QUANTIFIERS

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Chapter 1

Phenomena

Chapter 2

The Concept of a (Generalized) Quantifier

In this chapter we present our principal semantic tool, the concept of a (generalized) quantifier introduced by logicians in the mid-20th century.[1] But rather than writing down the formal definitions directly, we introduce the concept gradually, following the historical development right from its Aristotelian beginnings. This procedure has not only pedagogical merits, we believe. The evolution of notions of quantification is actually quite interesting, both from a historical and a systematic perspective. We hope this will transpire in what follows, even though our perspective in the text is mainly systematic. Indeed, we shall use these glimpses from the history of ideas not merely as a soft way to approach a technical concept, but also as occasions to introduce a number of semantic and methodological issues that will be recurring themes of this book.

2.1 Early History of Quantifiers

2.1.1 Aristotelian Beginnings

When Aristotle invented the very idea of logic some two thousand four hundred years ago, he focused precisely on the analysis of quantification. Operators like *and* and *or* were added later (by Stoic philosophers). Aristotle's syllogisms can be seen as a formal rendering of some important inferential properties, hence of important aspects of the meaning, of the expressions `all`, `some`, `no`, `not`

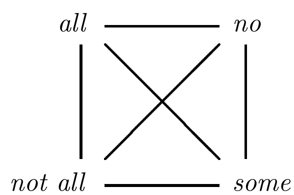
[1] *Terminological note:* Logicians call these objects 'generalized' quantifiers, since they were originally generalizations of the universal and the existential quantifiers from first-order logic. But once the naturalness and the ubiquity of the concept is appreciated, it becomes natural to drop the qualification, and just call them *quantifiers*. This is what we officially do. However, the original terminology, as well as abbreviations such as 'GQ theory', have become somewhat entrenched, so we occasionally insert 'generalized' within parentheses to remind the reader that we are talking about the general concept, not just $\forall$ and $\exists$.

`all`. These provide four prime examples of the kind of quantifiers that this book is about.

A *syllogism* has the form:[2]

$$\frac{\begin{array}{l} Q_1 AB \\ Q_2 BC \end{array}}{Q_3 AC}$$

where each of Q_1, Q_2, Q_3 is one of the four expressions above. Later on these expressions were often presented diagrammatically in the *square of opposition*:[3]



Positions in this square indicate logical relationships of 'opposition', which can be seen as forms of *negation*. Two expressions Q_i and Q_j vertically above one another in the square are each others 'contradictory' negations, or as we shall say, *outer* negations. The proposition Q_i `A's are B` is then simply the negation of Q_j `A's are B`, i.e., it is equivalent to `It is not the case that` Q_j `A's are B`. This propositional negation 'lifts' to the (outer) negation of a quantifier, and we can write $Q_i = \neg Q_j$ (and hence $Q_j = \neg\neg Q_j = \neg Q_i$).
When there is a horizontal line between Q_i and Q_j they are each others 'contrary', or *inner* negations: now Q_i `A's are B` is equivalent to Q_j `A's are not B`, which can be thought of as applying the inner negation $Q_j\neg$ to the denotations of `A` and `B`. Finally, diagonally opposed expressions in the square are each others *duals*, where the dual of Q_i is the outer negation of its inner negation (or vice versa): $Q_i^{\mathrm{d}} = \neg(Q_i\neg) = (\neg Q_i)\neg = \neg Q_i\neg$.
The square is *closed* under these forms of negation: applying any number of these operations to a quantifier in the square will not lead outside it. For example, $(no^{\mathrm{d}})\neg = \neg no\neg\neg = \neg no = some$. (It is often noted that except in one case (*not all*) the various negations of these four quantifiers are lexically expressed in languages like Greek and English; in fact this appears to hold for most languages.)
The square is not closed under all Boolean operations, e.g., the conjunction *some but not all* does not belong to it.

[2]This is the so-called *first figure* — three more figures are obtained by permuting AB or BC in the premisses. We are simplifying Aristotle's mode of presenting the syllogisms, but not distorting it. Observe in particular that Aristotle was the first to use *variables* in logic, and thus to introduce the idea of an *inference scheme*.

[3]Aristotle pointed out the various kinds of 'opposition' holding between these four quantifiers (cf. below), but it appears to have been Apuleios of Madaura (2nd century A.D.) who first presented them in the form of a square. The two top quantifiers — or rather the corresponding propositions — are called 'universal' and the two bottom ones 'particular'. In the classical square the two bottom quantifiers are actually reversed compared to the figure here. Then, the leftmost quantifiers (*all, some*) are 'affirmative', and the rightmost ones (*no, not all*) 'negative'.

Here are two typical examples of syllogistic inference schemes:

(2.1) $\texttt{all}\,AB$

$\underline{\texttt{no}\,BC}$

$\texttt{no}\,AC$

This scheme is clearly *valid*: no matter what the properties A, B, C are, it always holds that *if* the two premisses are true, then so is the conclusion.

(2.2) $\texttt{all}\,AB$

$\underline{\texttt{some}\,BC}$

$\texttt{some}\,AC$

This too has the stipulated syllogistic form, but it is *invalid*: one may easily choose A, B, C so as to make the premisses true but the conclusion false.

A syllogism is a particular instantiation of a syllogistic scheme:

All Greeks are sailors

<u>No sailors are scared</u>

No Greeks are scared

is a valid syllogism, instantiating the scheme in (2.1), and

All whales are mammals

<u>Some mammals have fins</u>

Some whales have fins

is an invalid one instantiating (2.2).

It was perfectly clear to Aristotle (though regrettably not always to his followers) that the *actual* truth or falsity of its premisses or conclusion is irrelevant to the (in)validity of a syllogism — except that no valid inference can have true premisses and a false conclusion. In particular, a valid syllogism can have false premisses. For example, in the first (valid) syllogism above all three statements involved are false, whereas in the second (invalid) one they are all true.

There are 256 syllogistic schemes. Aristotle characterized which ones of these are valid, not by enumeration but by an axiomatic method where all valid ones were deducible from (the valid ones in) the first figure (in fact from just two syllogisms from that figure). Apart from being the first example of a deductive system, it was an impressive contribution to the logico-semantic analysis of quantification.

However, it must be noted that this analysis does not exhaust the *meaning* of these quantifiers,[4] since there are many valid inference schemes involving these words which do not have the form of syllogisms, for example,

[4]Aristotle didn't claim it did; in fact he was well aware that there are other forms of logically valid reasoning.

John knows every professor

Mary is a professor

John knows Mary

No student knows every professor

Some student knows every assistant

Some professor is not an assistant

These inferences go beyond syllogistic form in at least the following ways: (i) names of individuals play an essential role; (ii) there are not only names of properties (such as adjectives, nouns, intransitive verbs) as in the syllogisms, but also names of binary relations (such as transitive verbs); (iii) quantification can be iterated (occur in both the sybject and the object of a transitive verb, for example). While none of these features may seem, at least in hindsight, like a formidable challenge, it is certainly much harder than in the syllogistic case to describe the logical structures needed to account for the validity of inferences of these kinds. At the same time, it is rather clear that a logic which cannot handle such inferences will not be able to account for, say, the structure of proofs in elementary geometry (like Euclid's), or, for that matter, much of everyday reasoning.

The failure to realize the limitations of the syllogistic form is part of the explanation why logic led a rather stagnant and unproductive existence after Aristotle, all the way up to the late 19th century. Only when the syllogistics is extended to modern predicate logic do we in fact get a full set of inference schemes which, in a particular but precise sense, capture *all* valid inferences pertaining to the four quantifiers Aristotle studied.

Proof-theoretic vs. Model-theoretic Semantics for Quantifiers

Approaching meaning via inference patterns is characteristic of a *proof-theoretic* perspective on semantics. In the case of our four quantifiers, however, it seems that whenever a certain system of such schemes is proposed — such as the syllogisms — we can always ask if these schemes are *correct*, and if they are exhaustive or *complete*. Since we understand these questions, and likewise what reasonable answers would be, one may wonder if there isn't some other more primary sense in which we know the meaning of these expressions. But, however this (thorny) issue in the philosophy of language is resolved, it is clear that in the case of Aristotelian quantifiers there is indeed a different and more direct way in which their meaning can be given. Interestingly, that way is also clearly present, at least in retrospect, in the syllogistics.

For, on reflection, it is clear that each of these four quantifier expressions stands for a particular binary relation between properties, or, looking at the matter more extensionally, a *binary relation between sets*. When A, B, C are arbitrary sets,[5] these relations can be given in standard set theoretic notation as follows:

$$\begin{aligned}
&\mathit{all}(A,B) \iff A \subseteq B\\
&\mathit{some}(A,B) \iff A \cap B \neq \emptyset\\
&\mathit{no}(A,B) \iff A \cap B = \emptyset\\
&\mathit{not\text{-}all}(A,B) \iff A - B \neq \emptyset.
\end{aligned}$$

So, for example,

(2.3) All Greeks are sailors

simply *means* that the set of Greeks stands in the inclusion relation to the set of sailors.

This observation fits with a *model-theoretic* perspective on meaning, rather than a proof-theoretic one. In a way it lays the foundation for the whole theory of (generalized) quantifiers that this book is built on, so the model-theoretic approach will be dominant here.

Quantifier Expressions and their Denotations

In this connection, let us note something that has been implicit in the above. Quantifier *expressions* are syntactic objects, different in different languages. On the present account, some such expressions 'stand for', or 'denote', or have as their 'extensions', particular relations between sets. So, for example, the English `no` and the Swedish `ingen` both denote the relation that holds between two sets if and only if they are disjoint. There is nothing language-dependent about these relations. But, of course, to talk about them we need to use language: a *meta-language* containing some set-theoretic terminology and, in the present book, English. In this meta-language, we sometimes use the handy convention that an English quantifier expression, in italics, names the corresponding relation between sets. Thus, *no* as defined above is the disjointness relation, and hence the relation denoted by both the Swedish expression `ingen` and the English expression `no`.

We note that on the present (Aristotelian) analysis each of the main words in a sentence like (2.3) has an extension; it denotes a set-theoretic object. Just as `sailor` denotes the set of sailors and `Greek` the set of Greeks, `all` denotes the inclusion relation. Such a 'name theory' of extension has by no means always been preferred in the history of semantics, not even by those who endorse a model-theoretic perspective.[6] As we will see presently, it was explicitly rejected

[5] Aristotle did not allow the empty set. We are disregarding this complication here, for simplicity. But note that it is still a moot point whether the *meaning* of quantifier expressions contains special provisos concerning non-emptiness. For example, does `All A's are B` imply `Some A's are B` (as Aristotle in fact claimed)? If there is no semantic implication here (as we would maintain), is there still a 'pragmatic' one?

[6] Indeed, it has often been ridiculed, under names such as the 'Fido'-Fido theory, as obviously inadequate.

by medieval logicians, and likewise by some of the founders of modern logic, like Bertrand Russell. The reason was, partly, that to these logicians there seemed to be no good candidates for the denotation of this kind of expressions, so they had to be dealt with in another way. However, Aristotle's account of the four quantifers mentioned so far surely points to one such candidate:

(*) Quantifier expressions denote relations between sets.

As it turns out, this simple idea resolves the problems encountered by earlier logicians, and it lays the foundation of a coherent and fruitful account of quantification.

2.1.2 Middle Ages

Medieval logicians and philosophers devoted much effort to the semantics of quantified statements, essentially restricted to syllogistic form. We shall not recount their efforts here, but merely note that they explicitly made the distinction between words that have an independent meaning, and words that don't. The former were called *categorematic*, the latter *syncategorematic*. Here is an illustrative quote.

> Categorematic terms have a definite and certain signification, e.g. this name 'man' signifies all men, and this name 'animal' all animals, and this name 'whiteness' all whitenesses. But syncategorematic terms, such as are 'all', 'no', 'some', 'whole', 'besides', 'only', 'in so far as' and such-like, do not have a definite and certain signification, nor do they signify anything distinct from what is signified by the categoremata. ... Hence this syncategoremata 'all' has no definite significance, but when attached to 'man' makes it stand or suppose for all men .... And the same is to be held proportionately for the others, ... though distinct functions are exercised by distinct syncatgoremata .... (William of Ockham, *Summa Logicae*, vol. I, around 1320; [Bocheński 1970], p. 157–8.)

It is reasonable to take having 'a definite and clear signification' here to mean having (at least) a denotation. A quantifier expression like `all` does not denote anything by itself, Ockham says, but combined with categorematic nouns like `man` it does denote something, apparently the set of all men. Likewise, (an utterance of) `some man` could be taken to denote a particular man. So quantifier words do not denote, but quantified noun phrases do. However, these ideas about *what* they denote become problematic, to say the least, when applied to other cases. Do we say that `no man` denotes the empty set, and thus has the same denotation as `no woman`? What does `not all cats` denote? As we will see, these problems are unsolvable and in fact show that this sort of account is on the wrong tack.[7]

[7] In Section XXX we give a formal proof that any attempt to interpret quantified noun phrases as sets of individuals must fail.

The next quote also begins with attempt to explain, along similar (hence equally problematic) lines, the signification of the Aristotelian quantifier expressions.

> The universal sign is that by which it is signified that the universal term to which it is adjoined stands copulatively for its suppositum (*per modum copulationis*) ... The particular sign is that by which it is signified that a universal term stands disjunctively for all its supposita. ... Hence it is requisite and necessary for the truth of this: 'some man runs', that it be true of some (definite) man to say that he runs, i.e. that one of the singular (propositions) is true which is a part of the disjunctive (proposition): 'Socrates (runs) or Plato runs, and so of each', since it is sufficient for the truth of a disjunctive that one of its parts be true. (Albert of Saxony, *Logica Albertucii Perutilis Logica, vol. III*, Venice, 1522; [Bocheński 1970], p. 234.)

The latter part of the quote, on the other hand, is a way of stating the truth conditions for quantified sentences, in terms of (long) disjunctions and conjunctions. This brings out the important point that it is perfectly *possible* to state adequate truth conditions for quantified *sentences* without assuming that the quantifier expressions themselves have an independent meaning (denotation). Indeed, this is how the Tarskian truth definition is usually formulated in current text books in first-order logic. Medieval logicians, and later Russell (see below) took the position that it is in fact *necessary* to proceed in this way. Generalized quantifier theory applied to natural languages, on the other hand, claims that it is in fact possible to have quantifier expressions as denoting, and that such an approach (which, as we have seen, can be traced back to Aristotle) has definite advantages: it identifies an important syntactic and semantic category, and it conforms better to the Principle of Compositionality, according to which the meaning of a complex expression is determined by the meanings of its parts and the mode of composition. More about this later.

Logicality vs. Syncategorematicity

While being on the subject of medieval logic, we take the opportunity to state another important point. It was common then to define *logic* as the study of the syncategorematic terms. And at first sight, it is indeed tempting to identify the syncategorematic/categorematic distinction with the distinction between logical and non-logical constants. But such an identification should be resisted, at least when one is concerned with more substantial fragments of natural languages. What makes a term merit the attribute 'logical' (or, for that matter 'constant') is one thing, having to do with particular features of its semantics; precisely what is involved here is a matter of some controversy, and we return to it in Chapter XXX. However, the fact that a word or morpheme does not have independent semantic status is quite another thing, which applies to many other expressions than those traditionally seen as logical.

Syncategorematic terms correspond roughly to what linguists nowadays call *grammatical morphemes*. English examples might be the progressive verb ending `-ing`, the word `it` in sentences such as `It is hard to know what he meant`, and the infinitive particle `to`: in `To lie to your teacher is bad`, the second `to` (the preposition) might be categorematic, but the first syncategorematic.

A grammatical morpheme may not belong in a dictionary at all, or if it does there could be a description of its phonological and syntactic features (e.g. valence features: which other words it combines with), but not of its meaning. This does not preclude, as pointed out above, that there might be a systematic account of what larger phrases containing the grammatical morpheme mean. But its meaning as it were arises via a grammatical rule; hence the name.

Now, it may be a matter of dispute whether our four quantifier expressions belong to this category. (We claim, of course, that this dispute can be settled.) But whether they do or not, this fact is surely quite distinct from the eventual fact of their logicality.

2.2 Quantifiers in Beginning Predicate Logic

Predicate logic was invented at the end of the 19th century. A number of philosophers or mathematicians had similar ideas, partly independently of each other, at about the same time, but the pride of place goes without a doubt to Gottlob Frege. Nevertheless it is interesting to see the shape these new ideas about quantification took with some other authors too, notably Peirce, Peano, and Russell.

One totally new concept concerns variable-binding: the idea of variables that could be bound by certain operators, notably the universal and existential quantifiers. This was no small feat — it is not something that can be 'copied' from natural languages, since it is not obvious that this sort of variable-binding occurs there at all — and the idea took some time to crystallize. In a way variable-binding belongs to the syntax of predicate logic, though it is of course crucial to the project to explain the truth conditions of sentences with bound variables (an explanation which took even longer to reach its correct form). In this respect (too), Frege was unique. His explanation, though perfectly correct and precise, did not take the form that was fifty years later given by Tarski's truth definition. Instead, he treated — no doubt because of his strong views about compositionality — the quantifier symbols as categorematic, standing for certain second-order objects. This is closely related to the idea of quantifiers as relations between sets that we have traced back to Aristotle, though in Frege's case combined with a much more expressive formal language than the syllogistics and containing the mechanism of variable-binding. Nothing even remotely similar can be found with the other early predicate logicians, so let us begin with them.

2.2.1 Peirce

Peirce in fact designed two systems of predicate logical notation. One was 2-dimensional and diagrammatic, employing so-called existential graphs. What the other one looked like is indicated in the following quote:

> ... the whole expression of the proposition consist[s] of two parts, a pure Boolean expression referring to an individual and a Quantifying part saying what individual this is. Thus, if k means 'he is king' and h, 'he is happy', the Boolean
>
> $$(\bar{k} + h)$$
>
> means that the individual spoken of is either not a king or is happy. Now, applying the quantification, we may write
>
> $$Any(\bar{k} + h)$$
>
> to mean that this is true of any individual in the (limited) universe .... .
>
> In order to render this notation as iconical as possible we may use Σ for *some*, suggesting a sum, and Π for *all*, suggesting a product. Thus $\Sigma_i x_i$ means that x is true of some one of the individuals denoted by i or
>
> $$\Sigma_i x_i = x_i + x_j + x_k + \text{etc.}$$
>
> In the same way, $\Pi_i x_i$ means that x is true of all these individuals, or
>
> $$\Pi_i x_i = x_i x_j x_k, \text{ etc.}$$
>
> ... It is to be remarked that $\Sigma_i x_i$ and $\Pi_i x_i$ are only *similar* to a sum and a product; they are not strictly of that nature, because the individuals of the universe may be innumerable. ([Peirce 1885], in [Bocheński 1970], p. 349.)

Thus k, h, x are formulas here. Quantified sentences are divided into a (Boolean) formula and a quantifier symbol, similarly to the modern notation. The quantifier symbols are chosen in a way to indicate their meaning (in terms of conjunction and disjunction), but Peirce does not yet have quite the notation for bound variables. In $\Sigma_i x_i$ it would seem that i is a variable whose occurrences in the formula x get bound by the quantification, but in $x_i + x_j + x_k + \ldots$ the i, j, k look more like names of individuals. One sees how Peirce has basically the right ideas, but that expressing them formally is still a non-trivial matter.

2.2.2 Peano

One feature of standard predicate logic is that the same variables that occur free in formulas can get bound by quantification. It appears that the first to introduce this idea was Peano:

> If the propositions a, b, contain undetermined beings, such as $x, y, \ldots$, i.e. if there are relationships among the beings themselves, then $a \supset_{x,y,\ldots} b$ signifies: whatever $x, y, \ldots$, may be, b is deduced from the proposition a. ([Peano 1889], in [Bocheński 1970], p. 350.)

Here we have a consistent use of variable-binding, albeit only for universally quantified conditionals.

