^{i\varphi} = \frac{Z_{\text{out}} - 1}{Z_{\text{out}} + 1} \tag{28}
\]

where \( Z_{\text{out}} = 1/Z_M \). A factor \( e^{i\varphi} \) has been added in order to impose a phase in \( r^{(\text{out})} \) equal to \( \pi \) in the zero frequency limit, which is the expected limit of a perforated plate open to the atmosphere (without back cavity). This leads to an impedance of the perforated plate \( Z_{\text{out}} \) equal to the inverse of \( Z_M \). The main advantage of the proposed model is that it is linear in the parameters \( k_j \), which is ideal for performing ordinary least squares regression.

Figure 5 (a, b) shows the values of \( Z_M \) associated to experiments, calculated from Eq. (28) as \( Z_M^{\exp} = -(r_{\text{exp}}^{\text{out}} - 1)/(r_{\text{exp}}^{\text{out}} + 1) \). It is highly encouraging to observe that the data collapse into samples following smooth and monotonic trajectories, which is evidence of the physical soundness of the transformation proposed. Subsequently, linear regression is applied to find optimal values for the coefficients \( k_j \). The coefficients \( k_2 \) and \( k_3 \) are dropped due to the negligible regression values assigned to them. It means that the impedance \( Z_M \) is well described by a function of \( 1/\omega \) in addition to a bias. The results shown in Fig. 5 suggest that the regression model is satisfactory as it adequately follows the trend of the experimental data across all frequencies under consideration.

The collection of data is assumed homoscedastic, which is one of the fundamental assumptions of ordinary linear regression. It means that the variance of \( Z_M(\omega) \) is equal for all frequencies \( \omega \), and computed as the variance of the data collection. The distribution of \( Z_M \) is unknown and cannot be assumed to follow any standard shape, e.g., Gaussian. Using the data, we can instead either compute the DD gPC polynomials directly from the sample moments, or first estimate the PDF by e.g. kernel density estimation, and then compute DD gPC. We use the latter approach here, as it
7.2. Uncertainties in the flame response

Similarly to the uncertainties in $r^{(out)}$, we consider two cases.

7.2.1. Case A

Case A accounts for a $n - \tau$ model written as $F(\omega) = ne^{-i\omega\tau}$, where both $n$ and $\tau$ are assumed independent of frequency, both with independent Gaussian distributions. We assign the same expected value and standard deviation for all frequencies, as shown in Table 2. Accordingly, we use Hermite polynomials for the associated gPC expansion.

7.2.2. Case B

We consider the coefficients $h_k$ (see Eq. (14)) to be uncertain and characterized by a covariance matrix directly obtained from system identification [18,43]. The propagation of these uncertainties into the gain $n(\omega)$ and phase $\phi(\omega)$ of the flame response is carried out by a Monte-Carlo method with negligible computational cost compared to the one associated with the solution of Eq. (24).

This is a very suitable way of propagating flame response uncertainties, because the uncertainties contained in sixteen input random variables (the sixteen $h_k$ coefficients) are recovered in just two random variables ($n(\omega)$ and $\phi(\omega)$) per frequency, thus reducing the stochastic complexity of the problem.

The expected value and standard deviation of $n(\omega)$ and $\phi(\omega)$ are shown in Fig. 8. The distribution at two different frequencies is shown in Fig. 7. Note that the distributions of $n(\omega)$ and $\tau(\omega)$ are not Gaussian (more evident in the case at 480 Hz) even if each coefficient $h_k$ is Gaussian distributed. This is due to the nonlinear transformation Eq. (14). We therefore construct a set of DD polynomial functions derived from the data of $n(\omega)$ and $\tau(\omega)$ at each frequency. Finally, we assume that $n(\omega)$ and $\phi(\omega)$ are not strongly correlated and consider them as independent random variables. This assumption was corroborated by Monte Carlo simulations with correlated and uncorrelated input data (not shown).

7.3. Input uncertainties in the gPC model

In this subsection we summarize the stochastic models and their implementation for the test cases A and B.

7.3.1. Case A

Four parameters are considered Gaussian and fully described by their expected value and standard deviation, as described in Table 1. Accordingly, Eq. (3) reduces to a two-term expansion for all the input random parameters as $c = c_1 \Psi_1 + c_2 \Psi_2$. The first coefficient refers to the expected value $c_1 = E(c)$, whereas the second represents the standard deviation $|c_2| = S(c)$. They are introduced in the Stochastic-SS model as follows:

1. The magnitude $|r^{(out)}|$ appears in the matrix $F$, as seen in Eq. (A.12). Accordingly, the expected value $E(|r^{(out)}|) = r_1$ and standard deviation $S(|r^{(out)}|) = r_2$ are introduced in $F_1$ and $F_2$, respectively.
2. The phase $\phi(\omega)$ of $r^{(out)}$ is related to a virtual length as per $\phi(\omega) = 2l_\omega / \ell_4$, where $l_\omega$ is independent of frequency. This virtual length is subsequently subtracted from the combustion chamber length $\ell_c = 0.336 - l_1$ (see Fig. 3). The expected value is given by $E(l_c) = 0.336 - E(l_1)$, where $E(l_1) = l_1$. The
standard deviation is assumed to be \( \bar{\mathcal{S}}(\xi) = l_{x,2} = 0.1\bar{\xi}_G/(2\omega) \) (which corresponds to \( \mathcal{S}(\psi^{\text{out}}) = 0.1 \)). Note also that the input parameter \( l_x \) is related to the matrix \( A^C \) and \( B^C \) via its inverse \( \bar{\xi}/\Delta \lambda = \bar{\eta}C/l_C \) (see Eq. (A.2)). To compute the gPC coefficients of \( l_C^{-1} \), we project the identity

\[
1 = l_C^{-1} = \sum l_{ij}^k \Psi_i \sum l_{jk}^{-1} \Psi_j,
\]

onto the basis functions, and solve the resulting linear system of equations \( G(l_C)^{-1} = \bar{e}_i = l_C^{-1} = [l_{C,1}^{-1}, \ldots, l_{C,n}^{-1}]^\top \). Finally, the matrices \( \hat{A}_k \) and \( \hat{B}_k \) are constructed from \( A_k^C \) and \( B_k^C \) (see Eq. (A.10)).

3. The input parameter \( \tau \) is related to the matrix \( A^F \) and \( B^F \) with its inverse \( 1/\tau \) (see Eq. (A.6)). Similarly to the variable \( l_C \), we solve the linear system of equations \( G(\hat{\tau})^{-1} = \bar{e}_i \) for \( \tau^{-1} = [\tau_{1,1}^{-1}, \ldots, \tau_{k,1}^{-1}]^\top \). Finally, the matrices \( \hat{A}_k \) and \( \hat{B}_k \) are assembled from \( A_k^F \) and \( B_k^F \) (see Eq. (A.10)).

4. The expected value \( \mathbb{E}(n) = n_1 \) and the standard deviation \( \mathcal{S}(n) = n_2 \) are introduced in the matrices \( C_k^F \) and \( C_k^F \), respectively (see Section A.2). In turn, these matrices are accounted for in \( C_1 \) and \( C_2 \) (see Eq. (A.11)).

7.3.2. Case B

Similarly to case A, four parameters are considered uncertain: \( \psi^{\text{out}}_1(n), \psi^{\text{out}}_2(n), \phi(n) \) and \( n(n) \). All of them are functions of frequency, and exactly represented by two-term expansions \( c(\omega) = c_1(\omega)\Psi_1 + c_2(\omega)\Psi_2 \), where \( \Psi \) is DD polynomials.

1. Similar to case A, the coefficients \( r_1^{(\text{out})}(\omega) \) and \( r_2^{(\text{out})}(\omega) \) are present in the matrix \( F \).

2. The phase \( \psi(n) \) is related to a virtual length \( l(n) = \psi(n)\bar{\xi}_G/(2\omega) \). This virtual length is subtracted from the combustion chamber length \( l = 0.336 \) - \( l \). In contrast to case A, we employ DD polynomials for \( l_C^{-1} \), which are directly derived from available data. This allows an accurate representation of \( l_C^{-1} \) by only two terms in the expansion, in contrast to the five terms in the Hermite polynomial expansion of case A. The matrices \( \hat{A}_k \) and \( \hat{B}_k \) are constructed from \( A_k^F \) and \( B_k^F \) (see Eq. (A.10)).

3. The inverse of the phase of \( \mathcal{F}(\omega) \) is associated to \( \tau(\omega)/\tau(\omega) \). Accordingly, we employ the two-term DD gPC expansion \( \tau^{-1}(\omega) = \tau_{1,1}^{-1}(\omega)\Psi_1 + \tau_{2,1}^{-1}(\omega)\Psi_2 \). As in case A, the matrices \( \hat{A}_k \) and \( \hat{B}_k \) are constructed from \( A_k^F \) and \( B_k^F \) (see Eq. (A.10)).

4. The gain \( n(n) \) is treated as in case A.

Note that for both case A and case B the matrices \( \hat{A}_k, \hat{B}_k, \hat{C}_k \) and \( F_k \) for \( k \geq 2 \) are composed of zero elements except at the entries where the uncertain variables \( (\bar{f}_k, \bar{l}_C^k, \bar{r}_k^{-1}, \bar{n}_k) \) are introduced.

8. Numerical results

8.1. Quantifying the uncertainty in the Sound Pressure Level

The right-hand side of Eq. (24) accounts for the frequency dependent source of combustion noise \( \bar{Q}_n \), as illustrated in Fig. 8 [8]. This source is contained in the vector \( u \), whose size is equal to the one of \( u \) (see Eq. (A.14)). All of its elements are zero except for the entry \( u^0 = [0, 0, \bar{p}(\bar{Q}_n)/Q]^\top \).