2.2.3 Russell

In his early philosophy Russell subscribed to an almost Meinongian idea about denotation: all expressions of a certain form had to denote *something*, and it was the logician's task to say what these denotations were. Here is a quote from 1903:

> In the case of a class a which has a finite number of terms [members] — say, $a_1, a_2, a_3, \ldots a_n$, we can illustrate these various notions as follows:
>
> (1) *All a's* denotes a_1 and a_2 and … and a_n.
>
> (2) *Every a* denotes a_1 and denotes a_2 and … and denotes a_n.
>
> (3) *Any a* denotes a_1 or a_2 or … or a_n, where *or* has the meaning that it is irrelevant which we take.
>
> (4) *An a* denotes a_1 or a_2 or … or a_n, where *or* has the meaning that no one in particular must be taken, just as in *all a's* we must not take anyone in particular.
>
> (5) *Some a* denotes a_1 or denotes a_2 or … or denotes a_n, where it is not irrelevant which is taken, but on the contrary some one particular a must be taken.
>
> ([Russell 1903], p. 59.)

This is an honest attempt to explain the denotation of (what we would now call) certain quantified noun phrases; nevertheless it is clear that the account is full of unresolved problems.[8] It shows how hard the problem really was, and how confusion as to the meaning of the quantifiers was possible among prominent logicians around 1900.

[8] Even though it would also seem to contain some linguistic insights, for example, that *every* is *distributive* in some way that *all* is not.

Later on, Russell's explications of the quantifiers became more similar to those of Peano, in terms of propositional functions, and 'real' (free) and 'apparent' (bound) variables.

It is easy to see that problems like these with specifying the denotation of quantified noun phrases might lead one give up this idea completely, and concentrate on precise statement of truth conditions for quantified sentences instead. Indeed, Russell came to deny most emphatically that phrases like `every man, a man, the man` were 'denoting expressions', and similarly for definite descriptions like `the woman, the present king of France`. Today's standard predicate logic essentially takes the same view.

Russell's change of opinion on this issue was not to him a matter of giving up anything; rather he saw it as a deep insight about logical form contra surface form, with far-reaching consequences for logic, epistemology (cf. knowledge by acquaintace vs. knowledge by description), and philosophy of language. Though his arguments were quite forceful, it would seem that later developments in formal semantics, and in particular the theory of (generalized) quantifiers, have seriously undermined them. Roughly, this theory provides a logical form which does treat the offending expressions as denoting, and which thus brings out a closer structural similarity between surface and logical structure. One may debate which logical form is the correct one (to the extent that this question makes sense), but one can no longer claim that no precise logical form which treats quantifier expressions or noun phrases as denoting is available.

2.2.4 Frege

Already in 1879 (*Begriffsschrift*), Frege was clear about the syntax as well as the semantics of quantifiers. But his two-dimensional logical notation did not survive, so below we use a modernized variant.

First-level (n-ary) *functions* take (n) objects as arguments and yield an object as value. *Second-level* functions take first-level functions as arguments, and so on; values are always objects. Frege was the first to use the trick — now standard in type theory[9] — of reducing predicates (*concepts*) to functions: an n-ary first-level predicate is a function from n objects to the *truth values* The True and The False (or 1 and 0), and similarly for higher-level predicates.

For example, from the sentence

> John is the father of Mary

we can obtain the two unary first-level predicates

> ξ is the father of Mary

and

> John is the father of η,

[9]In fact, a type theorist might not call this a trick at all, claiming instead that functions are more fundamental mathematical objects than sets or relations.

the binary first-level predicate

(2.4) ξ is the father of η,

as well as the unary second-level predicate

$$\Psi(\text{John}, \text{Mary}).$$

(Here Ψ stands for any binary first-level predicate. 'Mixed' predicates, like $\Psi(\xi, \text{Mary})$, were not allowed by Frege.) For example, (2.4) denotes the function which sends a pair of objects to The True if the first object is the father of the second, and all other pairs of objects to The False. Equivalently, we can say it denotes the relation 'father of'.

Now suppose

(2.5)
$$A(\xi)$$

is a syntactic name of a unary first-level predicate. According to Frege, the (object) variable ξ does not belong to the name; it just marks a place, and we could as well write

$$A(\cdot).$$

The sentence

(2.6)
$$\forall x A(x)$$

is obtained, according to Frege, by inserting the name (2.5) into the second-level predicate name

(2.7)
$$\forall x \Psi(x).$$

(2.7) is a primitive name denoting the *universal quantifier*, i.e., the unary second-level predicate which is true (gives the value The True) for precisely those first level (unary) predicates which are true of every object. (Again the (first-level) variable Ψ in (2.7) is just a place-holder.) So (2.6) denotes the *value* of the universal quantifier applied to the predicate (denoted by) (2.5), i.e., a truth value. The result is that

$$\forall x A(x) \text{ is true iff } A(\xi) \text{ is true for any object } \xi.$$

Other quantifiers can be given as primitive, or defined in terms of the universal quantifier and propositional operators. For example,

$$\neg\forall x \neg\Psi(x)$$

is the existential quantifier, and

$$\forall x(\Psi(x) \rightarrow \Phi(x))$$

is the binary quantifier (second-level predicate) *all* — one of the four Aristotelian quantifiers. Summarizing, we note

- Frege's clear distinction between names (formulas, terms) and their denotations;
- the distinction between free and bound variables (Frege used different letters whereas nowadays we usually use the same), and that quantifier symbols are variable-binding operators;
- quantifier symbols are not syncategorematic but denote well-defined entities, *quantifiers*, that is, second-order (second-level) relations.

2.3 The Emergence of (Generalized) Quantifiers in Logic

The notion of *truth* — as distinct from notions such as validity or provability — was not formally connected to predicate logic until Tarski's famous truth definition in [Tarski 1935]. In this definition, quantifier symbols are again syncategorematic; they do not denote anything. But since one or two quantifiers are enough (depending on whether one of $\forall$ and $\exists$ is defined in terms of the other) for the mathematical purposes for which the logic was originally intended, such as formalizing set theory or arithmetic, one may just as well have one clause for each quantifier in the truth definition, and this is still standard practice.

2.3.1 Absolute vs. Relative Truth

In an important respect Tarski's original truth definition is half-way between Frege's conception and a modern one. Frege's notion of truth is *absolute*: all symbols (except variables and various punctuation symbols) have a given meaning, and the universe of quantification is fixed (to the class of all objects). The modern notion, on the other hand, is that of *truth in a structure* or *model*. Tarski too, in 1935, considers only symbols with a fixed interpretation, and although he describes the possibility of relativizing truth to an arbitrary domain,[10] he does not really arrive at the notion of truth in a structure until his model-theoretic work in the 1950's.[11]

The model-theoretic notion of truth is relative to two things: an interpretation of the non-logical symbols, and a universe of quantification. Let us consider these in turn.

Uninterpreted Symbols

We are by now accustomed to thinking of logical languages as (possibly) containing certain uninterpreted or *non-logical* symbols, which receive a meaning, or rather an extension, by an *interpretation*.

[10] In connection with 'present day ... work in the methodology of the deductive sciences (in particular ... the Göttingen school grouped around Hilbert)' ([Tarski 1935], p. 199 in Woodger's translation).

[11] This switch from an absolute to a relative notion of truth is thus non-trivial and in itself quite interesting; for an illuminative discussion see [Hodges 1986].

The point to note in the present context is that this idea applies nicely to the semantics of quantified expressions, if we keep in mind that an interpretation (in this technical sense) *assigns extensions* to certain words. For it is characteristic of quantifier expressions, at least on a first analysis (which actually goes a long way), that only the extensions of their 'arguments' are relevant. That is, in contrast with many other kinds of phrases in natural languages, *quantifier phrases are extensional*. Furthermore, phrases that provide arguments to quantifiers, like nouns or adjectives or verb phrases, do have sets as denotations, and it is often natural to see these denotations as depending on what the facts are. The extension of `dog` or `blue` are certain sets; under different circumstances they would have been other sets. It makes sense, then, to have a model or interpretation assign the appropriate extensions to them. Quantifier phrases, on the other hand, do not appear to depend on the facts in this way, so their interpretation is constant — in a sense which remains to make precise, and which we will come back to.

Methodological digression. We are claiming that a first-order framework allows us to get at the meaning of quantifier expressions, but we are *not* making the same claim for, say, nouns. Here the claim is only that it provides all the needed denotations — but it may provide many more (e.g. uncountable sets), and it says nothing about how these denotations are constrained. For example, nothing in predicate logic prevents us from interpreting `bachelor` in such a way that it is disjoint from the interpretation of `man`.

But this is not a problem; it rather illustrates a general feature of all modeling.[12] We are modeling a linguistic phenomenon — quantification. The model should contain the relevant features of that phenomenon, but may disregard other features of language. Abstracting away from certain features of the world it allows us, if successful, to get a clearer view of others, by means of systemizations, explanations, even predictions that would otherwise not be available. By way of an analogy, to build a model that allows you to find out if a certain ship will float — before putting it to sea — you need to model certain of its aspects, like density, but not others, like the material of its hull. Provided, of course, that the material doesn't have effects on floatability that you didn't foresee; that it doesn't is part of the claim that the model is successful. *End of digression.*

Incidentally, the treatment of individual constants in predicate logic fits well with the use of proper names in natural languages. Here the reason is not that the denotation of a name depends on what the world is like, but that languages contain a limited number of names, most of which are used to each denote a large number of individuals. Selecting a denotation for such a name is a way of fixing linguistic content for a certain occasion, and it accords with certain aspects of how names are used, though, again, it does not constitute — and is

[12]Where 'model' is now used in the sense in which one can build a model of an aircraft to test its aerodynamical properties, not in the logical sense which is otherwise predominant in this book, namely, that of a mathematical structure or interpretation.

not intended to constitute — anything like a full-scale analysis of this use.

Universes

The second relativization of truth is to a *universe*. If mathematical structures are used as interpretations of sentences in a formal language, it is obvious that universes can vary. But also in natural language, *universes of discourse* are omnipresent, even if usually not marked in the syntax. For a trivial example, if I claim on a certain occasion that

(2.8) All professors came to the party,

it is understood that I am not talking about all professors in the world, but perhaps only those in my department. So the universe of discourse might be the set of people either working or studying at that department. Getting this universe right matters crucially to the truth or falsity of the sentence, and thus to what is being claimed by that use of it. But none of the words in it, nor its grammatical form, carries this information. Thus, the universe has to be provided from somewhere else.

It does not follow that the universe of discourse *is* the universe of the model, or even that it is provided by the model. Sometimes it is identical to the universe of the model, and a handy convention is that if nothing else is said, the two are identified. But often it is a subset of the universe of the model, or of some other set built from that universe. It depends on context and may change as the discourse goes along, even though the model stays the same.

Think of a model — a universe plus an interpretation — as representing what we want to keep fixed at a certain point. Other semantically relevant things may be allowed to vary. For example, in logic, *assignments* (of values to variables) do not belong to the model. In language, various sorts of context may provide crucial information, such as the universe of discourse, the reference of certain pronouns, or, to take a different kind of example, the 'thresholds' of certain quantifier expressions like `most`.[13] This said, there is no doubt that the idea of truth as truth in a model is useful not only for formal languages used to talk about mathematical structures, but also for the semantics of natural languages.

2.3.2 First-Order Logic vs. First-Order Languages

Predicate logic for Frege or Russell was not first-order; it was rather higher-order since one could (universally or existentially) quantify over individuals, as well as over properties of individuals, or properties of properties of individuals, etc. But the first-order version, i.e., the restriction to quantification over individuals, is particularly well-behaved and well understood, and today it forms the basic logical system, which we call *first-order logic* or simply FO. This logic has

[13] How many A's must be B in order for `Most A's are B` to be true? Sometimes any number more than half seems enough, but other times a larger percentage is required.

- atomic formulas, including identities
- propositional operators
- the usual universal and existential quantifiers.

But the attribute 'first-order' is not restricted to FO. New quantifiers can be added to FO, and while expressivity may thereby greatly increase, the result is still first-order, as long as these quantifiers too are taken to range over individuals. Thus, *first-order languages* extend far beyond FO.[14]

Moreover, for quantification, the first-order version is paradigmatic. This point, we think, is a principled one. Any non-empty collection of any objects might be chosen as universe.[15] And to explain the meaning of a quantifier expression, we (usually) need to make no assumptions at all about that universe.

Linguistic and other circumstances sometimes impose various constraints on the range of the quantifiers. In a mathematical case, we might restrict attention to universes with a built-in total order, or with some arithmetic structure. In the case of natural languages, we find on occasion that quantification is over n-tuples of objects rather than single ones (adverbial quantification). Or it might be over sets of objects, or a Boolean algebra, or some sort of lattice structure (plurals and collective quantification). Furthermore, the forms of implicit quantification we have mentioned involve adding additional structure to the things (like times or possible worlds) quantified over.

Such manoeuvres can change the expressivity of the language: new things may be expressible, others may get lost. We will see examples of this later on. But here is the bottom line: understanding of what the quantifier expressions *mean* does not appeal to any such constraints. That understanding comes from the case with no restrictions and an arbitrary universe. For example, the expression `all` stands for the inclusion relation, regardless of whether we are talking about people, or n-tuples of cats, or numbers, or sets, or possible worlds, or whatever — its meaning does not depend at all on what we happen to be talking about.

The first-order case is also the logically simplest one: more structure usually means more complications. Thus, from a meaning-theoretic as well as from a methodological point of view, we would claim that *for the study of quantification*, first-order languages is the right place to start. But we repeat that this in no way means confining attention to FO. Indeed, natural languages easily express many simple quantifiers that go far beyond the expressive powers of FO.

2.3.3 First-Order Logic (FO)

A first-order *model* thus consists of two things: a universe, and an interpretation which assigns the right sort of things to (non-logical) symbols of various kinds.

[14] This terminology is not quite settled. Sometimes 'first-order language' is taken to mean a language using only the resources of FO. In this book the locution is used in a wider sense which, we believe, is both natural and illuminating.

[15] So it is slightly misleading to talk about a universe of individuals; 'objects' or 'entities' is better.

We write

$$\mathcal{M} = (M, I)$$

where M is the universe and I the interpretation function.

The crucial semantic relation is the *satisfaction* relation

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n)$$

where $\varphi = \varphi(x_1, \ldots, x_n)$ is a formula containing at most the variables $x_1, \ldots, x_n$ free, $a_1, \ldots, a_n \in M$, and x_i is taken to denote a_i, $1 \leq i \leq n$. If φ has no free variables, i.e., if φ is a *sentence*, then

$$\mathcal{M} \models \varphi$$

says that φ is *true in* $\mathcal{M}$.

Writing $\bar{a}$ for a sequence $a_1, \ldots, a_n$, and forgetting for simplicity about individual constants and function symbols, the truth definition for FO can then be expressed as follows, by induction on formulas:

standard truth definition for FO

(2.9) $\mathcal{M} \models P(a_1, \ldots, a_n) \iff \langle a_1, \ldots, a_n \rangle \in I(P)$,

when $P(x_1, \ldots, x_n)$ is an atomic formula,

(2.10) $\mathcal{M} \models a_i = a_j \iff a_i = a_j$,

(Note that '=' stands for different things on the left and right hand sides here.)

(2.11) $\mathcal{M} \models \neg\varphi \iff \mathcal{M} \not\models \varphi$,

(2.12) $\mathcal{M} \models \varphi \wedge \psi \iff \mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$,

and similarly for the other connectives; finally,

(2.13) $\mathcal{M} \models \forall x \varphi(x, \bar{a}) \iff$ for all $b \in M$, $\mathcal{M} \models \varphi(b, \bar{a})$,

(2.14) $\mathcal{M} \models \exists x \varphi(x, \bar{a}) \iff$ for some $b \in M$, $\mathcal{M} \models \varphi(b, \bar{a})$.

The last two clauses in the truth definition explain the meanings of the universal and existential quantifiers, without assuming that they denote anything. (Similarly for (2.11) and (2.12), which only indirectly utilize the fact that $\neg$ and $\wedge$ denote truth functions.) Still, even from a purely logical point of view it is very natural to ask: What do these expressions have in common? What category or type do they belong to? What kinds of things are their meanings? What would other specimens of the same category look like? The first logician to look at these questions from a model-theoretic point of view was Mostowski in 1967.

We begin by a rephrasing of (2.13) and (2.14) which prepares the way for the desired generalizations.

2.3.4 Mostowski Quantifiers

A simple example of a quantifier distinct from $\forall$ and $\exists$ is, say, $\exists_{\geq 5}$, where

$$\mathcal{M} \models \exists_{\geq 5} x\varphi(x, \bar{a}) \iff \text{there are at least five } b \text{ in } M \text{ s. t. } \mathcal{M} \models \varphi(b, \bar{a}).$$

Once this example is appreciated, any number of others come to mind. To express the general idea, we introduce the following notation.

extension of a formula in a model

For a formula $\psi = \psi(x, y_1, \ldots, y_n) = \psi(x, \bar{y})$, and a sequence $\bar{a}$ of n objects in M, let

$$\psi(x, \bar{a})^{\mathcal{M},x} = \{b \in M : \mathcal{M} \models \psi(b, \bar{a})\}.$$

More generally, if $\psi = \psi(x_1, \ldots, x_k, y_1, \ldots, y_n) = \psi(\bar{x}, \bar{y})$,

$$\psi(\bar{x}, \bar{a})^{\mathcal{M},\bar{x}} = \{(b_1, \ldots, b_k) \in M^k : \mathcal{M} \models \psi(b_1, \ldots, b_k, \bar{a})\}.$$

Then (2.13) and (2.14) become

$$\mathcal{M} \models \forall x\varphi(x, \bar{a}) \iff \varphi(x, \bar{a})^{\mathcal{M},x} = M$$

$$\mathcal{M} \models \exists x\varphi(x, \bar{a}) \iff \varphi(x, \bar{a})^{\mathcal{M},x} \neq \emptyset$$

Likewise, writing $|X|$ for the cardinality of the set X,

$$\mathcal{M} \models \exists_{\geq 5} x\varphi(x, \bar{a}) \iff |\varphi(x, \bar{a})^{\mathcal{M},x}| \geq 5.$$

We can go on *ad lib.* For example, the following two quantifiers have been studied in model theory:

$$\mathcal{M} \models Q_0 x\varphi(x, \bar{a}) \iff \varphi(x, \bar{a})^{\mathcal{M},x} \text{ is infinite}$$

$$\mathcal{M} \models Q^C x\varphi(x, \bar{a}) \iff |\varphi(x, \bar{a})^{\mathcal{M},x}| = |M|.$$

(On finite universes, Q^C is identical to $\forall$, but not on infinite ones.)

We have a arrived at Mostowski's notion of a (generalized) quantifier. Here is a general formulation.

type $\langle 1 \rangle$ quantifiers

Let, for each universe M, Q_M be any set of subsets of M, and use at the same time (to simplify notation) 'Q' as a new symbol for a corresponding variable-binding operator. Then Q is a (generalized) *quantifier of type* $\langle 1 \rangle$, whose meaning is given by

(2.15)
$$\mathcal{M} \models Qx\varphi(x, \bar{a}) \iff \varphi(x, \bar{a})^{\mathcal{M},x} \in Q_M.$$

For the examples considered above we have:

- $\forall_M = \{M\}$
- $\exists_M = \{A \subseteq M : A \neq \emptyset\}$
- $(\exists_{\geq 5})_M = \{A \subseteq M : |A| \geq 5\}$
- $(Q_0)_M = \{A \subseteq M : A \text{ is infinite}\}$
- $(Q^C)_M = \{A \subseteq M : |A| = |M|\}$ (the Chang quantifier)

Two other examples, which will play some role in what follows, are

- $(Q_{even})_M = \{A \subseteq M : |A| \text{ is an even natural number}\}$
- $(Q^R)_M = \{A \subseteq M : |A| > |M - A|\}$ (the Rescher quantifier[16])

Actually, Mostowski imposed a constraint on the sets Q_M, one which is satisfied by the previous examples, but not by arbitrary sets of subsets of M. For example, the constraint does not hold for

- $Q_M = \{A \subseteq M : \text{ John} \in A \text{ and Mary} \in A\}$.

The condition is called *Isomorphism closure*; we will come back to this in Chapter 3.2.2.