After assembling each matrix block \( M_{jk} \) defined by Eq. (25), the linear system (24) is solved for each frequency \( s = \omega \). Finally, the acoustic field is computed as \( p(s) = (f + g)\bar{c} = \sum \Psi_k f_k + g_k \bar{c} \). Note that the vectors \( f_k \) and \( g_k \) are subvectors of the state vectors \( [x_k^T, x_k^M, x_k^F]^\top \) (see Eqs. (A.11) and (A.13)).

The Sound Pressure Level (SPL) corresponding to case A and case B is computed as \( |p| \) for all integer frequencies between 1 and 800 Hz. Figure 9 shows the value of \( |p| \), i.e. the SPL evaluated at the location of the microphone used in the experiments [30] (see Table 1). Figure 9 shows in continuous black line the solution of Eq. (9). Three major peaks are observed for case
A. The first two (around 100 Hz and 300 Hz) are associated with the ITA feedback loop as \( f_{\text{ITA}} = 1/(2\pi) \) and \( f_{\text{ITA}} = 3/(2\pi) \). The third peak around 400 Hz is associated with the quarter-wave mode of the combustion chamber. For case B, two peaks are observed. The first is associated with the ITA feedback loop and the second with the quarter-wave mode of the combustion chamber, as discussed in [8]. Figure 9 also shows 100 colored dots per frequency denoting the percentiles of the PDFs: percentiles reaching 50% are shown in yellow, whereas extreme percentiles (approaching 0 or 100%) are dark blue colored. It is observed that the median (yellow trajectory) does not necessarily overlap with the deterministic estimation of the SPL (black trajectory). For case A, the SPL regions around 100 Hz and between 300 Hz and 400 Hz are specially affected by the input uncertainties, as seen by the large distance between the lower and upper percentiles (the 1% and 99% percentiles represented by the blue dots). For case B, in contrast, the distance between lower and upper percentiles grows until 400 Hz, and remains approximately constant for higher frequencies.

Experimental measurements (gray line) from [30] are also presented, showing the good agreement of the modeling approach, although some discrepancies are also noticeable. At high frequencies (larger than 100 Hz for case A and than 400 Hz for case B) experimental measurements lay within the uncertain region bounded by the lower and upper percentiles. Consequently, one can explain the miss-match against experiments by the fact that (at least) four model input parameters \((\mathbb{P}_{\text{out}}(\omega), \mathbb{Q}_{\text{out}}(\omega), \tau(\omega), \mathbb{M}(\omega))\) are uncertain. At low frequencies (below 100 Hz) there exists an important difference between experimental measurements and modeling results. Especially for case B, the input uncertainties do not significantly influence the outcome \( \mathbb{G}_{\text{mic}} \). Accordingly, uncertainties in the model parameters are not an explanation for the miss-match at low frequencies. One may believe that uncertainties in the sources of combustion noise (not considered in our study) may be the explanation for the disagreement under discussion. Note however, that the strength of the source, which is proportional to the radiated acoustic waves, would have to be more than one order of magnitude larger, so that the acoustic modeling results match experimental measurements. Such a scenario is not likely to be true. It is possible, as reported in other studies of combustion noise for confined flames [8, 44, 45], that hydrodynamic fluctuations at walls, whose energy may be significant at low frequencies, affect the acoustic measurements. As a result, the measured fluctuations of pressure contain both acoustic and hydrodynamic contributions, which is highly noticeable at low frequencies. Additionally, a pressure node at the location of the microphone is visible for frequencies around 50 Hz in the numerical results for both case A and B. No pressure node is observed in the measurements of combustion noise. This observation may be evidence that additional contributions to acoustic fluctuations are contained in the measured pressure signals.

In Fig. 10 we display the PDF for six different frequencies for case A. Very good agreement is obtained in comparison with results from Monte Carlo simulations, where 100000 calculations of Eq. (9) were performed for each frequency for samples of \( n, \tau, \mathbb{P}_{\text{out}} \). Fig. 10 (out) and \( \mathbb{Q}_{\text{out}} \). We highlight that even cases where the PDFs are very skewed (210 Hz) or show bimodal distributions (580 Hz) are well recovered. Note that there are few cases where the agreement is not as good (as for example at 300 Hz). A further increase of the order \( P \) of the polynomials does not improve results, suggesting that Hermite polynomials are inadequate for these cases. The PDFs for case B have also been validated against Monte Carlo simulations with very good agreement. The PDFs obtained do not exhibit standard distributions, as shown in Fig. 11.

It is important to clarify that results from case A and case B are based on different models. While case A uses an \( n \rightarrow \tau \) model for the flame response and a simple model for the outlet reflection coefficient (see Fig. 4 a and b), case B is based on a frequency dependent flame response and a more sophisticated model for the acoustic reflection at the outlet (see Fig. 4 c and d). The difference in these two models is what mainly determines the difference in the SPL shown in Fig. 9. This can be understood by looking at the black line, which denotes the SPL for deterministic calculations, i.e. when assuming no uncertainties in the input variables.

A question may eventually arise: what is the correlation between \( r(\text{out}) \) and the amplitude of the first (\( \approx 100 \) Hz) and second peak (\( \approx 400 \) Hz)? We now want to understand whether a high value of \( r(\text{out}) \) or a low value of \( r(\text{out}) \) produce a high or low amplitude of the first and second peak. This kind of analysis can be performed by means of the gPC expansion (3) of the frequencies associated with the two peaks.

We note in passing that the statistical correlation between two stochastic variables \( u \) and \( v \) can be computed directly from their gPC coefficients as \( \text{corr}(u, v) = \langle \sum_{i=2}^{p} u_i v_i \rangle / \sqrt{\langle \sum_{j=2}^{p} u_j^2 \rangle \langle \sum_{k=2}^{p} v_k^2 \rangle} \). Here, we use a more graphical depiction of correlation. The computed gPC expansion of case A is used to generate a large number of solution samples at low computational cost. The same input samples are used for the two frequencies of interest, and the results are plotted in Fig. 12 (left), showing the dependency of the two amplitudes as a function of \( r(\text{out}) \). The two peaks are anticorrelated: high values for the first and the second peak are conse-
Fig. 10. Case A: Probability density function for six frequencies, using Monte Carlo and igPC for $q = 5, d = 4$. 

Fig. 11. Case B: PDFs for six frequencies, using Monte Carlo and igPC for $q = 4, d = 4$. 
quence of high and low values of $r^{(\text{out})}$. This is further clarified by Fig. 12 (right) for three deterministic simulations. The analysis may be useful in acoustics of open and enclosed systems, since it can deliver correlations of acoustic peaks to any variable characterizing the acoustic system.

8.2. Global sensitivity analysis (Sobol indices)

Global sensitivity analysis by means of Sobol indices provides a decomposition of the total variance in terms of contributions of all subsets of input random variables [9]. The variance of any quantity of interest is expressed as the sum of the partial variances due to each input random variable in isolation, the sum of all pairs of random variables, and so on. The Sobol indices ($S$ with index subscripts) are defined as the normalized partial variances:

$$
\sum_i S_i + \sum_{ij} S_{ij} + \sum_{ijk} S_{ijk} + \cdots = 1. \tag{29}
$$

One advantage of performing UQ through gPC is that all Sobol indices can be computed analytically from the gPC coefficients [46]. An example illustrating the connection between Sobol indices and the gPC expansion is provided in the supplementary material.

The Sobol indices of $p^{(\text{mic})}$ for the cases A and B are shown in Fig. 13. Note that the results are especially important for frequencies where the variance of $p^{(\text{mic})}$ is high (see white subplots in Fig. 13). We observe that the uncertainties in the outlet reflection $\varphi^{(\text{out})}$ and $q^{(\text{out})}$ contribute the most to the overall uncertainty in $p^{(\text{mic})}$ in case A and B for most of the frequency range of interest. Uncertainties in the flame response (through the gain $a$ and time delay $\tau$) highly contribute to the overall uncertainty in $p^{(\text{mic})}$ in case B, but only in the frequency band around 50 Hz and the first peak in the spectrum (≈ 100 Hz). For 50 Hz < $f$ < 100 Hz, only the uncertainties that are close to 100 Hz play a significant role, as the variance of $p^{(\text{mic})}$ around this frequency is considerable. This is in agreement with our expectations, since the strength of this peak, which is exclusively related to the ITA feedback loop, directly depends on the flame response $F(\omega)$: If no flame response is accounted for in the model, the aforementioned peak does not show up at all in the spectrum [8]. Uncertainties in the time delay $\tau$ for case A are important over the whole range of frequencies considered. This is because, in case A, uncertainties in $\tau$ are constant and induce uncertainties in the flame response phase $\phi$ that are proportional to frequency as per $\phi = \omega \tau$. The phase shift produced by $F(\omega)$ on the upstream and downstream travelling acoustic waves contributes to the way acoustic waves interfere with each other, which eventually reflects on the SPL along the combustor. The observations confirm that not only including the flame response $F(\omega)$ in combustion noise models for confined flames is essential, but also evaluating $F(\omega)$ with high accuracy is important in estimation of combustion noise. Additionally, it is worth to remark that global sensitivity analysis shows that the combined variance contribution of $\varphi^{(\text{out})}$ and $\tau$ (for case A), and of $q^{(\text{out})}$ and $\varphi^{(\text{out})}$ (for case B) is considerable for the overall variance of $p^{(\text{mic})}$ at some frequencies (see Fig. 13 for $S_{1,3}$ and $S_{1,2}$ for case A and B, respectively).