2.3.5 Lindström Quantifiers

Mostowskian quantifiers are, on each M, unary relations among subsets of M. We can generalize immediately to binary or n-ary relations among subsets. This turns out to increase expressivity considerably. For example, it was noted early by logicians that whereas the Rescher quantifier Q^R says of a set A that most elements of the universe belong to A, it cannot be used to say of two sets A and B that most elements of A belong to B. In other words, it cannot be used to express the relation

- $most_M(A, B) \iff |A \cap B| > |A - B|$.[17]

Notice by the way that, on finite universes, we get

- $(Q^R)_M(A) \iff |A| > 1/2|M|$
- $most_M(A, B) \iff |A \cap B| > 1/2|A|$,

[16] There is some terminological confusion in the literature about which quantifier is to be called 'the Rescher quantifier'. In this book, it is Q^R as defined above. That is also how Rescher defines it in [Rescher 1962] (he calls it 'M'), so the name is quite apt. See also the next footnote.

[17] Rescher distinguished Q^R from *most* and claimed (without proof) that 'It is readily shown that ... "Most A's are B's" cannot be defined by means of the usual resources of [FO]; not even when these are supplemented by [Q^R], or any other type of quantification, for that matter.' ([Rescher 1962], p. 374) We shall see precisely what this claim amounts to later on. The claim is true, but the proof is not trivial. The only published proof, to our knowledge, of the full claim (if taken to mean that *most* is not definable in terms of FO with any finite number of type $\langle 1 \rangle$ quantifiers added) appeared in [Kolaitis and Väänänen 1995].

so there *most* means 'more than half of', and Q^R means 'more than half of the elements of the universe'.[18] Here are some more examples of quantifiers as relations between subsets of M:

- $all_M(A, B) \iff A \subseteq B$
- $no_M(A, B) \iff A \cap B = \emptyset$
- $some_M(A, B) \iff A \cap B \neq \emptyset$
- $not\text{-}all_M(A, B) \iff A - B \neq \emptyset$

These are of course the four Aristotelian quantifiers from Section 2.1.1.

- $MO_M(A, B) \iff |A| > |B|$
- $I_M(A, B) \iff |A| = |B|$ (the Härtig, or equicardinality quantifier)
- $more\text{-}than_M(A, B, C) \iff |A \cap C| > |B \cap C|$.

All of these are of type $\langle 1, 1 \rangle$, except the last which is of type $\langle 1, 1, 1 \rangle$.[19]

What are the corresponding variable-binding operators? A type $\langle 1, 1 \rangle$ quantifier symbol Q will now apply to a *pair* of formulas (φ, ψ), and we may write,

$$Qx, y(\varphi, \psi),$$

where all free occurrences of x are bound in φ, and all free occurrences of y are bound in ψ. Actually, it is possible, and sometimes simpler, to use the same variable, writing instead

$$Qx(\varphi, \psi),$$

where all free occurrences of x are bound in both φ and ψ. Thus we have, for example,

$$\mathcal{M} \models most\, x(\varphi(x, \bar{a}), \psi(y, \bar{a})) \iff most_M(\varphi(x, \bar{a})^{\mathcal{M},x}, \psi(x, \bar{a})^{\mathcal{M},x}),$$

and similarly for the other examples. Let us summarize:

type $\langle 1, 1 \rangle$ quantifiers

A (generalized) *quantifier of type* $\langle 1, 1 \rangle$ associates with each universe M a binary relation Q_M between subsets of M. Using the same symbol as a variable-binding operator, the meaning of a quantified formula $Qx(\varphi, \psi)$, where $\varphi = \varphi(x, x_1, \ldots, x_n)$ and $\psi = \psi(x, x_1, \ldots, x_n)$ have at most the free variables shown, is given by

(2.16)
$$\mathcal{M} \models Qx(\varphi(x, \bar{a}), \psi(x, \bar{a})) \iff Q_M(\varphi(x, \bar{a})^{\mathcal{M},x}, \psi(x, \bar{a})^{\mathcal{M},x}).$$

[18] As noted earlier, this relation *most* is one meaning of the English word `most`, although in some contexts it could mean the same as, say, `at least 75% of`.

[19] *MO* was called *more* in [?], but we now find that name misleading. Unlike the Härtig quantifier, it has no established name in the literature. The word `more` occurs in various comparative constructions, but not properly as an English determiner expression. `more-than`, on the other hand, can be seen as a 2-place determiner, as we saw in Chapter 1.

Similarly for type $\langle 1,1,1\rangle$ quantifiers, etc. Observe that on the left hand side of (2.16) a syntactic expression — a formula in a formal language — is said to be true in a model $\mathcal{M}$ (or, more precisely, satisfied by a sequence $\bar{a}$ of elements of M). On the right hand side, by contrast, we are saying that two sets stand in a certain relation, namely the relation Q_M, to each other.

This generalization of Mostowski's notion of quantifier is due to [Lindström 1966]. Lindström further observed that there is no principled reason to restrict attention to relations between *sets*; the general concept is that of a relation between relations, or a second-order relation.[20] Here are three examples. For $A \subseteq M$ and $R, S \subseteq M^2$, define

- $W^{\mathrm{rel}}_M(A,R) \iff R$ is a well-ordering of A
- $\mathrm{Ram}_M(A,R) \iff \exists X \subseteq A$ [X is infinite and any two distinct elements of X are related by R] (a 'Ramsey quantifier')
- $(\mathrm{Res}^2(MO))_M(R,S) \iff |R| > |S|$ (the 'resumption' of MO to pairs).

The first two quantifier have type $\langle 1,2\rangle$ and the last $\langle 2,2\rangle$; the principle should be clear. The names of these quantifiers will be explained later.

As to the syntax of the formal language corresponding to such quantifiers, the variable-binding operator symbol now applies to an k-tuple of formulas if the second-order relation is k-place, and in each formula it binds the number of variables corresponding to the arity of that relation argument. Let us illustrate with a quantifier of type $\langle 2,3,1\rangle$.

type $\langle 2,3,1\rangle$ quantifiers

A (generalized) *quantifier of type* $\langle 2,3,1\rangle$ associates with each universe M a ternary relation Q_M between a binary relation, a ternary relation, and a unary relation (subset) over M. Using the same symbol as a variable-binding operator, the meaning of a quantified formula $Qxy, zuv, w(\varphi,\psi,\theta)$, where $\varphi = \varphi(x,y,x_1,\dots,x_n)$, $\psi = \psi(z,u,v,x_1,\dots,x_n)$, and $\theta = \theta(w,x_1,\dots,x_n)$ have at most the free variables indicated, and $a_1,\dots,a_n \in M$, is given by

(2.17)
$$\begin{aligned} & \mathcal{M} \models Qxy, zuv, w(\varphi(x,y,\bar{a}), \psi(z,u,v,\bar{a}), \theta(w,\bar{a})) \\ \iff & Q_M(\varphi(x,y,\bar{a})^{\mathcal{M},x,y}, \psi(z,u,v,\bar{a})^{\mathcal{M},z,u,v}, \theta(w,\bar{a})^{\mathcal{M},w}) \end{aligned}$$

Already in Chapter 1 we distinguished between monadic and polyadic quantifiers. Here is the formal definition:

[20] This is of course very similar to Frege's treatment of quantifiers as second-level relations which was sketched in Section 2.2.4. However, neither Mostowski nor Lindström seems to have been aware of Frege's definition. See also Section **??**.

monadic vs. polyadic quantifiers

A *monadic* quantifier is one of type $\langle 1, 1, \ldots, 1\rangle$. Semantically it is (on each M) a relation between a subsets of M. Syntactically it binds one variable in each of the formulas it applies to. A *polyadic* quantifier is of type $\langle n_1, \ldots, n_k\rangle$, where each n_i is a natural number ≥ 1. Semantically it is (on each M) a relation between k relations over M, where the i:th relation is n_i-ary. Syntactically it binds n_i variables in the i:th formula it applies to.

Logicians generalized the quantification available in FO partly as a 'local' and relatively well behaved means to increase its expressive power. There are other ways to strengthen FO, for example, by going to second-order logic. But with the generalized quantifier approach we can add the expressive means for a particular purpose that cannot be served in FO, while remaining in a first-order language.

Here is an example. Suppose you want to be able to say, of an ordering relation $<$, that each element has at most finitely many predecessors.[21] In second-order logic, you can say that a set A is infinite by a sentence of the following sort:

(2.18) There is an element a of A and a one-one function f from A to $A - \{a\}$.

But to use the machinery of second-order logic just for this purpose is perhaps a bit of overkill. Instead, add to FO the quantifier Q_0 from the previous section. Then you can say directly that a set is infinite, and now the sentence

$$\forall x \neg Q_0 y (y < x)$$

expresses the desired proposition. FO with Q_0 added is stronger than FO and therefore lacks some of its pleasant properties (such as a completeness theorem), but it is much weaker than second-order logic.

2.3.6 Branching Quantifiers

There is another natural way of generalizing FO quantification, which actually antedates generalized quantifiers by a few years. Think of formulas in prenex form, beginning with a quantifier prefix consisting of a sequence of expressions of the form $\forall x$ or $\exists y$. When a universal quantifier precedes an existential one, a *dependency* is created. In

(2.19)
$$\forall x \exists y \forall z \exists w \varphi(x, y, z, w)$$

[21] For example, given a Big Bang theory and a discrete time ordering, this would be true of that ordering, even if each time point has infinitely many successors.

y depends on x, and w depends on both x and z. Another order in the quantifier prefix may change the dependencies: in

(2.20)
$$\forall x \forall z \exists y \exists w \varphi(x, y, z, w)$$

both y and w depend on x and z. A convenient way of making the dependencies explicit is by means *Skolem functions*:[22] (2.19) and (2.20) are equivalent, respectively, to

(2.21)
$$\exists f \exists g \forall x \forall z \varphi(x, f(x), z, g(x, z))$$

(2.22)
$$\exists f \exists g \forall x \forall z \varphi(x, f(x, z), z, g(x, z)).$$

Now, another way to generalize ordinary quantification is to lift the restrictions that are created by linear dependencies, and allow, for example, a sentence with truth conditions

(2.23)
$$\exists f \exists g \forall x \forall z \varphi(x, f(x), z, g(z)).$$

This was proposed in [Henkin 1961], where also the following graphical way of writing (2.23) as a first-order sentence was introduced:

(2.24)
$$\begin{matrix} \forall x \ — \ \exists y \\ \forall z \ — \ \exists w \end{matrix} > \varphi(x, y, z, w)$$

The partial ordering of the quantifier prefix clearly shows that y depends only on x, and w only on z. This can be extended to all partially ordered prefixes of existential and universal quantifiers; the particular prefix in (2.24) is called the *Henkin prefix*.

It is not obvious that partially ordered or branching quantifiers extend the expressive power of FO, for example, that there is no FO sentence equivalent to (2.24). But this can be seen by the following simple but ingenious argument due to Ehrenfeucht. In FO, no sentence in two 1-place predicate symbols A and B expresses that $|A| \leq |B|$.[23] That is, no FO sentence can say that there is a 1-1 function f from A to B:

$$\exists f \forall x \forall z[(x = z \leftrightarrow f(x) = f(z)) \ \ \& \ \ (A(x) \rightarrow B(f(x)))].$$

[22]Actually, this needs some form of the *Axiom of Choice*, which guarantees that the required functions exist.

[23]This is equivalent to saying that the quantifier MO from Section 2.3.5 is not FO definable. We will see later how to prove such facts.

But this is equivalent to

$$\exists f \exists g \forall x \forall z[(x = z \leftrightarrow f(x) = g(z)) \ \& \ (A(x) \rightarrow B(f(x)))]$$

which can be written by means of the Henkin prefix:

$$\begin{matrix} \forall x \text{ — } \exists y \\ \forall z \text{ — } \exists w \end{matrix} \Big> [(x = z \leftrightarrow y = w) \ \& \ (A(x) \rightarrow B(y))]$$

So with the Henkin prefix we *can* say that $|A| \leq |B|$. Similarly, we can also express the sentence (2.18) from the previous section.

First-Order vs. Second-Order Again

The Henkin prefix may seem to make our notion of a first-order language from Section 2.3.2 problematic. For if the Henkin sentence is written as in (2.24) it looks first-order, but the version (2.23) quantifies over functions and looks second-order. So what is it?

Second-order logic, or SO, is like FO except that it in addition has variables and (universal and existential) quantification over sets, relations, or functions. We can make a similar distinction as before between SO on the one hand, and second-order *languages* on the other.

But now one may want also to use 'second-order' in an extended sense. If a *logic* is, roughly, a syntax (a language) plus a truth definition, it makes sense to say that such a logic is 'essentially second-order' if its truth definition essentially involves quantification over relations or functions. Or, a bit more precisely, if its expressive power is that of SO, or of some substantial fragment of SO.

Logic with branching quantifiers might be case in point. Its truth definition does seem to involve quantification over functions — essentially, in contrast with (2.21) and(2.22) where the use of Skolem functions was not essential but could be replaced by FO expressions. Also, its expressive power turns out to be exactly that of SO restricted to formulas whose prenex forms begin with a sequence of *existential* second-order quantifiers, followed by an FO formula, just as in (2.23).

We may note in this connection that the Henkin prefix trivially corresponds to a (generalized) quantifier: define Q^H of type $\langle 4 \rangle$ by

> $(Q^H)_M(R) \iff$ there are functions f, g on M such that for all $a, b \in M$, $R(a, f(a), b, g(b))$.

(2.24) would then be written

$$Q^H xyzw \, \varphi(x, y, z, w)$$

Again, this quantification is first-order in that it only employs individual variables, although the dependencies are no longer visible in the notation. Thus, the notion of a partially ordered quantifier is in this sense already subsumed under the concept of a (generalized) quantifier. But it could still be claimed that *logic* with the Henkin quantifier is essentially second-order.

However, this latter issue is controversial. Jaakko Hintikka has argued forcefully in recent years that not only is logic with partially ordered prefixes first-order in nature: it should in fact replace FO as *the* first-order logic; see [Hintikka 1996] and [Hintikka and Sandu 1997]. He has introduced a special linear syntax where *in*dependencies are marked explicitly, so that, for example, (2.24) can be written

(2.25)

$$\forall x \exists y \forall z (\exists w / \forall x) \varphi(x, y, z, w)$$

This is called IF ('independence-friendly') logic. Moreover, Hintikka gives the semantics for IF logic in game-theoretic terms, so that truth becomes the existence of a winning strategy in a certain game.

One might argue that this is still essentially second-order, since winning strategies are precisely a kind of functions, but Hintikka thinks not and in addition claims a host of other advantages for IF logic.

However, our simple point in Section 2.3.2 still stands, we think, and does not involve taking a position on these more difficult issues. The point is just that, in order to explain the meaning of a particular quantifier it is often not only possible but in fact advisable to use a a first-order language, for that amounts to quantification over an *arbitrary* domain, without making any assumptions abouts its structure. And, we claimed, that is how the meanings of quantifier expressions usually seems to be given to us. So in *this* (perhaps shallow but still illuminating) sense, the Henkin quantifier is undoubtedly first-order.

Branching Generalized Quantifiers

If $\forall$ and $\exists$ can be branched, what about other quantifiers? This question was first raised in [Barwise 1978], where it was pointed out that such branching seemed to give the correct truth conditions for certain English sentences.[24] For example, the simplest form of branching with *most* would be

$$\left.\begin{array}{l} \mathit{most}\ x\ A(x) \\ \mathit{most}\ y\ B(y) \end{array}\right> R(x, y)$$

which from two quantifiers of type $\langle 1, 1\rangle$ produces a polyadic quantifier of type $\langle 1, 1, 2\rangle$, saying that there is a subset X of A containing most of its elements, and a corresponding subset Y of B, such that each pair (a, b) where $a \in X$ and $b \in Y$ belongs to the relation R. This can be taken to be the form of the truth condition for a sentence like

[24]Hintikka had claimed earlier that the Henkin quantifier is needed for the semantics of English. This claim engendered some controversy, and Barwise pointed out that branching of *most* might be a clearer example. For references, see [Barwise 1978] [BETTER: GIVE SOME OF THESE REFERENCES].

(2.26) Most boys in your class and most girls in my class have all dated each other.

but it is not expressible by *most* alone (used linearly). We shall return to these matters in more detail in Chapter XXX.

Simple branching of *most* makes obvious sense. The reason seems to be that this quantifier is (right upward) monotone; cf. Section 3.2.4. For quantifiers with other forms of monotonicity, or with no monotonicity at all, it is more doubtful what branching should amount to. See [Barwise 1978] and [**?**] for a discussion, and [Sher 1997] for a proposal of a general definition.

2.3.7 Quantifers and Type Theory

TBW

2.3.8 Summary

TBW

Chapter 3

Basic Properties of Natural Language Quantifiers

In this chapter we introduce and discuss some characteristic properties of quantifiers. These properties are all relevant from a linguistic perspective, though several of them have a general mathematical-logical interest as well, and have been studied quite independently of any connection to natural languages.

Some of these properties can be stated irrespective of the *type* of the quantifier, whereas others make sense only for specific types. This will become clear from our presentation, though we shall spend the bulk of the chapter on type $\langle 1 \rangle$ and type $\langle 1, 1 \rangle$ quantifiers. These two types are by far the most prominent ones in a linguistic context; it will appear that most quantifiers of other types related to natural languages are in some sense reducible to them.

Before we begin, however, some preliminary terminological and methodological points will be made.

3.1 Preliminaries

3.1.1 Global and Local Quantifiers

A quantifier Q, then, is on each universe M a second-order relation Q_M (a relation between relations) over M. When it is necessary to make the distinction, we call such second-order relations *local* quantifiers, since they are local to a particular universe. The quantifier Q itself, on the other hand, i.e., the function which associates with each M a local quantifier Q_M on M, can then be called a *global* quantifier.

A property $\mathbf{P}$ of local quantifiers extends automatically to the corresponding global quantifier along the following scheme:

(G) Q is $\mathbf{P}$ $\Leftrightarrow_{\text{def}}$ For each M, Q_M is $\mathbf{P}$.

Most properties of quantifiers that we discuss are like this — they are intended for the global case but naturally defined first for the local case; indeed, via (G) the definition applies directly to both cases.

Some properties, however, are genuinely local in that the corresponding global property is uninteresting or even unintelligible. Here are two examples.

Trivial Quantifiers

We call a second-order relation *trivial* if it is either the universal relation or the empty relation. More precisely, for each type $\tau = \langle n_1, \ldots , n_k\rangle$:

the two trivial quantifiers of type τ on M

$\mathbf{1}_{\tau,M}$ is the universal relation of type τ over M. That is, whenever R_i is an n_i-ary relation on M, for $1 \leq i \leq k$,

$$\mathbf{1}_{\tau,M}(R_1, \ldots , R_k)$$

holds. Likewise, $\mathbf{0}_{\tau,M}$ is the empty relation of type τ over M, so $\mathbf{0}_{\tau,M}(R_1, \ldots , R_k)$ never holds. (When there is no risk of confusion, one or both of the subscripts $_\tau$ and $_M$ will be dropped.)

Now, one *could* lift this notion of triviality to Q itself via (G). Then there would be just two trivial global quantifiers, but that would be a coarse and uninteresting notion of triviality. More interestingly, Q might associate trivial quantifiers with some universes (say, with all universes of size less than a fixed number), and non-trivial ones with other universes. More fine-grained notions of (non-)triviality are called for in the global case, and would typically be adapted to a particular class of quantifiers or a particular investigative purpose.[1]

The Number of Quantifiers

Suppose you ask *how many* quantifiers (of a certain type) with a certain property there are. If you mean global quantifiers, this question risks being senseless. Such a quantifier associates with each set M a second-order relation over M, so there would normally be at least as many such quantifiers as there are sets, but there is no such thing as 'the number of all sets'. The class of all sets is not itself a set, and hence cannot be counted.[2]

If, on the other hand, you mean local quantifiers on a particular universe, the question has an answer that one might be able to figure out using set theory

[1] Examples are the condition VAR from [van Benthem 1984] for type $\langle 1, 1\rangle$ quantifiers, and what is called $(m_1, \ldots , m_k)$-nontriviality in [Westerståhl 1994] for certain quantifiers of type $\langle 1, 1, \ldots , 1, k\rangle$. The point of such global notions of non-triviality is to be able to formulate results or generalizations in the most perspicuous and illuminating way, without having to make exceptions or add extra clauses for various less interesting cases.