Direct comparison of Fig. 13(a) and (b) does not reveal whether the significant differences in variance and parameter sensitivities between cases A and B are due mainly to the differences in input uncertainties – Gaussian distribution modeled by Hermite polynomials or empirical distribution modeled by DD polynomials – or to the differences in the physical models. Comparing Gaussian and empirical distributions for the physical setup of case B leads to rather significant differences in the distributions of SPL, as shown in Appendix B for different frequencies. Interestingly, despite the differences in the SPL PDFs, the variance and Sobol indices as functions of frequency and shown in Fig. 13(b), are almost indistinguishable when employing respectively Gaussian and empirical distributions for this test problem. (Due to the similarity, only the case of empirical distributions has been included in the results section here). From these observations, we conclude that the differences in parameter sensitivities between case A and B can be attributed almost exclusively to the physical models for the parameter ranges considered.

Finally, note that the Sobol indices shown in Fig. 13 are independent of the source $b$ of Eq. (24), because we assume it to be deterministic. Consequently, the source $\tilde{Q}_b(\omega)$ (containing $b_1$) serves merely as a scaling factor of $x$ (as per $x = M^{-1}b$). A consequence is that the PDFs (c.f. Figs. 10 and 11) of the output $x$ can be easily scaled if the source is changed. For example, Eq. (24) may be solved for a generic value of $\tilde{Q}_b(\omega)$ and if this source changes (new data available), the previously computed PDF can easily be scaled according to the new $\tilde{Q}_b(\omega)$ without solving Eq. (24) again.

8.3. Comparison with NigPC

In terms of accuracy, igPC and NigPC yield identical results for problems that are linear in the input random variables [47]. For nonlinear problems this is not always the case, although both classes of methods should be expected to converge to the same solution as the number of expansion terms $P$ grows to infinity. In this work, the systems of equations are nonlinear in the input random variables but linear in the solution vector.

The comparison between igPC and NigPC in terms of computational cost is highly dependent on the type of problem and the number of stochastic dimensions. For the linearized Euler equations with a single stochastic variable, NigPC was demonstrated to outperform igPC in [48]. The relative efficiency of igPC com-

**Fig. 12.** Dependency of acoustic amplitude of first peak (100 Hz) and second peak (400 Hz) to values of $r_{\text{out}}$. $q = 5$, $d = 4$. Case A.

\(\dot{\phi} = 4.5\)

Fig. 13. Sobol indices as function of frequencies for $q = 5, d = 4$ (case A) and $q = 4, d = 4$ (case B). Indices of $S$ represent $1 \rightarrow |r^{(\text{out})}|$, $2 \rightarrow \theta^{(\text{out})}$, $3 \rightarrow \tau$ and $4 \rightarrow n$. $S_{\text{Remain}}$ accounts for $S_{1,4}, S_{2,3}, S_{2,4}, S_{3,4}$ and higher ($S_{ijk}$) combined contributions. White subplots display the variance of $p^{(\text{mic})}$ for the frequency range of interest. (a) case A, (b) case B.

(\text{b})

Fig. 14. (a) Number of nonzero elements $n_z$ and flops. (b) Computational times involved in igPC and NigPC. (c) Value of $P$ and $N_q$ given $q$ and $d$.

pared to NigPC with increasing number of random parameters was demonstrated for elliptic problems in [49]. Also within the context of preconditioned elliptic systems, a slight advantage for NigPC was demonstrated for larger relative errors, and a small advantage for igPC for smaller relative errors [50].

We perform a comparison between igPC and NigPC in terms of the computational costs of solving a stochastic SS system. The accuracy of the NigPC results is comparable to that of the igPC results (given that the expansion order $q$ in each dimension is the same for both approaches). Therefore, we do not compare the accuracy of the results here.

The objective in NigPC is to find the gPC expansion coefficients $c_k$ by computing Eq. (4). We will consider both the computational time $t_{\text{NigPC}}$ to obtain the gPC coefficients for a single point in physical space, and the time $t_{\text{NigPC-full}}$ of computing the same for the full spatial field. The former case is of interest to compute e.g.
$|p^{(\text{mic})}(\xi)|$, and the latter if we want the SPL everywhere in space. We consider case A where we use Hermite polynomials and use Eq. (4) to approximate $|p(\xi)|$ using Gauss-Hermite quadrature. The investigation is expected to give very similar results for case B in terms of computational cost as the DD polynomials and DD Gauss quadrature rules can be obtained in a preprocessing step. Assuming that a total-order $q$ basis yields a good approximation of the quantity of interest via Eq. (3), the number of quadrature points $N_q$ required to compute Eq. (4) is given by $N_q = (q + 1)^2$. Each sample of $|p^{(\text{mic})}(\xi)|$ corresponds to the outcome of a computational of Eq. (9) for values of $\alpha$, $r$, and $t^{(\text{out})}$ evaluated for all quadrature points. The computational time of NigPC can be estimated as

$$t_{\text{NigPC}} = t_{\text{SS}}N_q + t_{\text{integration}},$$

where $t_{\text{SS}}$ is the time required to solve one realization of the SS model (Eq. (9)). For simplicity, we normalize the cost and define $t_{\text{SS}} = 1$ unit of time. The computational time $t_{\text{integration}}$ to numerically integrate Eq. (4) is negligible in comparison to $t_{\text{SS}}N_q$ ($t_{\text{integration}} \approx 0.29$ units of time). The computational cost to obtain the field of gPC coefficients needs to take the integration of the full field into account, and is given by

$$t_{\text{igPC-full}} = t_{\text{igPC}} + t_{\text{integration-full}},$$

where $t_{\text{integration-full}} \approx L t_{\text{integration}}$.

The igPC method requires the solution of Eq. (24), with the system matrix $M$ of size $N \times N$ with $N = PL$ where $L = 350$ is the size of the matrix blocks $M_{jk}$ (and of $A$ in Eq. (9)). The full matrix $M$ is sparse because the tensors $E(\psi_1 \psi_2 \psi_3)$ and $E(\psi_1 \psi_2 \psi_n)$, from which the construction of $M_{jk}$ is based, are sparse. Accordingly, we estimate that the number of nonzero entries $nz$ of $M$ scales linearly with $N$. In the same way, we expect that the computational time $t_{\text{igPC-full}}$ to solve Eq. (24) scales linearly with $N$. We estimate that the computational time $t_{\text{igPC-full}}$ to build the matrix $M$ scales quadratically with $N$ because, in the most general case, the $M_{jk}$ blocks are assembled one by one. The solution of Eq. (24) provides the gPC coefficients for the full discrete spatial field. The computational cost of igPC is thus the sum of the time we are interested in a single point or the entire domain,

$$t_{\text{igPC}} = t_{\text{igPC-full}} + t_{\text{igPC-full}}.$$

Figure 14 shows the computational costs of igPC and NigPC for the problem under investigation. We solve Eq. (24) for different $P = (q + d)/(q!d!) = 6$ by varying the polynomial degree $q$ from 1 to 7 and the number of stochastic dimensions $d$ from 1 to 3. Subsequently, linear regression is performed on the obtained data (nz, flops, $t_{\text{igPC-full}}$, $t_{\text{igPC-full}}$) to fit the scaling with respect to $N$. Figure 14 (left) shows that the number of nonzero entries $nz$ and flops scales linearly with $N$, as expected (exponents are close to unity). Figure 14 (center) shows, on the one hand, that $t_{\text{igPC-full}}$ scales (quasi) linearly with $N$. On the other hand, it shows that $t_{\text{igPC-full}}$ scales almost quadratically with $N (N^{1.87})$, as expected. The time $t_{\text{igPC}} = t_{\text{igPC-full}}$ is also shown in Fig. 14.

Consider the case $d = 3$ and $q = 6$, corresponding to $P = 84$ and $N_q = 343$, as illustrated in Fig. 14 (right). This corresponds to $t_{\text{igPC-full}} \approx 343$ (Fig. 14 (right)) and $t_{\text{igPC-full}} \approx 446$ units of time, respectively. On the other hand, for gPC the time required to assemble $M$ is approximately $t_{\text{igPC-full}} \approx 174$ units of time, and $t_{\text{igPC-full}} \approx 30.6$ units of time. The overall time required for igPC is, therefore, $t_{\text{igPC}} \approx 0.6 t_{\text{igPC-full}}$, or $t_{\text{igPC}} \approx 0.46 t_{\text{igPC-full}}$. A similar analysis for $d = 2$ and $q = 2$ (and accordingly $P = 10$ and $N_q = 9$) gives $t_{\text{igPC}} \approx 0.3 t_{\text{igPC-full}}$, and $t_{\text{igPC}} \approx 0.073 t_{\text{igPC-full}}$. If the quantity of interest depends on a single point in space only, it is questionable whether igPC is worth the effort, given that the (simpler) NigPC method delivers results of comparable quality at only slightly higher computational cost. In contrast, if the quantity of interest relies on the entire spatial field, NigPC requires repeated computation of Eq. (4) for every point in the spatial domain, and igPC is significantly faster.

The examples above pertain to $L = 350$, despite the fact that realistic configurations (e.g., in three spatial dimensions) would require thousands of degrees of freedom. A more realistic case of $L = 23000$ has been investigated. Results were qualitatively similar, and are not reported here.

This section has shown that igPC outperforms NigPC for the solution of a Stochastic-SS system if: i) the gPC order $P$ is small so that $t_{\text{igPC}}$ remains within reasonable limits, ii) if a more efficient way of building $M$ is developed so that $t_{\text{igPC}}$ does not scale quadratically with $N$; or if iii) a field of stochastic quantities (and not a mere point-wise variable) is of interest.