[2] One way turning the question into a meaningful ones is to (a) restrict attention to quantifiers satisfying ISOM (see Section 3.2.2), and (b) fix a cardinal number κ and ask instead how many quantifiers there are on a universe of cardinality at most κ.

in the infinite case, or some piece of finite combinatorics in case the universe was finite. For example, suppose your question was: How many distinct type $\langle 1, 1\rangle$ quantifiers are there in all on a given universe with just 2 elements? (The answer is: $2^{16} = 65.536$ (!)). How many with property $\mathbf{P}$? Now there are definite answers, though they may be easy or hard to find.

Two Perspectives on Language and Logic

The global/local contrast concerns not just different kinds of properties but different perspectives on quantifiers, and more generally on language and on the use of logic for linguistic purposes. Very roughly, one may say that the more linguistically oriented literature on quantification has tended towards a local perspective, where discussion takes places within a particular universe of discourse which is taken for granted or presupposed, whereas the more logically oriented literature has used a global perspective. But there really is no need to make a choice here. What is needed, rather, is to always be clear about whether a particular claim or generalization or definition or result is to be taken globally or locally.

As a rule of thumb, we prefer global versions whenever possible, not because our orientation is more logical than linguistic, but simply because local versions then follow by instantiation. But in many cases, the relationship between global and local is more complex. It may require work or non-trivial reformulation to extract a global statement from a local fact. And sometimes there may be more than one global version, or, as we have seen, none at all.

One area where this contrast is particularly acute is expressive power. To say that a certain quantifier is expressible or definable by such and such means may be true in local sense but false globally. Therefore, broad claims about expressive power can be seriously ambiguous if not carefully stated. And more than just resolving the ambiguity, one needs to reflect carefully on which concept of expressivity is the correct one for the occasion at hand. We will return to these questions in Chapter **??**, but clarity about the global/local distinction is a theme that runs throughout this book.

3.1.2 Natural Language Quantifiers

We shall use 'natural language quantifier' for such quantifiers as are likely to be useful in an account of the semantics of natural languages. Such a description might seem intolerably vague, but let us at least try to make our idea clear. In earlier work, the first author used this locution in a narrower and more operational sense, namely, for those quantifiers which are denoted by some determiner expression in some natural language.[3] However, many quantifiers that crop up in a linguistic context are not denotations of determiners. They might be denotations of noun phrases, or of certain adjectives, or they might arise in more complex constructions without necessarily being denotations of particular parts of these. Therefore, a broader notion also makes sense.

[3] In particular, in [Westerståhl 1989] and [Väänänen and Westerståhl 2001].

As an example, consider the two type $\langle 1,1\rangle$ quantifiers *most* and MO from Chapter 2.3.5. It is an interesting fact, and a fact which can be explained (cf. Section 3.3 below), that only the first of these is likely to be the denotation of a determiner in any language, even though from a logical point of view they are equally simple and natural; indeed MO seems more basic. However, MO still turns up in linguistic contexts. For example, the meaning of the phrase `more-than`, in constructions such as

(3.1) More students than teachers came to the opening.

appears to be derived in a uniform way from the simple comparison of cardinals given by MO; this is clear from the definition of the type $\langle 1,1,1\rangle$ quantifier *more-than* (which *is* a determiner denotation):

$$\textit{more-than}_M(A,B,C) \iff |A\cap C| > |B\cap C|$$

Therefore, in this book we count MO among the natural language quantifiers. The observation referred to above of course remains, but it can be made using 'determiner denotation' instead.

The general idea is this. The concept of a (generalized) quantifier is a mathematical one. It can be used for linguistic purposes, but it is likely that there is a huge number of quantifiers with purely mathematical definitions which will never turn up in the semantics of natural languages. Among type $\langle 1,1\rangle$ quantifiers consider, for example,

$$Div_M(A,B) \iff |A\cap B| \text{ divides } |A|$$

$$Trt_M(A,B) \iff |A\cap B| > \sqrt[3]{|A|}$$

Even though these two satisfy standard properties of determiner denotations (see Section 3.3), it seems likely they will never turn up as such, or in any other linguistic constructions. Such examples can of course be multiplied endlessly.

Quantifiers can be classified by descriptions of the form: 'being the denotation of expressions (in some language) of category X'. Other descriptions may be in terms of a specific construction Y without mentioning denotation. Such classifications can be quite useful. If one enumerated all the relevant categories and constructions, one would arrive at a definition of the concept of a natural language quantifier that we are after. We don't have such an enumeration, hence no definition. But we hope that the above explanations at least point in the direction of one.

3.1.3 Boolean Operations on Quantifiers

For any fixed type τ, we can define Boolean combinations of quantifiers of type τ in the obvious way:

- $(Q\wedge Q')_M(R_1,\ldots,R_k) \iff Q_M(R_1,\ldots,R_k)$ and $Q'_M(R_1,\ldots,R_k)$

- $(Q \vee Q')_M(R_1, \dots, R_k) \iff Q_M(R_1, \dots, R_k)$ or $Q'_M(R_1, \dots, R_k)$
- $(\neg Q)_M(R_1, \dots, R_k) \iff$ not $Q_M(R_1, \dots, R_k)$

Note that these become definitions of global quantifiers $Q \wedge Q'$, $Q \vee Q'$, and $\neg Q$, by (G) above.

Discussing the Aristotelian square of opposition in Chapter 2.1.1, we also found a natural notion of *inner* negation $Q\neg$ for those quantifiers. Then one is not negating the quantified proposition but replacing one argument by its complement. This of course presupposes that a particular argument has been selected. For type $\langle 1, 1\rangle$ quantifiers denoted by determiner expressions it is the second argument — naturally, since it corresponds to the verb phrase which is precisely what one is negating when forming the 'contrary' of a given sentence.[4]

Only for quantifiers of type $\langle n\rangle$ is there no choice, and we have, for $R \subseteq M^n$,

- $(Q\neg)_M(R) \iff Q_M(M^n - R)$

In the general case of a type $\langle n_1, \dots, n_k\rangle$ quantifier, assuming the i'th argument has been selected, we define

- $(Q\neg)_M(R_1, \dots, R_k) \iff Q_M(R_1, \dots, R_{i-1}, M^{n_i} - R_i, R_{i+1}, \dots, R_k)$

Then, as before, the *dual* of Q is

- $Q^{\mathrm{d}} = \neg(Q\neg)$

Since $\neg(Q\neg) = (\neg Q)\neg$ we may omit parentheses and write $Q^{\mathrm{d}} = \neg Q \neg$.

There are often interesting relations between properties of a quantifier and properties of its dual. This motivates the following terminology:

co-properties

Let **P** be a property of quantifiers of a type τ for which the notion of dual is well-defined. Then we say that

(3.2) Q is *co*-**P** iff Q^{d} is **P**.

For a trivial example, let **P** be the property of belonging to the square of opposition. Then **P** and co-**P** are the same properties. Less trivial examples will follow.

Inner negation is a Boolean operation too. Taking the complement of a set (or a set of n-tuples) is the same kind of operation as negating a sentence — they are both the complement operation in the corresponding Boolean algebra. So it is to be expected that Boolean-type 'laws' hold for inner negations and duals as well, for example the following ones.

3.1.1 Fact

[4] Other terms in the literature for inner negation in this case are *VP negation* and *post-complement*.

(a) $\neg(Q_1 \wedge Q_2) = \neg Q_1 \vee \neg Q_2$; $(Q_1 \wedge Q_2)\neg = Q_1\neg \wedge Q_2\neg$; $(Q_1 \wedge Q_2)^d = Q_1^d \vee Q_2^d$

(b) $\neg(Q_1 \vee Q_2) = \neg Q_1 \wedge \neg Q_2$; $(Q_1 \vee Q_2)\neg = Q_1\neg \vee Q_2\neg$; $(Q_1 \vee Q_2)^d = Q_1^d \wedge Q_2^d$

(c) $\neg\neg Q = Q\neg\neg = (Q^d)^d$

Proof. (a1) is just propositional logic. (a2) can be directly verified using only the definitions of inner negation and conjunction of quantifiers. (a3) follows from the definition of duals, using (a1) and (a2). (b) and (c) are similar. □

3.1.4 Relational vs. Functional View on Quantifiers

We have defined quantifiers (locally) as second-order relations, but from a type-theoretic perspective they would be viewed as functions instead. Recall that Frege defined all relations, including quantifiers, in terms of their characteristic functions. So for example, a (local) type $\langle 1 \rangle$ quantifier Q_M would be a function from $\wp(M)$, the powerset (set of subsets) of M, to the set of truth values $\mathbf{2} = \{0, 1\}$, and instead of $Q_M(A)$ one would write

$$Q_M(A) = 1$$

(The global Q then associates with each universe M such a function.) The set of all such functions is sometimes written[5]

$$[\wp(M) \longrightarrow \mathbf{2}]$$

Likewise, a type $\langle 1, 1 \rangle$ quantifier on M could be seen as a function from $\wp(M)$ to $[\wp(M) \longrightarrow \mathbf{2}]$.[6]

Whether one adopts a functional or a relational perspective on quantifiers is, we think, largely a matter of taste. In principle, it is easy to translate from one to the other. There is a certain austere elegance to a type-theoretic framework, where everything is a function. Also, linguists have often preferred a functional view since it allows an intuitively pleasing formulation of compositionality in terms of function application. This too goes back to Frege, and in modern times is especially prominent in the tradition from Montague. In particular, the use of Boolean functions of various kinds, not just quantifiers, makes possible an elegant and uniform treatment account of large tracts of semantics; cf. [Keenan and Faltz 1985].[7]

However, when one focuses on the theory of (generalized) quantifiers, the relational perspective is, to our minds, logically and intuitively somewhat simpler, and so that perspective is the one usually preferred in this book.

[5] Another notation is $\mathbf{2}^{\wp(M)}$.

[6] Equivalently, it could be seen as a function from $\wp(M) \times \wp(M)$ to $\mathbf{2}$.

[7] Further remarks on the advantages of a functional view of quantifiers can be found in [Keenan and Westerståhl 1997], in particular p. 858.

3.2 Type ⟨1⟩ Quantifiers

Type ⟨1⟩ quantifiers are, on each universe, sets of subsets of that universe. The terminology here varies somewhat in the literature; at times they are called ‘unary’ quantifiers (then type ⟨1, 1⟩ quantifiers are binary). But ‘unary’ can also refer to the arity of the arguments or the second-order relation, and the word is also used for the case when all of the arguments are 1-place, i.e., for what are here called *monadic* quantifiers.[8] In that case type ⟨1⟩ has been called ‘simple unary’. There is also some diversity as regards the type notation.[9] However, we shall stick to the original notation and terminology from [Lindström 1966] introduced in Chapter 2.3.5.

3.2.1 Examples

Some Logical Quantifiers

Most of the following examples of type ⟨1⟩ quantifiers were defined in Chapter 2.3.4:

(3.3) $\forall$, $\exists$, $\exists_{\geq 5}$, $\exists_{\leq 7}$, $\exists_{=3}$, Q_{even}, Q_0, Q^C, Q^R.

On finite universes, Q^R is a typical example of a *proportional* quantifier: we have $(Q^R)_M(A) \Leftrightarrow |A| > 1/2|M|$. Let us introduce the following notation.

(3.4) For natural numbers p, q such that $0 < p < q$ and a finite universe M, the type ⟨1⟩ quantifier (p/q) is defined by:[10]

1. $(p/q)_M(A) \iff |A| > p/q \cdot |M|$

   Similarly,

2. $[p/q]_M(A) \iff |A| \geq p/q \cdot |M|$

So $Q^R = (1/2)$. In addition, we have Boolean combinations, including inner negations and duals, of the quantifiers above (as well as those mentioned below). Here are some examples:

$$\forall^{\mathrm{d}} = \exists$$

$$(\exists_{=3}\neg)_M(A) \iff |M - A| = 3$$

$$(\exists_{\geq 2} \wedge \exists_{\leq 7})_M(A) \iff 2 \leq |A| \leq 7$$

$$(Q^R)^{\mathrm{d}}_M(A) \iff |A| \geq 1/2|M|; \quad \text{in general:}$$

[8] E.g. in [Väänänen 1997].

[9] For example, type ⟨1, 1, 1⟩ was written ⟨⟨1, 1⟩, 1⟩ in [Keenan and Westerståhl 1997] (to emphasize the special role of the first two ‘noun’ arguments for quantifiers like *more-than*) and (1; 3) in [Väänänen 1997].

[10] Some quantifiers only ‘make sense’ for finite universes, so when discussing them we will tacitly assume that universes are finite. To make a definition like (3.4) fully general one may stipulate, for example, that $(p/q)_M = \mathbf{1}_M$ when M is infinite.

(3.5) $(p/q)^{\mathrm{d}} = [(q-p)/q]$

These are all logical quantifiers in a sense to be made precise in Section 3.2.2 below. In a natural language context, many non-logical type $\langle 1 \rangle$ quantifiers arise. Here is a first example.

Montagovian Individuals

In the treatment of proper names introduced in [Montague 1969], a name like `John` referring to an individual j is interpreted as the set of subsets containing j.[11] More precisely, define, for each individual j,

(3.6) $(I_j)_M = \{A \subseteq M : j \in A\}$

The motive is to obtain a uniform treatment of all noun phrases as (locally) sets of subsets of the universe. Since proper names can be conjoined with quantified noun phrases into more complex noun phrase, as in `John and most students`, the uniformity resulting from a Montagovian treatment of names is substantial.

Note that even if I_j is non-logical, there is nothing 'local' about its definition as we have formulated it here. It is a global quantifier, which with every universe M associates a local quantifier, given by (3.6). In particular, the definition does not presuppose that $j \in M$ (although this would be presupposed for most *applications* of this quantifier in concrete cases). If $j \notin M$, it follows from the definition that $(I_j)_M$ is the trivial quantifier $\mathbf{0}_M$ on M.[12]

One may still claim that quantifiers of the form I_j really are independent of the universe in a strong sense. If `John is a professor` is true (false), it remains so if the universe of discourse is extended (provided the set of professors is the same). The same holds for `Somebody is a professor`, but not, for example, for `Everybody is a professor`. Universe-independence in this sense is a property of (some) global quantifiers. It will be made precise and discussed in Section 3.2.3.

In a context where it is clear that `John` denotes j, we may write I_j simply as *John*. And we may go on:

(3.7) $\textit{John and Mary} = I_j \wedge I_m$

(3.8) $\textit{John or Mary} = I_j \vee I_m$

Then a set $A \subseteq M$ belongs to $(\textit{John and Mary})_M$ (to $(\textit{John or Mary})_M$) iff $j \in A$ and $m \in A$ ($j \in A$ or $m \in A$), so using these quantifiers will give correct truth conditions for both of

(3.9) John and Mary used to like pizza.

---

[11] So in a sense we have two interpretations of proper names. To separate these, one may let the *lexical item* `John` denote j, and the *noun phrase* `John` denote the corresponding set of subsets.

[12] This is for simplicity; it is not intended to convey an insight about the semantics of names.

(3.10) John or Mary used to like pizza.

Of course, this analysis will not work for all uses of `John and Mary`, for example,

(3.11) John and Mary met at the party.

We come back to such examples in Chapter XXX.

Noun Phrase Denotations

Proper names are one kind of noun phrase, but perhaps the most obvious candidates for type $\langle 1 \rangle$ quantifiers in a natural language context are those interpreting quantified noun phrases like

> every cat, most sailors, at least three students, not more than five professors, ...

A proper analysis of these will involve the interpretation of the corresponding determiner expressions, for which type $\langle 1, 1 \rangle$ quantifiers are used. We come to this in Section 3.3 below, but in fact it turns out that in a precise sense type $\langle 1 \rangle$ quantifiers is the right place to start. We therefore make the following definition here, which begins to explore the fact that in such noun phrases, quantification is *restricted* to the set provided by the noun.

restricted type $\langle 1 \rangle$ quantifiers

Let Q be type $\langle 1 \rangle$ quantifier and A a fixed set. The quantifier $Q^{[A]}$ is defined as follows, for any M and any $B \subseteq M$:

(3.12) $(Q^{[A]})_M(B) \iff Q_{M \cap A}(A \cap B)$

For example, let *cat* be the set of cats, $cat \subseteq M$. Then $(\exists_{=3}^{[cat]}\neg)_M$ is the set of subsets B of M such that $(\exists_{=3}\neg)_{cat}(cat \cap B)$, i.e., such that $|cat - B| = 3$. So if *hungry* is the set of hungry things in M (including, say, hungry cats, hungry dogs, hungry people, etc.), $(\exists_{=3}^{[cat]}\neg)_M(hungry)$ holds if and only if the sentence

(3.13) All but three cats are hungry.

is true. Thus, $(\exists_{=3}^{[cat]}\neg)_M$ is a suitable interpretation of the noun phrase `all but three cats`. In this way, restrictions of the quantifiers exemplified in (3.3) above provide a plethora of type $\langle 1 \rangle$ quantifiers that turn up in linguistic contexts.

Above we assumed that $cat \subseteq M$, but this is not essential. Whether there are cats outside the universe considered depends not only on the facts but on methodological choices; in any case the restriction in definition (3.12) is to *cats in* M. A sentence like (3.13) normally is not about all cats but only those in a local universe of discourse, say, the set X of living things in my house. We may effect this restriction either by taking $M = X$, or by restricting $\exists_{=3}\neg$ in (3.13) not to *cat* but to $A = X \cap$ *cat*. In the latter case, $A \subseteq M$.

[Barwise and Cooper 1981] used the following suggestive terminology for a characteristic trait of restricted quantifiers:

- Suppose Q is a type $\langle 1 \rangle$ quantifier, M a universe, and $A \subseteq M$. Then Q_M *lives on* A iff, for all $B \subseteq M$,

  $Q_M(B) \iff Q_M(A \cap B)$

  (This is a property of local quantifiers, not global ones.)

Here are some facts about this property:

(i) Q_M always lives on M, but need not live on any proper subset of M.

> [The first claim is obvious. To see that the second claim is true, let, for example, $a \in M$, $A = M - \{a\}$, $B = \{a\}$. Then $\exists_M(B)$ but $A \cap B = \emptyset$, so $\neg\exists_M(A \cap B)$. Hence, $\exists_M$ does not live on any proper subset of M.]

(ii) If $A \subseteq M$, $(Q^{[A]})_M$ lives on A. It can also live on some proper subset of A.

> [The first claim follows directly from the definitions of $(Q^{[A]})_M$ and 'live on'. The second is a consequence of the example in (iii) below.]

(iii) If M is infinite, there need be no *smallest* subset A of M such that Q_M lives on A.

> [Let $Q_M(B)$ iff $M - B$ is finite, so Q means 'all but finitely many things in the universe'. Take an infinite universe M. Then Q_M lives on exactly the *co-finite* subsets of M, i.e., the sets $A \subseteq M$ such that $M - A$ is finite. (This follows using the fact that $M - (A \cap B) = (M - A) \cup (M - B)$, which is finite iff both $M - A$ and $M - B$ are finite.) But there is no smallest co-finite subset of M: if A is such a subset it has to be infinite, and so if $a \in A$, $A - \{a\}$ is a proper subset of A which is also co-finite with respect to M.]

This ends our list examples of type $\langle 1 \rangle$ quantifiers, but some more will appear in the following sections.

3.2.2 Isomorphism Closure

Mostowski imposed a condition on what we call type $\langle 1 \rangle$ quantifiers, namely, that they "should not allow us to distinguish between elements [of M]." ([Mostowski 1957], p. 13). Lindström used the same condition for the general case, and it is usually taken for granted by logicians. The condition, which we shall *Isomorphism Closure* or simply ISOM, expresses the idea that in logic only structure counts, not individual objects, sets, or relations. This means that if a sentence in a logical language is true in one model, it is true in all isomorphic models (cf. below).