9. Conclusions

Intrusive gPC has been used for UQ of combustion noise, assuming a generic thermoacoustic SS model with uncertain acoustic boundary conditions and flame response model. The methodology is promising for UQ in thermoacoustic systems, since it presents several advantages and novelties:

1. Statistics of interest, including PDFs, can be obtained at a significantly reduced cost compared to Monte Carlo simulation.
2. Although the propagation of input uncertainties to the outputs is highly nonlinear, the igPC method as proposed in this work permits the formulation of a linear system of equations.
3. Since the expanded system of equations is linear, igPC is computationally more efficient than sampling techniques such as NigPC, provided that the extended igPC matrix is efficiently constructed.
4. It becomes computationally affordable to perform global sensitivity analysis by means of variance decomposition.
5. Statistical outputs are not limited to scalars but instead are fields of statistical quantities, because they are given directly by the state vector of the Stochastic-SS model. In this work we have concentrated only on the pressure at a single point. If desired, we could perform UQ analysis on the entire pressure field without any additional cost.

Additionally, this study delivers four conclusive physical observations:

1. Acoustic reflection uncertainties at the outlet (both magnitude and phase) highly contribute to the overall combustion noise uncertainty at all frequencies of interest.
2. The contribution of uncertainties in the flame response (magnitude and phase) to the overall combustion noise uncertainty is particularly important within the frequency range, where the peak associated with the ITA feedback loop appears.
3. The different parameter sensitivities displayed in the test cases A and B (Fig. 13) can be attributed almost exclusively to the physical model for the parameter ranges considered, and not to the input distribution type of the input random variables. This conclusion may be problem specific.
4. The computed Sobol indices are independent of the source of combustion noise (when considered deterministic). A consequence of this is that PDFs of the output can be easily scaled if the source changes.

The SS-igPC method developed in this work is not limited to the thermoacoustic model proposed, as it can be used for UQ studies of any system that can be properly described within the state-space formalism.
Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. State space system

The system under investigation is divided into ducts (subsystem D), one reference velocity (subsystem R), one flame response (subsystem F) and two acoustic scattering of cross section change with and without flame (subsystem CS). In the following the notation \( Z_{m,n} \) is used to describe a zero matrix of size \( m \times n \).

A1. Subsystem D

We discretize the Duct element by \( \eta \) nodes. The state vector \( \mathbf{x}^D \) of size \( 2\eta \times 1 \), input vector \( \mathbf{u}^D \) and output vector \( \mathbf{y}^D \) of size \( 2 \times 1 \) are given by

\[
\mathbf{x}^D = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{\eta+1} \\ g_1 \\ \vdots \\ g_{\eta} \end{bmatrix}, \quad \mathbf{u}^D = \begin{bmatrix} B_d \end{bmatrix}, \quad \mathbf{y}^D = \begin{bmatrix} B_d \end{bmatrix}.
\]

The matrix \( A^D \), \( B^D \), \( C^D \) of size \( 2\eta \times 2\eta \) and \( D^D \) of size \( 2\eta \times 2 \eta \) read

\[
A^D = \begin{bmatrix} A_d & \alpha_{\zeta} \\ \eta & A_0 \end{bmatrix}, \quad B^D = \begin{bmatrix} B_d \\ B_0 \end{bmatrix}, \quad C^D = \begin{bmatrix} \zeta/\Delta x \\ \eta/\tau \end{bmatrix}, \quad D^D = \begin{bmatrix} Z_{\eta,1} & Z_{\eta,2} \\ Z_{\eta,1} & Z_{\eta,2} \end{bmatrix},
\]

where the submatrices \( A_d, A_0 \), of size \( \eta \times \eta \) and \( B_d, B_0 \), of size \( \eta \times 1 \) are operators that describe the downstream 'd' propagation of the acoustic wave \( f \) and upstream 'u' propagation of the acoustic wave \( g \). The quantity \( \Delta x \) is the distance between two adjacent nodes and \( \zeta \) denotes the speed of sound. The matrix \( C^D \), of size \( 2 \times 2 \eta \) is given by

\[
C^D = \begin{bmatrix} Z_{\eta,1} & 1 \\ Z_{\eta,2} & 1 \end{bmatrix},
\]

whereas the matrix \( D^D = Z_{2,2} \). The Plenum, mid-section and combustion chamber elements are represented by ducts of different lengths and different characteristic speed of sound \( \zeta \).

A2. Subsystem R

The reference velocity is modeled by means of one feedthrough matrix \( D^R \) only. Therefore, no state vector is considered. The input and output read

\[
\mathbf{u}^R = \begin{bmatrix} \hat{u}^{\text{ref}} \end{bmatrix}, \quad \mathbf{y}^R = \begin{bmatrix} \hat{u}^{\text{ref}} \end{bmatrix},
\]

where the matrices are given by \( B^R = Z_{1,2}, \quad C^R = Z_{1,1}, \quad D^R = [1, -1] \).

A3. Subsystem F

The subsystem \( F \), which models the flame response to upstream acoustic perturbations, is modeled by

\[
\mathbf{x}^F = \begin{bmatrix} \hat{v}_2 \\ \vdots \\ \hat{v}_{\eta+1} \end{bmatrix}, \quad \mathbf{u}^F = \begin{bmatrix} \hat{v}^{\text{ref}} \end{bmatrix}, \quad \mathbf{y}^F = \begin{bmatrix} \hat{v}^{\text{ref}} \end{bmatrix}.
\]

The matrices \( A^F \) of size \( \eta \times \eta \) and \( B^F \) of size \( \eta \times 1 \) read

\[
A^F = A_d \eta/\tau \quad \text{and} \quad B^F = B_d \eta/\tau.
\]

A4. Subsystem CS

The system CS, which models the acoustic Scattering of a cross-section change with (and without) Flame, is described by means of one feedthrough matrix \( D^CS \) only. Therefore, no state vector is considered. The input and output read

\[
\hat{u}^CS = \begin{bmatrix} f_u^{(u)} \\ g_u^{(u)} \end{bmatrix}, \quad \hat{y}^CS = \begin{bmatrix} f_u^{(u)} \\ g_u^{(u)} \end{bmatrix}.
\]

The matrix \( D^CS \) reads

\[
D^CS = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \end{bmatrix},
\]

where

\[
S_{11} = \frac{2\alpha\zeta}{\zeta + \alpha}, \quad S_{12} = \frac{\zeta - \alpha}{\zeta + \alpha}, \quad S_{13} = \frac{\theta\zeta\alpha}{\zeta + \alpha}, \quad S_{21} = \frac{\alpha - \zeta}{\zeta + \alpha}, \quad S_{22} = \frac{2}{\zeta + \alpha}, \quad S_{23} = \frac{\theta\alpha}{\zeta + \alpha}.
\]

A5. Interconnected System

In this section, the interconnected matrices \( \bar{A}, \bar{B}, \bar{C}, \bar{D} \) and \( \bar{F} \) are presented. Indices \( P \) (Plenum), \( M \) (MidSection), \( C \) (Combustion chamber) represent ducts based on the subsystem D. Indices \( S \) and \( ST \) refer to a cross-section change without \( (\theta = 0, \ S_{13}, S_{23} = 0 \) and \( \zeta = 1 \)) and with temperature jump and flame \( (\theta > 0, \ S_{13}, S_{23} > 0 \) and \( \zeta > 1 \)), respectively, and are described by the subsystem CS. The interconnected matrix \( \bar{A} \) of size \( 7\eta \times 7\eta \) and \( \bar{B} \) of size \( 7\eta \times 15 \) read

\[
\bar{A} = \begin{bmatrix} A^P & A^M & A^F \\ A^C & A^F & A^C \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B^P & B^S B_M & B^R B^F B^{ST} & B_C \end{bmatrix}.
\]

where \( B^{ST} = Z_{1,3}, \quad C^{ST} = Z_{2,2} \) and \( B^S = Z_{2n,3}, \quad C^S = Z_{2n,2} \). The matrices \( \bar{C} \) and \( \bar{D} \) are of size \( 12 \times 7\eta \) and \( 12 \times 15 \), respectively. They are given by
\[
\begin{bmatrix}
C^p \\
C^s \\
C^M \\
C^R \\
C^F \\
C^{ST} \\
C^C
\end{bmatrix}
\]

\[
\begin{bmatrix}
D^p \\
D^s \\
D^M \\
D^R \\
D^F \\
D^{ST} \\
D^C
\end{bmatrix}
\]

The matrix $F$, which relates outputs to inputs of each subsystem, is of size $15 \times 12$ and reads

\[
F = \begin{bmatrix}
0 & r^{(in)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(A.12)

Appendix B. Case B with Hermite Polynomials

In this section we show the PDF of the sound pressure level associated with Case B for: i) input random variables characterized by an empirical distribution (DD polynomials in Fig. B.15 as also shown in Fig. 11) and ii) input random variables characterized by a Gaussian distribution (Hermite polynomials in Fig. B.15). Despite the difference in the SPL PDFs, the variance and Sobol indices as function of frequency (as shown in Fig. 13(b)) are almost indistinguishable when employing respectively Gaussian and empirical distributions for this test problem.

Figure B.16 show the SPL for all frequencies investigated for the two aforementioned cases. Only small differences are perceived between the two cases due to the fact that the expected value and variance associated with the corresponding PDFs is very similar. The most remarkable difference is perhaps the skewness of the PDFs for frequencies larger than 400 Hz: whereas the skewness of the PDFs associated with Hermite polynomials is almost imperceptible, the skewness of the PDFs associated with DD polynomials is noticeable in this frequency region.

![Fig. B1. Case B: PDFs for six frequencies using igPC for $q=4$, $d=4$. Validation against Monte Carlo is not shown to enhance clarity.](Image)
Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.combustflame.2020.03.010.

References