We first give a definition of ISOM for type ⟨1⟩ quantifiers, however, which does not mention isomorphic models but uses the notion of two sets having the same size, i.e., the same cardinal number.

ISOM type ⟨1⟩ quantifiers

A type ⟨1⟩ quantifier Q satisfies ISOM if whether $Q_M(A)$ holds or not depends only on the cardinality of A and $M - A$. In other words, for any universes M, M' and any $A \subseteq M$, $A' \subseteq M'$:

(3.14) If $|A| = |A'|$ and $|M - A| = |M' - A'|$, then $Q_M(A) \Leftrightarrow Q_{M'}(A')$.

It follows that Q can be identified with a *binary relation between cardinal numbers*, namely, those pairs $(|M - A|, |A|)$ for which $Q_M(A)$ holds. This relation is also called Q. It is thus defined as follows, for any cardinal numbers κ, λ:

(3.15) $Q(\kappa, \lambda) \iff$ there are M and $A \subseteq M$ such that $|M - A| = \kappa, |A| = \lambda,$ and $Q_M(A)$

Conversely, given any binary relation Q between cardinal numbers, we get back the corresponding type ⟨1⟩ quantifier (also called Q) by

$$Q_M(A) \iff Q(|M - A|, |A|)$$

Thus, ISOM type ⟨1⟩ quantifiers only care about 'quantities', or sizes of sets, not the sets themselves. The condition can be extended to all monadic quantifiers (and is sometimes called QUANT in the literature).

All the quantifiers mentioned in (3.3) – (3.5) satisfy ISOM. We can see this by identifying the corresponding relations between numbers. Here are a few cases:

$$\forall(\kappa, \lambda) \iff \kappa = 0$$

$$\exists(\kappa, \lambda) \iff \lambda \neq 0$$

$$\exists_{\geq 5}(\kappa, \lambda) \iff \lambda \geq 5$$

$Q_{even}(\kappa, \lambda) \iff \lambda$ is an even natural number

$Q_0(\kappa, \lambda) \iff \lambda$ is infinite

$$Q^C(\kappa, \lambda) \iff \lambda = \kappa + \lambda$$

$$Q^R(\kappa, \lambda) \iff \kappa > \lambda$$

$$[p/q](m, k) \iff k \geq p/q(m + k)$$

Furthermore, the following is easily established.

3.2.1 Fact
The class of ISOM type $\langle 1 \rangle$ quantifiers is closed under Boolean operations, including inner negations and duals. In particular, using the terminology introduced in Section 3.1.3, co-ISOM = ISOM. Moreover, viewed as relations between numbers, the inner negation of a quantifier is simply its converse, i.e.,

$$Q\neg(\kappa, \lambda) \iff Q(\lambda, \kappa)$$

However, if a quantifier involves a particular individual or a particular set, ISOM usually fails. For example:

3.2.2 Fact
Quantifiers of the form I_a are never ISOM.

Proof. Intuitively, this is obvious, but let us check with the definition. Take two distinct objects a, b, and a universe M_0 which contains them both. Then $|\{a\}| = |\{b\}| = 1$, and $|M_0 - \{a\}| = |M_0 - \{b\}| = |M_0| - 1$. On the other hand, $(I_a)_{M_0}(\{a\})$ holds, since $a \in \{a\}$, but $(I_a)_{M_0}(\{b\})$ fails, since $a \notin \{b\}$. Thus, we have found a counter-instance to (3.14) (with $M = M' = M_0$, $A = \{a\}$, and $A' = \{b\}$), so ISOM is not satisfied. $\square$

Likewise, quantifiers of the form $Q^{[A]}$ are usually not ISOM, even if Q is ISOM. Intuitively, this is again clear. Even if it is a fact that

> At least four students smoke.

and also that the number of smokers (in a given finite universe of discourse) is the same as the number of drinkers, it certainly does not follow that

> At least four students drink.

But it *would* follow, if the quantifier $\exists_{\geq 4}^{[student]}$ were ISOM. The next proposition extracts the idea of this argument and gives a very general sufficient condition.

3.2.3 Proposition
Suppose $A \neq \emptyset$, and that Q_A is not trivial. Then $Q^{[A]}$ is not ISOM.

Proof. Since Q_A is not trivial, there are subsets C_1, C_2 of A such that $Q_A(C_1)$ but not $Q_A(C_2)$. We now claim that by possibly adding some new elements to A, we can form a universe M' and subsets B_1, B_2 of M' such that

- $|B_1| = |B_2|$ and $|M' - B_1| = |M' - B_2|$
- $A \cap B_1 = C_1$ and $A \cap B_2 = C_2$

Thus, $Q_A(A \cap B_1)$ and $\neg Q_A(A \cap B_2)$, i.e., (since $A \subseteq M'$) $(Q^{[A]})_{M'}(B_1)$ and $\neg(Q^{[A]})_{M'}(B_2)$, which contradicts ISOM for $Q^{[A]}$.

To verify the claim, suppose $|C_1 - C_2| = k_1$, $|C_2 - C_1| = k_2$, $|C_1 \cap C_2| = n$, and $|A - (C_1 \cup C_2)| = m$. Furthermore, suppose that $k_1 \leq k_2$ (the case when $k_2 \leq k_1$ is symmetric). We then add a set D of sufficiently many new elements to A, so that $B_1 = C_1 \cup D$ has exactly $k_2 + n$ elements. (If k_2 is a finite number then $|D| = k_2 - k_1$; otherwise $|D| = k_2$.) Also, we let $B_2 = C_2$ and $M' = A \cup D$. Now $|B_1| = k_1 + n + (k_2 - k_1) = k_2 + n = |B_2|$, and $|M' - B_2| = k_1 + m + (k_2 - k_1) = k_2 + m = |M' - B_1|$. (These calculations hold also when k_2 is infinite and $k_2 - k_1$ is replaced by k_2.) Thus, the claim is proved. □

We may conclude that noun phrase denotations of the form $Q^{[A]}$ are usually not ISOM. The conclusion, however, should be stated with some care. In Proposition 3.2.3, A is a fixed set. In a linguistic context, on the other hand, a universe of discourse M is given, and a noun denotation A such that (usually) $A \subseteq M$. Thus, for the argument of the above proof to go through there has to be enough elements in $M - A$ to make the required extension. (The proof builds a subuniverse $M' = A \cup D$ such that $A \subseteq M' \subseteq M$ and provides a counter-example to ISOM with universe M'. This automatically gives a counter-example to ISOM with universe M.[13])

Two more observations are in order. First, for some noun phrases, like

everything, something, nobody

the noun denotation is as it were indexically set to M itself (think of `thing` as denoting the universe of discourse). Thus, given M, the noun phrase denotation is $(Q^{[M]})_M$, which is just Q_M. But this means that these noun phrases can actually be interpreted as the quantifiers $\forall$, $\exists$, $\neg\exists$, respectively, which are of course ISOM. However, among quantified noun phrases they are exceptions rather than the rule.

Second, though noun phrase denotations $Q^{[A]}$ are usually not ISOM, the underlying quantifier Q usually is. In Section 3.3 below we will see that a type $\langle 1, 1\rangle$ quantifier Q^{rel}, the relativization of Q, is the proper interpretation of the corresponding determiner phrase, and Q^{rel} is ISOM if Q is. This underscores that ISOM, far from being a condition that only interests logicians, is highly relevant for natural language semantics too.

ISOM for Arbitrary Quantifiers

How does one generalize ISOM from type $\langle 1\rangle$ to quantifiers of arbitrary type? The condition in (3.14) generalizes easily to other *monadic* quantifiers; this will be described in Section 3.3. But for arbitrary, i.e., polyadic, quantifiers, one needs another formulation, in terms of the notion of isomorphism between structures or models.

A *structure* in general consists of a universe M, some relations over M, some distinguished elements of M, and some operations on M. Thus it is essentially

[13] It is also a matter of what a local version of Proposition 3.2.3 should look like. A first suggestion might be that if $\emptyset \neq A \subsetneq M$ and Q_A is not trivial, then $(Q^{[A]})_M$ is not PERM_M (cf. Section 3.2.2). But this is not quite true since, as noted, $M - A$ must be sufficiently big for the above proof to go through.

what we in Chapter 2.3.3 called a *model* of a first-order language (with corresponding non-logical predicate symbols, individual constants, and function symbols). If there are no relations, such structures are usually called *algebras* in mathematics. Here, on the other hand, we restrict attention to *relational* structures, which have only relations. These structures can be classified (typed) in the same way as quantifiers.[14]

> **isomorphic structures**
>
> A (relational) structure of type $\tau = \langle n_1, \ldots, n_k \rangle$ has the form $\mathcal{M} = (M, R_1, \ldots, R_k)$, where each R_i is a n_i-ary relation over M. Two such structures $\mathcal{M}$ and $\mathcal{M}' = (M', R'_1, \ldots, R'_k)$ are *isomorphic*, in symbols,
>
> $$\mathcal{M} \cong \mathcal{M}'$$
>
> if there is a 1-1 function f from M onto M' (a *bijection* from M to M') such that, for each i between 1 and k, and for all $a_1, \ldots, a_{n_i} \in M$,
>
> $$R_i(a_1, \ldots, a_{n_i}) \iff R'_i(f(a_1), \ldots, f(a_{n_i}))$$
>
> f is then called an *isomorphism* from $\mathcal{M}$ to $\mathcal{M}'$.

Now, the connection with our first version of ISOM is the following

3.2.4 Fact
If $\mathcal{M} = (M, A)$ and $\mathcal{M}' = (M', A')$ are type $\langle 1 \rangle$ structures, then

(3.16)
$$\mathcal{M} \cong \mathcal{M}' \iff |A| = |A'| \textit{ and } |M - A| = |M' - A'|$$

Thus, ISOM for a type $\langle 1 \rangle$ quantifier Q amounts to the condition that whenever $\mathcal{M} \cong \mathcal{M}'$, $Q_M(A) \Leftrightarrow Q_{M'}(A')$.

Proof. By definition, two sets have the same cardinality iff there is a bijection between them. Now if $\mathcal{M} \cong \mathcal{M}'$ via a bijection f, then f restricted to A is a bijection from A to A', and f restricted to $M - A$ is a bijection from $M - A$ to $M' - A'$. To see this, note that we already know that f is 1-1, and that $a \in A$ iff $f(a) \in A'$ by (3.16). Also, if $b \in A'$ ($b \in M' - A'$), $b = f(a)$ for some unique a, and, again by (3.16), $a \in A$ ($a \in M - A$). So it follows that $|A| = |A'|$ and $|M - A| = |M' - A'|$.

Conversely, if there is a bijection g from A to A', and a bijection h from $M - A$ to $M' - A'$, then the union $f = g \cup h$ of the two (viewed as sets of

[14] There is a familiar ambiguity of the English word 'structure'. On the one hand it refers to mathematical structures as above. On the other hand one often says that if $\mathcal{M} \cong \mathcal{M}'$ they have the 'same structure', or (as we did in the beginning of this section) that in logic only structure counts, not particular objects or relations. A more precise terminology for the latter usage, which avoids the ambiguity, is to say that isomorphic structure have the same *isomorhism type*.

ordered pairs) is a bijection from M to M' such that for all $a \in M$, $a \in A$ iff $f(a) \in A'$. Thus, $\mathcal{M} \cong \mathcal{M}'$. □

It is now immediate what ISOM should amount to for an arbitrary quantifier.

ISOM for arbitrary quantifiers

A quantifier Q of type $\tau = \langle n_1, \ldots, n_k \rangle$ satisfies ISOM iff whenever $(M, R_1, \ldots, R_k)$ and $(M', R'_1, \ldots, R'_k)$ are isomorphic structures of type τ,

$$Q_M(R_1, \ldots, R_k) \iff Q_{M'}(R'_1, \ldots, R'_k)$$

In other words, ISOM quantifiers cannot distinguish between isomorphic structures. It should be clear that this is a precise way of expressing Mostowski's idea that they do not allow us to distinguish between elements of the universe. Furthermore, we can now also see that instead of defining (global) quantifiers as functions associating with each universe a certain second-order relation over that universe, we could simply have defined them as classes of relational structures of a given type. In fact, that is how they were defined in [Lindström 1966]. This is not much more than a notational variant: instead of writing

$$Q_M(R_1, \ldots, R_k)$$

we may write

$$(M, R_1, \ldots, R_k) \in Q$$

Then ISOM is literally closure under isomorphism: if $\mathcal{M} \in Q$ and $\mathcal{M} \cong \mathcal{M}'$, then $\mathcal{M}' \in Q$.

Finally, we note that there is a slightly different way of making precise Mostowski's idea. With ISOM one cannot even distinguish between elements of different universes, if the corresponding 'structure' is the same. Another version restricts attention to elements of one universe at a time.

- An *automorphism on* $\mathcal{M}$, where $\mathcal{M}$ is of type τ, is an isomorphism from $\mathcal{M}$ to itself. The corresponding condition on a type τ quantifier Q of being closed under automorphisms is sometimes called PERM, or *Permutation Closure*,[15] since a bijection from a set M to itself is often called a permutation on M. ISOM implies PERM, but not (always) vice-versa.

- Note that on a given universe M, Permutation Closure can be stated as a condition on Q_M. If we call this local version PERM_M, then PERM is the corresponding global version (cf. (G) in Section 3.1.1). ISOM has no such local version.

We can see that PERM is weaker than ISOM by considering again (3.14). The corresponding version of PERM says that, for all M and all $A, A' \subseteq M$:

[15] Or PI, or *Permutation Invariance*.

(3.17) If $|A| = |A'|$ and $|M - A| = |M - A'|$, then $Q_M(A) \Leftrightarrow Q_M(A')$.

Now suppose that Q is defined by

$$Q_M = \begin{cases} \exists_M & \text{if } |M| < 10, \\ (\exists_{\geq 2})_M & \text{if } |M| \geq 10 \end{cases}$$

Then Q satisfies PERM but not ISOM: PERM is clear, but if M has, say, 9 elements, a unit set $\{a\}$ is in Q_M for $a \in M$, whereas if M is extended to M' by adding just one new element, $\{a\}$ is not in $Q_{M'}$, contradicting ISOM.

3.2.3 Extension

Looking at the list of examples of type $\langle 1 \rangle$ quantifiers in Section 3.2.1, in particular the ISOM ones which can be seen as binary relations between numbers, it is conspicuous that for some of them, whether $Q_M(A)$ (alternatively, $Q(\kappa, \lambda)$) holds or not depends only on A (on $|A| = \lambda$). For others, it depends only on $M - A$ (on $|M - A| = \kappa$), and for the rest it depends on both arguments. The first of these properties is a kind of *universe-independence*, which in the literature goes by the name of EXT or *Extension*. This terminology derives from the first version of the following definition.

EXT for type $\langle 1 \rangle$ quantifiers

A quantifier Q of type $\langle 1 \rangle$ satisfies EXT if and only if

(3.18) $A \subseteq M \subseteq M'$ implies $Q_M(A) \Leftrightarrow Q_{M'}(A)$

That is, extending the universe has no effect. An equivalent formulation is:

(3.19) If A is a subset of both M and M', then $Q_M(A) \Leftrightarrow Q_{M'}(A)$

Clearly (3.19) implies (3.18). Conversely, if $A \subseteq M, M'$, let $M'' = M \cap M'$. Since $A \subseteq M'' \subseteq M$ and $A \subseteq M'' \subseteq M'$, we get, using (3.18) twice, $Q_M(A) \Leftrightarrow Q_{M''}(A) \Leftrightarrow Q_{M'}(A)$.

For ISOM type $\langle 1 \rangle$ quantifiers, EXT has yet another useful formulation.

3.2.5 Fact
If Q is ISOM, EXT is equivalent to the following condition:

(3.20) *If $Q(\kappa, \lambda)$, then for all cardinal numbers κ', $Q(\kappa', \lambda)$.*

This means that Q can be seen as a class S of cardinal numbers, i.e., the cardinal numbers of those sets which satisfy the condition of the quantifier:

$$\begin{aligned} (3.21)\ \lambda \in S &\iff \textit{for some } \kappa, Q(\kappa, \lambda) \\ &\iff \textit{for some } M, \textit{ and some } A \subseteq M \textit{ s.t. } |A| = \lambda, Q_M(A) \end{aligned}$$

In other words, Q is EXT iff there is a class S such that (3.21) holds, or (equivalently), for all M and all $A \subseteq M$,

$$Q_M(A) \iff |A| \in S$$

Proof. Suppose EXT holds and that $Q(\kappa, \lambda)$. So for some M and some $A \subseteq M$ such that $|A| = \lambda$ and $|M - A| = \kappa$, $Q_M(A)$. Now if κ' is any number, take a universe M' such that $A \subseteq M'$ and $|M' - A| = \kappa'$. This is always possible, and it follows by (3.19) that $Q_{M'}(A)$, i.e., that $Q(\kappa', \lambda)$. Thus, (3.2.5) holds.

Conversely, given (3.2.5), suppose that $A \subseteq M \subseteq M'$. Then $Q_M(A) \Leftrightarrow Q(|M - A|, |A|)$, and $Q_{M'}(A) \Leftrightarrow Q(|M' - A|, |A|)$. But then (3.2.5) implies that $Q_{M'}(A) \Leftrightarrow Q_M(A)$.

We have shown that when Q is seen as a relation between cardinal numbers, EXT means that the first argument of this relation is immaterial. Thus, Q can be identified with the class of numbers that occur as the second argument (i.e., with the range or co-domain of the relation). □

The following are examples of ISOM and EXT quantifiers:

$$\exists,\ \exists_{\geq 5},\ \exists_{\leq 7},\ \exists_{=3},\ Q_{even},\ Q_0$$

Thus, by Fact 3.2.5, $\exists$ can be identified with the class of cardinal numbers > 0, $\exists_{\geq 5}$ with those ≥ 5, $\exists_{\leq 7}$ with $\{0, 1, \ldots, 7\}$, $\exists_{=3}$ with $\{3\}$, Q_{even} with $\{0, 2, 4, \ldots\}$, etc.

Among quantifiers which are not ISOM, we may note that

- All quantifiers of the form I_a are EXT.

  [Since $a \in A$ or not regardless of the surrounding universe.]

- If Q is EXT, so is $Q^{[A]}$.

  [Follows from (3.19) and the definition (3.12) of $Q^{[A]}$ in Section 3.2.1.]

Another easily verified observation is:

- The class of EXT type ⟨1⟩ quantifiers is closed under conjunction, disjunction, and (outer) negation.

This gives us a host of EXT quantifiers, also odd ones such as $\exists_{=3} \vee I_a$, which holds of a set A iff A either has exactly three elements or contains a.

Here is one more example, that we have not mentioned before. A sentence like

Nobody but John smokes.

expresses that John smokes but nobody else (in the discourse universe) does. This means that the noun phrase `nobody but John` denotes the quantifier NB_j (assuming that `John` denotes j) defined by

(3.22) $(NB_j)_M(A) \iff A = \{j\}$

This is EXT, since the truth condition is quite independent of M. NB_j is an *exception* quantifier; we will have more to say about these in Section ZZZ. Not all of them are EXT; for example,

Everybody but John smokes.

gives the exception quantifier

(3.23) $(EB_j)_M(A) \iff A = M - \{j\}$

where M plays a crucial role. Note that EB_j is the inner negation of NB_j.

If $Q_M(A)$ 'depends' only on A, then $Q\neg_M(A)$ and $Q^{\mathrm{d}}_M(A)$ 'depend' only on $M - A$. In other words, they are co-EXT. So taking inner negations and duals of the previous examples we get co-EXT type $\langle 1 \rangle$ quantifiers, where the earlier condition on A is replaced by the corresponding condition, or its negation, on $M - A$. The foremost example is

$$\forall$$

Likewise,

$$(\exists_{\leq 7}\neg)_M(A) \iff |M - A| \leq 7$$

$$(\exists^{\mathrm{d}}_{\leq 7})_M(A) \iff |M - A| > 7$$

etc. are co-EXT. As in Fact 3.2.5, co-EXT and ISOM type $\langle 1 \rangle$ quantifiers can also be identified with classes of cardinal numbers; this time the sizes of the sets $M - A$ satisfying the condition of the quantifier.

Proportional quantifiers are typical examples of quantifiers that are neither EXT nor co-EXT, i.e., where both A and $M - A$ are essential. As one might expect, we will see that such quantifiers are more complex in various senses than the EXT or co-EXT ones. We can also get examples from Boolean combinations of EXT and co-EXT quantifiers, such as

$$\exists \wedge \neg\forall$$

which we can read as `something but not everything`. Then $(\exists \wedge \neg\forall)_M(M)$ is false, but if M is a proper subset of M', $(\exists \wedge \neg\forall)_{M'}(M)$ is true. This shows that EXT fails, and by a similar argument co-EXT fails too.

As we have said, EXT expresses a notion of universe-independence. This notion is not in any way tied to type $\langle 1 \rangle$ quantifiers, so it is immediate how to formulate EXT for abitrary types. Here is the version corresponding to (3.18).

EXT for arbitrary quantifiers

A quantifier Q of type $\langle n_1, \ldots, n_k \rangle$ satisfies EXT iff the following holds:

(3.24) If $R_i \subseteq M^{n_i}$ for $1 \leq i \leq k$, and $M \subseteq M'$, then $Q_M(R_1, \ldots, R_k) \Leftrightarrow Q_{M'}(R_1, \ldots, R_k)$

What is the significance of EXT? We are not quite sure if there is a simple answer to this question. In Section 3.3 we shall find that in the type $\langle 1 \rangle$ case, it corresponds exactly to a significant class of natural language quantifiers of type $\langle 1,1\rangle$. But in the general case, what we would like to say is the following: EXT is a criterion of *constancy* of expressions that denote quantifiers.

A quantifier associates a second-order relation (of a certain type) with each universe. In principle, it can associate wildly different relations with different universes. With universes containing John and at least three dogs it could associate the universal quantifier, with those containing John but fewer than three dogs it could associate I_j, and with other universes it could associate $\exists_{\geq 7}^{[cat]}$. Nothing in the general definition of a quantifier prevents this. Even if ISOM is imposed, Q_M could still be completely different for different *sizes* of M. But in practice such examples do not seem to occur 'naturally', and in particular not in natural language contexts.

So EXT is a substantive constraint, which rules out global quantifiers that associate wildly different local quantifiers with different universes. Contraposing, we would like to say that EXT allows only those global quantifiers that associate the *same* quantifier with each universe. But of course that can at best be a metaphorical or preliminary way of speaking. If $M \neq M'$, Q_M and $Q_{M'}$ are usually *different*.[16] So we need another sense of 'same quantifier' here, and one suggestion, then, is that EXT provides precisely this sense.

We would like to say this, but there is a problem: in the type $\langle 1\rangle$ case, EXT does not seem to give the right notion of 'same quantifier'. To take the most obvious example, the universal quantifier $\forall$ intuitively appears to be the same on each universe, but it is not EXT. Obviously, if a subset of a universe contains everything in it (i.e., if it is identical to that universe), it cannot contain everything in a larger universe. The same holds for quantifiers like *all but at most three things*.

These examples are co-EXT (their duals are $\exists$ and $\exists_{\geq 4}$, respectively), so we might try to capture our idea of constancy with 'EXT or co-EXT' instead. But even the Rescher quantifier, and other proportional type $\langle 1\rangle$ quantifiers, surely are not 'wildly different', but instead quite uniform, over different universes. Something else must be going on here. Perhaps the significance of EXT is different for type $\langle 1\rangle$ than for other types. We come back to the question in Section 3.3.4.

We end this section with two more remarks about EXT. First, since EXT says precisely that the universe is irrelevant, one may drop the subscript indicating the universe for such quantifiers, and write simply

$$Q(R_1, \dots, R_k)$$

rather than

$$Q_M(R_1, \dots, R_k)$$

More formally, we can *define*

[16] For example, if M and M' are disjoint then Q_M and $Q_{M'}$ are always different except in the one case when both are the empty set.

- $Q(R_1, \ldots, R_k)$ iff for some M such that $R_i \subseteq M^{n_i}$ for $1 \leq i \leq k$, $Q_M(R_1, \ldots, R_k)$.

It then follows that this is well-defined (independent of the choice of M) if Q satisfies EXT.

Second, at the end of the last section we saw that PERM is a weaker constraint than ISOM. In the presence of EXT, however, they are equivalent. The verification of the following fact is left as an exercise.[17]

3.2.6 Fact
Let Q be a quantifier (of any type) satifying EXT. If PERM holds for Q, then so does ISOM.

3.2.4 Monotonicity

Monotonicity is a phenomenon that turns up in all kinds of contexts, linguistic as well as mathematical. Abstractly, it says of some function F that it is *increasing*: relative to an ordering of the arguments and an ordering (the same or a different one) of the values: if $x \leq y$, then $F(x) \leq F(y)$. Such functions are well-behaved in ways that make them have interesting properties, and they occur frequently. Leaving abstract formulations aside, we begin with monotonicity of type $\langle 1 \rangle$ quantifiers.

monotonicity for type $\langle 1 \rangle$ quantifiers

Let Q be of type $\langle 1 \rangle$. Q_M is (monotone) *increasing* iff the following holds:

(3.25) If $A \subseteq A' \subseteq M$, then $Q_M(A)$ implies $Q_M(A')$.

Q_M is (monotone) *decreasing* iff

(3.26) If $A' \subseteq A \subseteq M$, then $Q_M(A)$ implies $Q_M(A')$.

These local notions extend immediately to the global case: Q is monotone increasing (decreasing) if each Q_M is.

Monotone increasing quantifiers are often called just *monotone* in the literature.[18] Most of the quantifiers we have seen so far are increasing or decreasing; here are the relevant observations:

[17]Hint: First show that under EXT, PERM implies that whenever $(M, R_1, \ldots, R_k)$ and $(M', R'_1, \ldots, R'_k)$ are isomorphic structures such that M and M' are *disjoint*,

$$Q_M(R_1, \ldots, R_k) \iff Q_{M'}(R'_1, \ldots, R'_k)$$

Then show that the general case follows from this.

[18]Thus, these quantifiers can be called 'monotone increasing', or just 'increasing', or just 'monotone'. It is also common to let 'monotone' stand for 'either increasing or decreasing', so that monotonicity is what increasing and decreasing quantifiers have in common, whereas non-monotone quantifiers are neither increasing nor decreasing. Further, the terms *upward* and *downward* monotone are also used. These slight differences in terminology should not

- Of the quantifiers exemplified in Section 3.2.1, (3.3) – (3.5),
  - $\forall$, $\exists$, $\exists_{\geq n}$, Q_0, Q^C, Q^R, (p/q), $[p/q]$, $(p/q)^{\mathrm{d}}$ are monotone increasing,
  - $\exists_{\leq n}$ is monotone increasing,
  - $\exists_{=n}$, $\exists_{=n}\neg$, $\exists_{\geq n} \wedge \exists_{\leq m}$, Q_{even} are neither increasing nor decreasing (if $n \leq m$).
- There might seem to be fewer decreasing than increasing ones, but that is just an impression created by what may appear most natural to take as primitive, as the following observations illustrate:
  - Q is increasing iff $\neg Q$ is decreasing.

    [Since if $Q_M(A)$ implies $Q_M(A')$, then $\neg Q_M(A')$ implies $\neg Q_M(A)$.]
  - Q is increasing iff $Q\neg$ is decreasing.

    [Since if $A \subseteq A'$, then $M - A' \subseteq M - A$; hence:]
  - Q is increasing (decreasing) iff Q^{d} is increasing (decreasing).
- The trivial quantifiers $\mathbf{1}_M$ and $\mathbf{0}_M$ are both increasing *and* decreasing. But on a given universe M they are the only quantifiers with this property.

  [For, if $Q_M \neq \mathbf{0}_M$ there is some $A \subseteq M$ such that $Q_M(A)$. Take any $B \subseteq M$. Then $Q_M(A \cap B)$ since Q_M is decreasing, hence $Q_M(B)$ since Q_M is increasing. So $Q_M(B)$ holds for any B, i.e., $Q_M = \mathbf{1}_M$.]
- Montagovian individuals I_a are monotone increasing.
- If Q is monotone increasing (decreasing), so is $Q^{[A]}$.

The last observation[19] explains why many noun phrase denotations are increasing or decreasing. For example, `at most four students` is decreasing, which is illustrated by facts such that if

(3.27) At most four students came to the opening.

is true, then

(3.28) At most four students came to the opening wearing a tie.

must also be true. Likewise, if

(3.29) Most professors came to the funeral.

cause confusion.

Note also that with the functional view of (local) type ⟨1⟩ quantifiers (Section 3.1.4) as functions in $[\wp(M) \longrightarrow \mathbf{2}]$, monotonicity is precisely that $A \leq A'$ implies $Q_M(A) \leq Q_M(A')$, where the (partial) ordering on subsets of M is the inclusion order, and the ordering on the values $0, 1$ is the usual one.

[19] Note that this last observation (in contrast with most of the other ones above) is a global but not a local fact: If one wants to conclude that $(Q^{[A]})_M$ is increasing, it does not help to know that Q_M is increasing; one needs to know that $Q_{M \cap A}$ is increasing.

is true, then

(3.30) Most professors came to the funeral or attended the service.

is also true, witnessing the fact that `most professors` is monotone increasing.

By means of such examples one can devise *tests* for monotonicity, for example, that sentences of the form `At least five A are B` should imply `At least five A are B or C`. Of course, this is not a test that the quantifier $\exists^{[A]}_{\geq 5}$ is increasing — we already know that it is. Rather, it is a test that we have interpreted noun phrases of the form `at least five A` correctly, or at least in a way that does not violate linguistic intuitions. Naturally such tests are not in themselves conclusive. The hypothesis tested concerns meaning, which is about all possible cases, whereas we can only test a finite number of them. And even if we should find a counterexample, the blame may be put elsewhere than on the meaning of the noun phrase. For example, if we should find that English speakers easily envisage situations where they would judge `John is at school` to be true, but `John is at school or at home` to be false, then we ought to conclude that something is seriously amiss with our semantics, but it need not be Montague's treatment of proper names — it could be something to do with `or` instead. Luckily, we do not find such intuitions prevalent among English speakers,[20] and this then is a test of (among other things) that treatment of names, which does interpret them by means of an increasing quantifier.

It is a fact that among the non-monotone quantifiers (in the sense of being neither increasing nor decreasing) that occur 'naturally', a great many are nevertheless Boolean combinations of monotone ones. Thus, *between three and six dogs* is the conjunction of *at least three dogs* and *at most six dogs*, and *all but five* is the inner negation of the conjunction of *at least five* and *at most five*. But a notable exception to this pattern is the quantifier Q_{even}, and its cognates like *an odd number of cards*, or *all but an even number of balls*. These are not increasing or decreasing, nor are they Boolean combinations of such quantifiers. There is no reason for the parity of a set to be preserved when one adds to or deletes from it an arbitrary number of elements. Of course this claim about Q_{even} — that it is not expressible as a Boolean combination of monotone quantifiers — is so far very preliminary: though seemingly plausible when one thinks about it, it needs to be (a) made precise, and (b) verified (or disproved). We return to such matters of expressive power, and in particular to the above claim, in Chapter **??**.

To extend the idea of monotonicity to arbitrary quantifiers is an easy matter. Naturally, if a quantifier as a second-order relation has more than one argument, we must specify which argument we are talking about. This leads to the following definition.

[20] Again, we must take care to distinguish pragmatic facts, such that if one *knows* that John is at school it may feel *odd to say* that he is at school or at home, from simple judgements about truth and falsity.

monotonicity for arbitrary quantifiers

A quantifier Q_M of type $\langle n_1, \ldots, n_k \rangle$ is (monotone) *increasing in the i:th argument* iff the following holds:

(3.31) If $Q_M(R_1, \ldots, R_k)$, and if it holds that $R_i \subseteq R'_i \subseteq M^{n_i}$, then $Q_M(R_1, \ldots, R_{i-1}, R'_i, R_{i+1}, \ldots, R_k)$.

Similarly for the (monotone) *decreasing* case.

Of course a quantifier can be increasing in one argument and decreasing in another; many such combinations occur naturally as we will see, beginning in Section 3.3 for the type $\langle 1, 1 \rangle$ case. But there are a few more things we want to say about type $\langle 1 \rangle$ quantifiers.

Linguistic Aspects of Monotonicity

Monotonicity is a pervasive trait in natural languages, so it is perhaps not surprising that it is involved in various generalizations and explanations of linguistic phenomena. [Barwise and Cooper 1981] give a number of such possible generalizations, for example, that noun phrases with *syntactically simple* determiners in natural languages denote either increasing or decreasing quantifiers, or conjunctions of such quantifiers. Another example is that *positive* (*negative*) *strong* natural language quantifiers are monotone increasing (decreasing), where Q is positive (negative) strong iff for every M, $Q_M(M)$ (for every M, not $Q_M(M)$).

Such 'linguistic universals' are sometimes nothing but proposed empirical generalizations across languages, but sometimes they can be explained along more principled lines, or figure themselves in explanations of other linguistic phenomena. As an example, Barwise and Cooper started exploring the idea that monotone quantifiers are *easier to process* than non-monotone ones; this was developed (a lot) further by van Benthem; cf. [van Benthem 1986], ch. 10, and also ch. 8. But probably the most striking use of monotonicity to explain a linguistic phenomenon concerns the distribution of so-called negative polarity items.

Monotonicity and Polarity

TBW

Monotonicity under ISOM

A monotone increasing quantifier Q associates with a universe M an idea of 'bigness': 'big' subsets of M are in Q_M in the sense that if A is in Q_M so are all bigger sets. Of course quite small sets may also be 'big'; the standard associated with M is arbitrary, and in the extreme case when $\emptyset \in Q_M$, all subsets of M are 'big' (so $Q_M = \mathbf{1}_M$).

In general there need be no smallest among the ‘big’ sets; smallest in the sense that no proper subset is ‘big’. For example, consider Q_0^{d}, the dual of *infinitely many things*, i.e., *all but finitely many things*. This is monotone increasing, since Q_0 is increasing, or, more directly, since if $M - A$ is finite and $A \subseteq A' \subseteq M$, then $M - A'$ is also finite. But if M is infinite there is no smallest A such that $M - A$ is finite. If $N = \{0, 1, 2, \dots\}$, then $N - \{0\} \supseteq N - \{0, 1\} \supseteq N - \{0, 1, 2\} \supseteq \dots$ are all ‘big’ by the standards of $(Q_0^{\mathrm{d}})_N$.

Note however that there is always a smallest *size* among the ‘big’ sets. This is because any class of cardinal numbers has a smallest element, so indeed for any quantifier Q_M there is a smallest size of the sets it contains, regardless of whether it is monotone or not. But when Q is monotone increasing and ISOM, this intuitive idea takes a definite and concrete shape. Then the standard of ‘bigness’ depends only on $|M|$, and is a cardinal number $\kappa \leq |M|$, such that $Q_M(A)$ holds iff $|A| \geq \kappa$. We shall introduce some very useful notation for this in the special case of *finite* universes.

So assume Q is monotone increasing and satisfies ISOM. There is then a corresponding function f which with each finite universe of size n associates a natural number $f(n)$, which is the smallest size of sets in Q_M, when $|M| = n$. Thus, $0 \leq f(n) \leq n$. This covers all cases except one, namely, when no sets at all are in Q_M (so $Q_M = \mathbf{0}_M$). But if we stipulate that $f(n) = n + 1$, it holds in general that if $A \subseteq M$,

$$Q_M(A) \iff |A| \geq f(|M|)$$

(In particular, if $f(|M|) = |M|+1$, $|A| \geq f(|M|)$ is always false.) We summarize this in the following definition and observation:

the quantifiers Q_f

If f is any function from natural numbers to natural numbers such that for all n, $0 \leq f(n) \leq n + 1$, the type $\langle 1 \rangle$ quantifier Q_f is defined as follows (on finite universes): For any finite M and any $A \subseteq M$,

(3.32) $(Q_f)_M(A) \iff |A| \geq f(|M|)$

Then Q_f is ISOM and monotone increasing. Moreover, any ISOM and monotone increasing quantifier Q is equal to Q_f (on finite universes) for some f, which we may call the *monotonicity function* for Q.

Here are some of our previous examples written in this format:

- If $f_1(n) = 1$ for all n, then $Q_{f_1} = \exists$. More generally, if $g(n) =$ a constant k for all n, then $Q_g = \exists_{\geq k}$.

- If $f_2(n) = n$ for all n, then $Q_{f_2} = \forall$. If $h(n) = n - k$ for a fixed k, then Q_h is *all but at most k things*.

- If $f_3(n) =$ the smallest natural number $> n/2$, then $Q_{f_3} = Q^R$.

We noted earlier that Q is increasing iff its dual Q^{d} is increasing. Now observe that, with $|M| = n$,

$$\begin{aligned}(Q^{\mathrm{d}}_f)_M(A) &\iff |M - A| < f(|M|)\\ &\iff n - |A| < f(n)\\ &\iff |A| > n - f(n)\\ &\iff |A| \geq n - f(n) + 1\end{aligned}$$

Thus, we have the following

3.2.7 Fact
Define

$$f^d(n) = n - f(n) + 1$$

for all n. Then f^d is the monotonicity function for the dual of Q_f, that is, $Q_{f^d} = (Q_f)^d$.

3.3 Type $\langle 1,1\rangle$ Quantifiers

The significance of the class of type $\langle 1,1\rangle$ quantifiers for natural language semantics is that most determiner expressions (simple or complex) can be taken to denote in this class. They are, on each universe, binary relations between subsets of the universe. However, not all such quantifiers are eligible as determiner interpretations, but only those which are so-called *relativizations* of type $\langle 1\rangle$ quantifiers. This is a fundamental observation, which goes a long way to explaining the characteristic semantic behavior of determiners. The concept of relativization (familiar from logic long before generalized quantifier theory was applied to natural language) is precisely what one needs for a precise account of the idea that determiners *restrict the domain of quantification* to the noun argument. We call this argument the *restriction* of the quantifier, and the second argument its *scope*.[21] After giving some examples, we define the relevant notion of relativization and the idea of domain restriction, and then discuss the consequences for the semantics of determiners.

3.3.1 Examples of Determiners

Here is a list of examples of English determiner expressions, of increasing complexity.[22]

(3.33) a. some, a, all, every, no, several, most, neither, the, both, this, these, my, John's, many, few, enough, a few, a dozen, ten,

[21] These terms are usually applied to the corresponding syntactic parts of a corresponding quantified sentence, as their names indicate. As we will see, 'restriction' is adequate also semantically. The (semantic) scope is sometimes called the *verb argument*.

[22] The list is essentially as in [Keenan and Westerståhl 1997].

b. the ten, John's ten, at least/more than/ fewer than/ at most/exactly ten, all but ten, half of the, half of John's, infinitely many, at most finitely many, about two hundred, nearly a hundred, most of John's, an even number of, at least two thirds of the, less than ten percent of the, between five and ten, exactly three quarters of the,

c. not all, not every, not many, no more than ten, not more than half of the, some/most but not all, at least two and no more than ten, neither John's nor Mary's, either fewer than five or else more than a hundred, no child's, most male and all female, not one of John's, more of John's than of Mary's, no/every ... but John, all but finitely many.

The list is intended to show the richness of English determiner expressions that *can* be interpreted as type $\langle 1 \rangle$ quantifiers. In various cases one might argue for another treatment. For example, it has been claimed that the indefinite `a` or the definite `the`, or numerals like `ten`, should not be treated as quantifiers. But our point here is just that they *can* be so treated, and that it is then fairly obvious which quantifiers they denote. For example,

$$\mathit{the}_M(A, B) \iff A \subseteq B \text{ and } |A| = 1$$

(this is the singular *the*; for the plural case the second conjunct is $|A| > 1$);

$$\mathit{ten}_M(A, B) \iff |A \cap B| = 10$$

(another reading that might be possible is *at least ten*);

$$\mathit{John's\ ten}_M(A, B) \Leftrightarrow |A \cap \{a : R(j, a)\}| = 10 \text{ and } A \cap \{a : R(j, a)\} \subseteq B$$

(where R is a contextually given relation, cf. `John's ten bikes`, `John's ten exams`, `John's ten friends`, `John's ten toes`).

$$\mathit{no} \ldots \mathit{but\ John}_M(A, B) \iff A \cap B = \{j\}$$

$$\mathit{every} \ldots \mathit{except\ John}_M(A, B) \iff A - B = \{j\}$$

In some of the above cases, like `about two hundred`, *vagueness* may be an obstacle to giving an interpretation. But this is the problem of finding an adequate semantics for vague expressions in general, and not specific to determiners.

Some of the determiners above require input from the *context* for their interpretation. As already noted `most` sometimes means *more than half of the* — this is the default interpretation we use here — but sometimes the 'threshold' is higher. There is no problem thinking of context as providing this. In other cases, context-dependence is so strong that one may doubt if extensional (generalized) quantifiers are adequate as interpretations. Consider

(3.34) a. Too many doctors attended the reception.

b. Too many lawyers attended the reception.

Even in a situation when the set of lawyers at the reception was identical to the set of doctors at the reception, these two sentences could have different truth values. One may thus decide to treat `too many` as *intensional* and exclude it from the present treatment. But one may also maintain that the nouns of the respective sentences form part of the context, so that different standards for 'too many' are generated. In the literature, this question has been discussed especially for `many` and `few`; some authors exclude them on the ground of intensionality whereas others attempt a context-dependent extensional interpretation.[23] Again, the issue is part of the more general one of how to deal with context-dependence and intensionality. We shall not need to take a stand on it here, but will return to the interpretation of `many` and `few` in Section 3.3.4.

The above list of examples shows that in many cases complex determiners are Boolean combinations of simpler determiners. Idealizing a bit, it is sometimes practical to assume that the class of determiner denotations is *closed* under Boolean operations. In reality, this is probably not true. Note, however, that

(a) the issue is not when expressions of the form `Det and Det` or `not Det` are clumsy or longwinded and therefore hardly ever used (as is no doubt often the case), but when they are meaningful, and

(b) even if such an expression is judged not meaningful, there might still be *another* (possibly clumsy) determiner expression which denotes the quantifier in question, and this is all that is required. For example, `not most` may be judged ill-formed and hence meaningless, but the negation of *most* is expressed by `at most half of the`, and is thus a determiner denotation.

Some attempts to describe more accurately the behavior of the class of determiner denotations under Boolean operations, in particular under outer and inner negation, can be found in [Barwise and Cooper 1981], pp. PPP, and [Westerståhl 1989], pp. 70–74.

3.3.2 Relativization

A type $\langle 1 \rangle$ quantifier specifies a property of sets, which — unless it satisfies EXT — depends on the universe considered. But then we can form a corresponding type $\langle 1, 1 \rangle$ quantifier, where the first argument plays the role of that universe. The 'action' of the resulting *relativized quantifier* is thus restricted to this first (set) argument; as a consequence, the relativized quantifier itself automatically satisfies EXT. The operation of relativization is not restricted to type $\langle 1 \rangle$ quantifiers but applies to any type. Here is the definition.

[23] Cf. [Barwise and Cooper 1981], [Keenan and Stavi 1986], [Westerståhl 1985], [Fernando and Kamp 1996], [Cohen 2001], among others.

definition of $Q^{\mathbf{rel}}$

If Q is of type $\langle 1\rangle$, the type $\langle 1,1\rangle$ quantifier Q^{rel} is defined as follows: for any M and any $A, B \subseteq M$,

(3.35) $(Q^{\mathrm{rel}})_M(A,B) \iff Q_A(A \cap B)$

More generally, if Q is of type $\langle n_1, \ldots, n_k\rangle$, Q^{rel} of type $\langle 1, n_1, \ldots, n_k\rangle$ is defined, for $A \subseteq M$ and $R_i \subseteq M^{n_i}$, $1 \le i \le k$, by

$$(Q^{\mathrm{rel}})_M(A, R_1, \ldots, R_k) \iff Q_A(R_1 \cap A^{n_1}, \ldots, R_k \cap A^{n_k})$$

Here we shall mainly be interested in the type $\langle 1\rangle$ case, so if we write 'Q^{rel}' it is assumed unless otherwise stated that Q is of type $\langle 1\rangle$.

The following examples of relativized quantifiers show that the relativization operation is in fact quite familiar.

- $\forall^{\mathrm{rel}}$ = *every*
- $\exists^{\mathrm{rel}}$ = *some*
- $\exists^{\mathrm{rel}}_{\ge 4}$ = *at least four*
- $\exists^{\mathrm{rel}}_{=5}$ = *exactly five*
- $(\exists_{=5}\neg)^{\mathrm{rel}}$ = *all but five*
- Q_0^{rel} = *infinitely many*
- $(Q^R)^{\mathrm{rel}}$ = *most*
- $(p/q)^{\mathrm{rel}}$ = *more than p q'ths of the*
- $[p/q]^{\mathrm{rel}}$ = *at least p q'ths of the*
- Q^{rel}_{even} = *an even number of*

These are all ISOM quantifiers, but relativization applies also to the non-ISOM case:

- If NB_j and EB_j are the type $\langle 1\rangle$ quantifiers *nobody but John* and *everybody but John*, respectively (that is, $(NB_j)_M(B) \Leftrightarrow B = \{j\}$ and $(EB_j)_M(B) \Leftrightarrow B = M - \{j\}$), then
  - $(NB_j)^{\mathrm{rel}}$ = *no ... but John*
  - $(EB_j)^{\mathrm{rel}}$ = *every ... except John*
- If JT is the type $\langle 1\rangle$ quantifier *John's things* defined by $JT_M(B) \Leftrightarrow \{a \in M : R(j,a)\} \subseteq B$ (where R is a contextually given 'ownership' relation), then

– JT^{rel} = *John's*

Similarly, if $JT_{=10}$ is *John's ten things*, then

– $JT^{\mathrm{rel}}_{=10}$ = *John's ten*

So it seems that all the interpretations of familiar determiners are in fact relativizations of type $\langle 1\rangle$ quantifiers; we shall elaborate on this in Section 3.3.3 below. But although all type $\langle 1\rangle$ quantifiers can be relativized, not all such relativizations are possible interpretations of natural language determiners. Apart from relativizations of purely mathematical type $\langle 1\rangle$ quantifiers, such as

$$D_M(A) \iff |A| \text{ divides } |M|$$

whose relativization is the quantifier *Div* from Section 3.1.2, we have cases like the following:

- If we relativize a Montagovian individual I_j (Section 3.2.1), we obtain

  $$(I_j^{\mathrm{rel}})_M(A,B) \iff j \in A \cap B$$

  But there appear to be no determiners `John-d` in natural languages such that, say (in the English case),

  John-d students smoke

  means that John is a student who smokes, i.e., such that *John-d students* means 'some student who is identical with John'. So these relativized quantifiers do not turn up in a natural language context.

- We may also relativize restricted type $\langle 1\rangle$ quantifiers of the form $Q^{[A]}$ ((3.12) in Section 3.2.1), obtaining

  $$\begin{aligned} (Q^{[A]})^{\mathrm{rel}}_M(B,C) &\iff Q^{[A]}_B(B\cap C) \\ &\iff Q_{A\cap B}(A\cap B\cap C) \\ &\iff Q^{\mathrm{rel}}_M(A\cap B, C) \end{aligned}$$

  which might be expressed loosely by saying that relativizing $Q^{[A]}$ to B is the same as relativizing Q to $A\cap B$. But the linguistic point here is another, namely, that $Q^{[A]}$ is already a kind of relativized quantifier, or more precisely obtained by first relativizing Q and then freezing the restriction argument; we come back to this in Section 3.3.3 below (Fact 3.3.3).

Let us also note that the right hand side of definition (3.35) (or the definition for arbitrary types) does not mention the universe at all, so it is clear that the following holds.

3.3.1 Fact
Relativized quantifiers satisfy EXT.

As stated at the end of Section 3.2.3, for EXT quantifiers one may drop the subscript indicating the universe. So we shall often write

$$Q^{\mathrm{rel}}(A, B)$$

rather than

$$(Q^{\mathrm{rel}})_M(A, B)$$

Empty Universes

For definition (3.35) to work properly, one needs to observe the following stipulation:

- Quantifiers are also defined on the empty universe. For most familiar quantifiers, the truth conditions in that case are obvious: they are obtained by simply applying the usual defining condition to $\emptyset$. For example,
  - $\exists_\emptyset(\emptyset)$ is false, since $\emptyset \neq \emptyset$ is false.
  - $\forall_\emptyset(\emptyset)$ is true, since $\emptyset = \emptyset$ is true.
  - $(Q^R)_\emptyset(\emptyset)$ is false, since it is false that $0 = |\emptyset| > |\emptyset - \emptyset| = 0$. But $(Q^R)^{\mathrm{d}}_\emptyset(\emptyset)$ is true.
  - Etc.

These stipulations may seem gratuitous in the type $\langle 1 \rangle$ case, but they are indispensable — or rather, they avoid totally unnecessary complications — for relativized quantifiers, since even if one often assumes that the universe of discourse is non-empty, the *arguments* of a quantifier must be allowed to be empty.

Still, one might feel that these stipulations, even if practical, are somewhat momentous, and one may also wonder if they are the 'right' ones. But it should be noted that in the case of relativized quantifiers it is just a matter of choosing between one of two possibilities. In general, the truth of

$$Q_M(\emptyset, B)$$

may vary with B. But when the quantifier is of the form Q^{rel}, there are only two cases; it follows immediately from (3.35) that:

- Either $Q^{\mathrm{rel}}_M(\emptyset, B)$ holds for all $B \subseteq M$ (namely, when $Q_\emptyset(\emptyset)$ is true), or it holds for no such B (when $Q_\emptyset(\emptyset)$ is false).

Thus, the stipulations are in fact not so momentous. Are they 'correct'? To the extent that this question has an answer, it depends on various intuitive judgements of speakers. As always, care must be taken to distinguish the fact that if one *knows* that the noun `A` has empty denotation, it would often be *odd* *to utter* a sentence of the form `Q A's are B`, from facts about the perceived falsity (or truth) of the sentence in this case. So one way to be careful is to use sentences where it clearly can be unknown whether the noun denotation is empty or not, such as the following:

(3.36) All solutions to this system of equations are integers.

It seems unproblematic that this sentence could be acceptable (for example, provable in a certain theory) regardless of whether there were any solutions at all to the system of equations in question, or at least regardless of whether one has any knowledge about the existence of such solutions.

A perhaps more convincing (because more widely applicable) strategy is to choose examples where an eventual presupposition of non-emptiness is explicitly canceled. For example, speakers presumably find the following acceptable:

(3.37) It is true that no graduate students at the party were drunk, because there were in fact no graduate students at the party.

Even the following seems acceptable:

(3.38) It is true that all graduate students at the party were drunk, because there were in fact no graduate students at the party.

But they would find the following unacceptable:

(3.39) It is true that at least two graduate students at the party were drunk, because there were in fact no graduate students at the party.

(3.40) It is true that no graduate students at the party except John were drunk, because there were in fact no graduate students at the party.

The following seems unacceptable too:

(3.41) It is true that more than half of the graduate students at the party were drunk, because there were in fact no graduate students at the party.

These observations would then support the corresponding stipulations. It should be noted, however, that for other determiners such judgements are less clear, e.g., for `most`, where the stipulation makes `Most A's are B` false when `A` is empty, or `at least half of the`, where the corresponding sentence would be true. A lot more can be said about this (cf. the discussion in [Barwise and Cooper 1981]), but here our main points have just been that

(a) definition (3.35) works properly only if quantifiers are defined also on the empty universe, and

(b) stipulations as to *how* they are defined on the empty universe are not arbitrary but directly related to the question of how we judge the truth value of sentences where a determiner phrase applies to an empty noun.

3.3.3 Conservativity, Extension, and Relativization

Conservativity

That determiner denotations satisfy EXT was (we believe) first noted by van Benthem; see [van Benthem 1986], ch. 1. Earlier on it had been observed[24] that they also satisfy another condition:

> **conservativity for type $\langle 1,1\rangle$ quantifiers**
>
> A type $\langle 1,1\rangle$ quantifier Q is called *conservative* (CONSERV) if, for all M and all $A, B \subseteq M$,
>
> (3.42) $Q_M(A,B) \iff Q_M(A, A\cap B)$

Barwise and Cooper expressed this property in terms of the 'live on' property (Section 3.2.1). Seeing a determiner D as denoting (on each M) a function $||D||_M$ associating with noun denotations (sets) $A \subseteq M$ a type $\langle 1\rangle$ quantifier, their claim was that $||D||_M(A)$ always lives on A. If we define the type $\langle 1,1\rangle$ Q by

$$Q_M(A,B) \iff B \in ||D||_M(A)$$

this is precisely the claim that Q is conservative. The term 'conservativity' is from [Keenan 1981].

First Determiner Universal

Putting together the above observations we obtain the following important *determiner universal*:

(DU-1) Type $\langle 1,1\rangle$ quantifiers that interpret determiners in natural languages satisfy CONSERV and EXT.

A few remarks about this claim are in order.

First, observe that (DU-1) places real constraints on the eligible the type $\langle 1,1\rangle$ quantifiers. In principle, nothing would seem to prevent some determiner in some language to denote a quantifier meaning *some* on universes with less than 10 elements, and *at least four* on larger universes. That quantifier is CONSERV but not EXT, but it just seems that no language has such determiners. Likewise, many mathematically natural type $\langle 1,1\rangle$ quantifiers are (EXT but) not CONSERV, for example, $MO_M(A,B) \Leftrightarrow |A| > |B|$, or the Härtig quantifier $I_M(A,B) \Leftrightarrow |A| = |B|$. At first sight at least, nothing in principle would seem to exclude these as determiner denotations. But in fact, they are excluded.

Second, in the case of CONSERV, clear intuitions about logical equivalent sentences in, for example, English, support (DU-1). It seems evident that the sentences in the following pairs are equivalent — indeed, the second sentence is just a more clumsy way of expressing the first.

[24]In [Barwise and Cooper 1981], [Keenan 1981], and [Higginbotham and May 1981].

(3.43) a. At least four students smoke.

b. At least four students are students who smoke.

(3.44) a. More than half of the balls are red.

b. More than half of the balls are red balls.

(3.45) a. All but five teams made it to the finals.

b. All but five teams are teams that made it to the finals.

(3.46) a. John's two bikes were stolen.

b. John's two bikes are bikes that were stolen.

But, third, it would clearly be unsatisfactory to see (DU-1) merely as an empirical generalization based on examples such as the above.[25] Rather, (DU-1) should in turn be *explained* by the notion that one function of determiners is to restrict the domain of quantification to the denotation of the corresponding noun. Indeed, this was the idea behind CONSERV and EXT all along. And we can now see how that idea can be made precise with the help of the concept of relativization.

The Relativization Characterization

The relevant observation is expressed in the following result.

3.3.2 Proposition
Quantifiers of the form Q^{rel} satisfy CONSERV and EXT. Conversely, each CONSERV and EXT type $\langle 1,1\rangle$ quantifier is equal to Q^{rel} for some type $\langle 1\rangle$ quantifier Q.

Furthermore, the operation $\cdot^{rel}$ is a bijection between the class of type $\langle 1\rangle$ quantifiers and the class of CONSERV and EXT type $\langle 1,1\rangle$ quantifiers, a bijection which in fact is an isomorphism in that it preserves a number of features of these quantifiers, among them ISOM, monotonicity, and Boolean combinations.

Proof. The verifications of these claims are all straightforward, but in view the importance of the result we go through them in some detail. We have already observed (Fact 3.3.1) that Q^{rel} satisfies EXT. It also satisfies CONSERV, since

$$\begin{array}{lll} (Q^{\mathrm{rel}})_M(A,B) & \Longleftrightarrow & Q_A(A\cap B) \quad \text{[by definition (3.35)]} \\ & \Longleftrightarrow & (Q^{\mathrm{rel}})_M(A,A\cap B) \quad \text{[again by (3.35), since} \\ & & A\cap(A\cap B)=A\cap B] \end{array}$$

[25]Even if it is true that, in the case of CONSERV, systematic sampling supports (DU-1) (for discussion of one or two possible exceptions, see [Keenan and Stavi 1986]), and moreover, familiar constructions of complex determiners from simpler ones preserve CONSERV: [Keenan and Westerståhl 1997], pp. 854-855.

Conversely, suppose Q_1 is a type $\langle 1,1\rangle$ quantifier which is CONSERV and EXT. Then define a type $\langle 1\rangle$ quantifier Q by

$$Q_M(B) \iff (Q_1)_M(M,B)$$

for all M and all $B \subseteq M$. It follows that

$$\begin{aligned}
(Q^{\text{rel}})_M(A,B) &\iff Q_A(A\cap B)\\
&\iff (Q_1)_A(A,A\cap B) \quad \text{[by definition of } Q\text{]}\\
&\iff (Q_1)_M(A,A\cap B) \quad \text{[since } Q_1 \text{ is EXT]}\\
&\iff (Q_1)_M(A,B) \quad \text{[since } Q_1 \text{ is CONSERV]}
\end{aligned}$$

This means that $Q_1 = Q^{\text{rel}}$, and we have proved the first part of the proposition.

For the second part, we have just seen that the operation $\cdot^{\text{rel}}$ is onto (surjective), so to prove that it is a bijection we only need to establish:

$$Q_1^{\text{rel}} = Q_2^{\text{rel}} \implies Q_1 = Q_2$$

But if $Q_1^{\text{rel}} = Q_2^{\text{rel}}$ and M and $B \subseteq M$ are arbitrary, then

$$\begin{aligned}
(Q_1)_M(B) &\iff (Q_1^{\text{rel}})_M(M,B) \quad \text{[by definition (3.35)]}\\
&\iff (Q_2^{\text{rel}})_M(M,B) \quad \text{[by assumption]}\\
&\iff (Q_2)_M(B)
\end{aligned}$$

Thus, $Q_1 = Q_2$.

Next, that $\cdot^{\text{rel}}$ preserves ISOM means that

$$Q \text{ is ISOM} \iff Q^{\text{rel}} \text{ is ISOM}$$

This too is straightforward, but we come back to a fuller discussion of the role of ISOM here in Section 3.3.5 below. Likewise, the preservation of monotonicity, or more precisely that

$$Q \text{ is increasing} \iff Q^{\text{rel}} \text{ is increasing in the 2nd argument}$$

is taken up in Section **??**. Finally, preservation of Boolean operations means that the following holds:

(a) $(Q_1 \wedge Q_2)^{\text{rel}} = Q_1^{\text{rel}} \wedge Q_2^{\text{rel}}$

(b) $(Q_1 \vee Q_2)^{\text{rel}} = Q_1^{\text{rel}} \vee Q_2^{\text{rel}}$

(c) $(\neg Q)^{\text{rel}} = \neg Q^{\text{rel}}$

(d) $(Q\neg)^{\text{rel}} = Q^{\text{rel}}\neg$, and hence, $(Q^{\text{d}})^{\text{rel}} = (Q^{\text{rel}})^{\text{d}}$

We check (d); the others are similar. For any M and any $A, B \subseteq M$ we get, using (3.35) and the definition of inner negation (Section 3.1.3),

$$\begin{aligned}
(Q\neg)^{\text{rel}}_M(A,B) &\iff (Q\neg)_A(A\cap B)\\
&\iff Q_A(A-B) \quad \text{[since } A-(A\cap B) = A-B\text{]}\\
&\iff Q^{\text{rel}}_M(A,M-B) \quad \text{[since } A\cap(M-B) = A-B\text{]}\\
&\iff (Q^{\text{rel}}\neg)_M(A,B)
\end{aligned}$$

The second claim in (d) now follows from the first claim and (c), via the definition of duals. □

This proposition, then, clarifies the import of the properties of conservativity and extension as aspects of domain restriction, since the CONSERV and EXT type $\langle 1,1\rangle$ quantifiers are precisely the relativizations of the type $\langle 1\rangle$ quantifiers. It gives substance to the universal (DU-1). And it explains why, although determiner denotations are irrevocably binary relations between sets,[26] the class of type $\langle 1\rangle$ quantifiers is still fundamental for their interpretations. Proposition 3.3.2 shows how the whole class of type $\langle 1\rangle$ quantifiers is mirrored in a subclass of the type $\langle 1,1\rangle$ quantifiers, and we shall explore this correspondence on several occasions in the sequel.

One property which is *not* preserved by $\cdot^{\mathrm{rel}}$ is EXT. That Q^{rel} is EXT does not imply that Q is EXT; indeed, Q^{rel} is *always* EXT. So it is natural to ask which property among CONSERV and EXT type $\langle 1,1\rangle$ quantifiers that EXT for type $\langle 1\rangle$ quantifiers corresponds to. That question too has a simple but interesting answer, which we will come back to in Section 3.3.4.

Restricted Quantifiers Once Again

We thus see that for type $\langle 1,1\rangle$ quantifiers that interpret determiner expressions in natural languages, there is a significant difference between the first argument, or restriction, which restricts the domain of quantification, and the second argument, i.e., the scope, which comes from the verb phrase. Furthermore, if we freeze the restriction, we obtain a type $\langle 1\rangle$ quantifier which serves as the interpretation of the noun phrase consisting of the determiner and the corresponding noun. This 'freezing' is quite useful, as a mechanism for reducing (CONSERV and EXT) type $\langle 1,1\rangle$ quantifiers to type $\langle 1\rangle$ quantifiers. Formally it can be described as follows.

freezing the restriction

If Q is any type $\langle 1,1\rangle$ quantifier, and A is any set, the type $\langle 1\rangle$ quantifier Q^A is defined, for all M and all $B \subseteq M$, by

(3.47) $(Q^A)_M(B) \iff Q_M(M \cap A, B)$

Now, as the reader will recall, we already defined, in Section 3.2.1, a notion of the restriction $Q^{[A]}$ of a *type* $\langle 1\rangle$ quantifier Q, and claimed that such restricted quantifiers were appropriate interpretations of noun phrases. Fortunately, the restriction of Q to A in this sense and the freezing of the restriction argument

[26] Or n-ary relations for some $n > 1$. We have already seen that determiner expressions like `more ... than` can be seen as taking *two* restrictions, and are thus of type $\langle 1,1,1\rangle$. In view of this, we shall need to consider, in Section 3.4, how the universal (DU-1) and the considerations about domain restriction made here can be generalized to that case.

to A for the corresponding relativized quantifier amount to exactly the same thing.

3.3.3 Fact
Let Q be any type $\langle 1\rangle$ quantifier, and A any set. Then

$$(Q^{rel})^A = Q^{[A]}$$

Proof. Take any M and any $B \subseteq M$. Then

$$\begin{aligned}((Q^{\mathrm{rel}})^A)_M(B) &\iff (Q^{\mathrm{rel}})_M(M\cap A, B) \quad \text{[by definition (3.47)]}\\ &\iff Q_{M\cap A}(A\cap B) \quad \text{[by definition (3.35)]}\\ &\iff (Q^{[A]})_M(B) \quad \text{[by definition (3.12)]}\end{aligned}$$

□

The idea of freezing is familiar from the literature, but there is actually a small twist as to how it should be defined, which is related to the global vs. local perspective issue (Section 3.1.1). We end this section by commenting on this; these remarks can be skipped if one is not interested in such methodological fine tuning.

Methodological digression. Often, freezing the restriction of a type $\langle 1,1\rangle$ quantifier Q is defined a bit 'sloppily' by considering only the case when the universe M is given and the restriction set A is a subset of M. That is, one defines a *local* quantifier Q^A on M by

$$Q^A(B) \iff Q_M(A,B) \text{ for all } B \subseteq M$$

and ignores how this could be turned into a global definition.[27] For most applications, and in particular for the important application of freezing in the definition of *iteration* of quantifiers (Section ZZZZ), the 'sloppy' version is sufficient since only the case when $A \subseteq M$ is used.

Since we prefer global definitions whenever possible, we used (3.47) above instead, which works for any A but of course reduces to the 'sloppy' definition when $A \subseteq M$. But another global version is possible as well. One might decide to include $A \subseteq M$ in the defining condition, either by reading

$$(Q^A)_M(B) \iff Q_M(A,B)$$

in this way, so that $(Q^A)_M(B)$ becomes *false* whenever $A \not\subseteq M$,[28] or by making this explicit, and thus defining, given Q and A,

$$(Q^A)_M(B) \iff A \subseteq M \text{ and } Q_M(A,B), \text{ for all } B \subseteq M$$

[27] This is done in [Keenan and Westerståhl 1997], [Westerståhl 1996], and essentially also in [Barwise and Cooper 1981].
[28] This route is followed in [Westerståhl 1994].

Again, there is no difference when $A \subseteq M$. But in the general case there is small difference. Note that exactly the same choice turns up for the definition of $Q^{[A]}$ (for a type $\langle 1\rangle$ Q): we used (3.12), i.e.,

$$(Q^{[A]})_M(B) \iff Q_{M\cap A}(A \cap B)$$

but could instead have chosen

$$(Q^{[A]})_M(B) \iff A \subseteq M \text{ and } Q_A(A \cap B)$$

It is easily seen that if we had taken the second alternative in *both* cases, then Fact 3.3.3 would still hold. So is there a reason for our choice of the first alternative here?

It is perhaps not a big deal, but we have two (small) arguments. First, we find it more satisfactory that restriction to a noun denotation A (say, the set of dogs) works even when you are in a smaller universe of discourse M (say, the set of things in my house): so you restrict to $M \cap A$ (the set of dogs in my house). Second, there are results which hold for our version of the definition but fail for the other case. For example, we noted in Section 3.2.3 that EXT is preserved when going from Q to $Q^{[A]}$; this is no longer true if '$A \subseteq M$' is included in the defining condition (since if M is extended to M', it can be the case that $A \not\subseteq M$ but $A \subseteq M'$). *End of digression.*

3.3.4 Symmetry

It is a notable fact that some determiners in natural languages (and their corresponding denotations) are symmetric and others are not. The (a) and (b) sentences in the following examples imply each other, but the (c) and (d) sentences do not.

(3.48) a. Some lawyers are crooks.

b. Some crooks are lawyers.

c. All lawyers are crooks.

d. All crooks are lawyers.

(3.49) a. No lawyers are crooks.

b. No crooks are lawyers.

c. Most lawyers are crooks.

d. Most crooks are lawyers.

(3.50) a. Several lawyers are crooks.

b. Several crooks are lawyers.

c. Every lawyer is a crook.

d. Every crook is a lawyer.

(3.51) a. Three lawyers are crooks.

b. Three crooks are lawyers.

c. All but two lawyers are crooks.

d. All but two crooks are lawyers.

(3.52) a. More than six lawyers are crooks.

b. More than six crooks are lawyers.

c. Two thirds of the lawyers are crooks.

d. Two thirds of the crooks are lawyers.

(3.53) a. At most four lawyers are crooks.

b. At most four crooks are lawyers.

c. The lawyers were crooks.

d. The crooks were lawyers.

(3.54) a. A woman was a witness.

b. A witness was a woman.

c. The woman was a witness.

d. The witness was a woman.

Thus, `no`, `some`, `three` and other numerals, `more than six`, `at most four` and other modified numerals, are symmetric, but `all`, `every`, `all but two`, and `most` and other proportionals, are not. Aristotle devoted a good part of his syllogistics to codifying the different inference characteristics of these two classes of quantifiers.

There are some initially puzzling challenges to symmetry, like the intuitive contrast between (3.55a) and (3.55b) below:

(3.55) a. Some men are bachelors.

b. Some bachelors are men.

(3.55a) seems informative whereas (3.55b) is trivial. But this puzzle can be satisfactorily resolved with a pragmatic theory about the point of utterances, such as Grice's theory of conversational implicature. (3.55a) normally carries the implicature (3.56a), and (3.55a) would carry (3.56b).

(3.56) a. Not all men are bachelors.

b. Not all bachelors are men.

The self-contradictory nature of (3.56b) explains the oddness of (3.55b). Conversational implicature is also pertinent to the feeling that symmetry pairs like those above are not fully synonymous. For example, (3.48a) has the implicature (3.57a), and (3.48b) has the non-equivalent implicature (3.57b).

(3.57) a. Not all lawyers are crooks.

b. Not all crooks are lawyers.

A similar observation is the following. Determiners have, as we have noted, the property of restricting the domain of quantification to what is accordingly called the restriction argument. This is a semantic property, codified in CONSERV and EXT. But it is natural to assume that there are also pragmatic aspects of domain restriction, such as making the extension of the corresponding noun the topic at that point of an utterance. If so, the respective sentences in a symmetry pair have different topics, and are already for this reason not fully synonymous.

In any case, aside from these pragmatic aspects, there is an obvious semantic property of generalized quantifiers denoted by symmetric determiners, namely, the symmetry, in the usual sense, of the corresponding second-order relations.

symmetry

A type $\langle 1,1 \rangle$ quantifier Q is *symmetric* (SYMM) if, for all M and all $A, B \subseteq M$,

(3.58) $Q_M(A,B)$ implies $Q_M(B,A)$

Clearly the quantifiers corresponding to the symmetry pairs above are all symmetric in this sense, whereas those in the other pairs are not symmetric. But looking at these examples, we can see that there seems to be another way of expressing the characteristic semantic properties of the quantifiers in the symmetry pairs. This is the property that only the intersection of the restriction and the scope matters for whether the sentence is true or not. For example, the only thing that matters for whether `Three lawyers are crooks` is true or not is the number of lawyers that are crooks. The number of honest lawyers, say, has no influence at all. This property (in this case of the quantifier *three*) has been called *intersectivity*. We give a precise definition, and then state exactly how it relates to symmetry.

intersectivity

A type $\langle 1, 1\rangle$ quantifier Q is *intersective* (INT) if the truth value of $Q_M(A, B)$ only depends on $A \cap B$. More precisely, for all M and all $A, B, A', B' \subseteq M$,

(3.59) If $A \cap B = A' \cap B'$, then $Q_M(A, B) \Leftrightarrow Q_M(A', B')$.

A different way of expressing the same property is the following: For all M and all $A, B \subseteq M$,

(3.60) $Q_M(A, B) \iff Q_M(A \cap B, A \cap B)$.

(3.59) implies (3.60), since if we take $A' = B' = A \cap B$ in (3.59), $A \cap B = A' \cap B'$, and so (3.60) follows. Conversely, if $A \cap B = A' \cap B'$ and (3.60) holds, two applications of this property gives the conclusion of (3.59). An advantage of expressing intersectivity as in (3.59) is, as we will see in Section 3.4, that it generalizes to determiner denotations of other types than $\langle 1, 1\rangle$.

It is also clear that intersectivity implies symmetry: in order to conclude $Q_M(B, A)$ from $Q_M(A, B)$, we let $A' = B$ and $B' = A$ in (3.59). The converse implication does not hold in general, but it does hold if Q is conservative:

3.3.4 Fact
Under CONSERV, SYMM and INT are equivalent.

Proof. It remains to verify that symmetry implies intersectivity under CONSERV. Suppose $A, B \subseteq M$. Then

$$\begin{array}{rcll} Q_M(A, B) & \iff & Q_M(A, A \cap B) & \text{[by CONSERV]} \\ & \iff & Q_M(A \cap B, A) & \text{[by SYMM]} \\ & \iff & Q_M(A \cap B, A \cap B) & \text{[by CONSERV]} \end{array}$$

Thus, (3.60) holds. □

We have claimed that determiner denotations are both CONSERV and EXT, and so the intuition that the quantifiers in symmetry pairs are both symmetric and intersective is justified. Note that under EXT, (3.60) can be written

(3.61) $Q_M(A, B) \iff Q_{A \cap B}(A \cap B, A \cap B)$.

The fact that determiner denotations are CONSERV and EXT is explained by their being relativizations of type $\langle 1\rangle$ quantifiers (Proposition 3.3.2). So what is the property of type $\langle 1\rangle$ quantifiers that corresponds to symmetry of their relativizations? The simple answer is given in the following

3.3.5 Proposition
Q^{rel} is symmetric if and only if Q satisfies EXT (for type $\langle 1\rangle$ quantifiers).

Proof. Suppose first that Q^{rel} is symmetric. Let A be a subset of both M and M'. Then

$$\begin{array}{lcl} Q_M(A) & \Longleftrightarrow & Q^{\mathrm{rel}}_M(M,A) \quad \text{[by (3.35)]} \\ & \Longleftrightarrow & Q^{\mathrm{rel}}_A(A,A) \quad \text{[by (3.61)]} \\ & \Longleftrightarrow & Q^{\mathrm{rel}}_{M'}(M',A) \quad \text{[by (3.61)]} \\ & \Longleftrightarrow & Q_{M'}(A) \quad \text{[by (3.35)]} \end{array}$$

Thus, Q satisfies EXT. Conversely, suppose that Q satisfies EXT, and that $A, B \subseteq M$. Then

$$\begin{array}{lcl} Q^{\mathrm{rel}}_M(A,B) & \Longleftrightarrow & Q_A(A\cap B) \quad \text{[by (3.35)]} \\ & \Longleftrightarrow & Q_{A\cap B}(A\cap B) \quad \text{[by EXT]} \\ & \Longleftrightarrow & Q^{\mathrm{rel}}_M(A\cap B, A\cap B) \quad \text{[by (3.35)]} \end{array}$$

So (3.60) holds for Q^{rel}, and hence it is symmetric. □

So we see that there is a substantial difference between EXT for type $\langle 1\rangle$ quantifiers and EXT for type $\langle 1,1\rangle$ quantifiers, in a natural language context — cf. the discussion at the end of Section 3.2.3. The latter property holds of all determiner denotations. But the EXT type $\langle 1\rangle$ quantifiers single out a particular class of determiner denotations, namely, the symmetric ones.

...

If closure under isomorphism (ISOM) holds, we can further characterize symmetry. Keenan calls a type $\langle 1,1\rangle$ quantifier Q *cardinal* (CARD) if the following holds, for all M and all $A, B \subseteq M$:

(3.62) If $|A\cap B| = |A'\cap B'|$, then $Q_M(A,B) \Leftrightarrow Q_M(A',B')$.

This is a stronger property than intersectivity (since the antecedent of (3.59) implies the antecendent of (3.62)), i.e., CARD implies INT. The precise relation between the two notions is given in the following.

3.3.6 Fact

For any type $\langle 1,1\rangle$ quantifier Q, CARD is equivalent to ISOM + INT, on finite universes. Hence, under CONSERV, it is equivalent to ISOM + SYMM on finite universes. Under CONSERV and EXT, it is equivalent to ISOM + SYMM on arbitrary universes.

Proof. Suppose Q satisfies CARD. We already saw that INT follows. Also, ISOM for type $\langle 1\rangle$ quantifiers in the form (3.14) in Section 3.2.2 generalizes for type $\langle 1,1\rangle$ quantifiers to a similar condition on the cardinalities of the four sets $A\cap B$, $A - B$, $B - A$, and $M - (A\cup B)$. (This is spelled out in Section 3.3.5 below.) But CARD says that the desired conclusion follows already from just one part of this condition, namely the one on $A\cap B$, and so *a fortiori* ISOM holds.

This implication is true whether the universe is finite or not. Now suppose, conversely, that Q satisfies ISOM and INT, that $A, B, A', B' \subseteq M$ where M is finite, and that $|A \cap B| = |A' \cap B'|$. It follows that $|M - (A \cap B)| = |M - (A' \cap B')|$ (but this is not necessarily true if M is infinite). Then

$$\begin{array}{lll} Q_M(A,B) & \Longleftrightarrow & Q_M(A \cap B, A \cap B) \quad \text{[by (3.60)]} \\ & \Longleftrightarrow & Q_M(A' \cap B', A' \cap B') \quad \text{[by ISOM]} \\ & \Longleftrightarrow & Q_M(A', B') \quad \text{[by (3.60)]} \end{array}$$

Thus CARD holds.

The claim about what happens under CONSERV follows immediately from this and Fact 3.3.4. And if EXT also holds, the cardinalities of $M - (A \cap B)$ and $M - (A' \cap B')$ are just irrelevant, so the above argument to infer CARD goes through using only the assumption that $|A \cap B| = |A' \cap B'|$. □

We observed in Section 3.2.3, Fact 3.2.5, that under ISOM, a type $\langle 1 \rangle$ quantifier satisfying EXT is characterized simply by a class of cardinal numbers. Combining this with the fact that CONSERV and EXT quantifiers are relativizations of type $\langle 1 \rangle$ quantifiers, and with Proposition 3.3.5, we obtain a final characterization of symmetric quantifiers:

3.3.7 Corollary
Under ISOM, a CONSERV and EXT type $\langle 1, 1 \rangle$ quantifier Q is symmetric if and only if there is a class S of cardinal numbers such that for all M and all $A, B \subseteq M$,

$$Q_M(A,B) \iff |A \cap B| \in S$$

On the Symmetry of *many* and *few*

It has been observed that (3.63a) intuitively seems not to entail (3.63b), and (3.64a) not to entail (3.64b).

(3.63) a. Many plutocrats are Americans.

b. Many Americans are plutocrats.

(3.64) a. Few scoundrels are Nobel Laureates.

b. Few Nobel Laureates are scoundrels.

This time it is not a matter of total synonymy or pragmatic considerations, since the very truth conditions appear to differ. One might conclude simply that *many* and *few* are not symmetric. However, the issue merits a bit more attention.

We noted (towards the end of Section 3.3.1) that the interpretation of the determiners `many` and `few` as quantifiers is strongly context-dependent, indeed so much that some authors prefer not to treat them as quantifiers at all. Let us however try to see what such an interpretation would have to amount to. For simplicity, we mostly restrict attention to `many`, but similar remarks hold for `few`.

All uses of `many` bound the cardinality of the intersection of the quantifier's restriction and scope. `Many` A`'s are` B says that $A \cap B$ is 'big', i.e., that

$$|A \cap B| > n$$

for some number n (smaller than the size of the universe). But unlike *more than a hundred* and *fewer than ten*, which specify their respective (lower and upper) bounds quite precisely, and even proportional quantifiers like *at least two thirds of the* and *most*, `many` and `few` permit quite a broad variety of factors to influence their bounds. They sometimes allow the bound to be set with reference to the size of the restriction (as with proportional quantifiers); this appears to be the case for the sentences in (3.63) and (3.64). But in (3.65) and (3.66) it seems to be the size of the scope that matters.

(3.65) Many Scandinavians have won the Nobel Prize.

(3.66) Few cooks applied.

For example, (3.66) could say on a given occasion that the number of cooks that applied is smaller than one fifth of the total number of applicants, but hardly that it is smaller than one fifth of the total number of cooks.

For an extensional treatment of `many`, there appear to be two possibilities. One could interpret it as saying that the size of the intersection of the restriction and the scope is greater than some fixed number, given by context for each use of the word. That makes the quantifier satisfy conservativity, extension, and symmetry. But such a perhaps extremely context-dependent treatment misses the uniformities of interpretation that after all seem to hold in cases like (3.63) – (3.66). An alternative treatment, then, would be to think of `many` as ambiguous between a proportional reading

$$|A \cap B| > k \cdot |A|$$

and an 'anti-proportional' reading

$$|A \cap B| > k \cdot |B|$$

and perhaps even other similar readings where further sets are used to determine the relevant sort of comparison. (The number k also has to be provided by context, but this is no different than with *most*, where the 'threshold' can vary with context.) Note, however, that whereas the proportional reading is conservative but not symmetric, the 'anti-proportional' one loses conservativity too, thus threatening our first determiner universal (DU-1) (Section 3.3.3). This might be considered a higher methodological price than one can afford.

The question of the correct treatment of `many` is both theoretical and empirical.

...

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TBW

3.3.5 ISOM and the Number Tree

3.4 Other Monadic Types

3.5 Some Polyadic Quantifiers

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