Emerging Quadrature Lattices of Kerr Combs

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Abstract
A quadrature lattice is a coupled array of squeezed vacuum field quadratures that offers new avenues in shaping the quantum properties of multimode light [1–3]. Such lattices are described within the framework of non-Hermitian, non-dissipative physics and exhibit intriguing lattice phenomena such as lattice exceptional points, edge-states, entanglement and non-Hermitian skin effect, offering fundamentally new methods for controlling quantum fluctuations [1, 4].

Nonlinear resonators are suitable for studying multimode pair-generation processes and squeezing which are non-dissipative in $\chi^{(2)}$ and $\chi^{(3)}$ materials [5–12], but observing non-Hermitian lattice phenomena in photonic quadrature lattices was not achieved. Remarkably, in dissipative Kerr microcombs [13], which have revolutionized photonic technology, such lattices emerge and govern the quantum noise that leads to comb formation. Thus, they offer a unique opportunity to realize quadrature lattices, and to study and manipulate multimode quantum noise which is essential for any quantum technology. Here, we experimentally study non-Hermitian lattice effects in photonic quadrature lattices for the first time. Our photonic quadrature lattices emerge at Kerr microcomb transitions, allowing us to observe fundamental connections between dispersion symmetry, frequency-dependent squeezed supermodes, and non-Hermitian lattice physics in an integrated setup. Our work unifies two major fields, quantum non-Hermitian physics and Kerr combs, and opens the door to utilizing dissipative Kerr combs to experimentally explore rich non-Hermitian physics in the quantum regime, engineer quantum light, and develop new tools to study the quantum noise and formation of Kerr combs.
1 Introduction

In recent years, there has been growing interest in photonic quadrature lattices and their quantum properties [1]. Unlike conventional photonic lattices, where the lattice sites are coherent field amplitudes, in photonic quadrature lattices, each discrete site consists of two optical quadratures that are coupled to other quadratures. Since the quadrature representation forms the natural basis for continuous variable (CV) quantum states, photonic quadrature lattices enable the understanding of rich multimode systems from the perspective of squeezing and deterministic entanglement generation. Notably, Hermitian quadratic bosonic lattices translate to non-Hermitian lattices in quadrature space. Thus, the rich physics of non-Hermitian lattices, which includes a plethora of effects such as optical exceptional points [15], the non-Hermitian skin effect [16], and beyond, manifests in quadrature space with novel implications for squeezing and entanglement [2, 4, 16, 17].

In contrast to traditional non-Hermitian optical lattices, where the interplay between gain and loss is governed by photon absorption, quadrature lattice dynamics are governed by parametric processes (i.e., energy-conserving) such as nonlinear optical pair creation and annihilation. Thus, the realization of a bosonic quadrature lattice requires both multimode parametric gain and loss, and the coupling or transfer of bosons between lattice sites. In recent experiments, non-Hermitian lattice effects in quadrature lattices were observed in optomechanics [2, 3, 18] by coupling several parametrically pumped opto-mechanical resonator modes together. In photonics, a fruitful approach for observing non-Hermitian lattice physics is along the synthetic frequency dimension of a single optical cavity, in which the optical modes are coupled by modulation at the free spectral range (FSR) frequency [19–21]. Such lattices can also be realized in a nonlinear environment [22], and even in the non-dissipative quantum regime by coupling photons generated across multiple frequency modes by second-order pair production with electro-optic modulation [8]. However, observing quadrature lattices requires detecting the multimode quadrature states with a local oscillator [10, 11, 23, 24], and probing the non-Hermitian lattice effects, which has not been achieved so far.

Remarkably, Kerr combs [13] (including dissipative Kerr solitons [25] and soliton crystals [26]) generated by third-order nonlinear resonators, naturally provide both pair production [27, 28] and coupling between the frequency modes through the process of Bragg scattering [5, 29–31]. These properties make Kerr combs a unique system to study multimode quantum phenomena [7, 32] and squeezing [33]. Interestingly, these Kerr combs also function as a multi-frequency pump (in contrast to a single pump [9]) which produces transport and amplification of the quantum fluctuations in the resonator, giving rise to “below-threshold” phase-locked combs [7]. The complexity of these squeezed combs, and their quadratic bosonic Hamiltonian, which includes parametric gain and coupling terms, has only recently begun to be treated theoretically [33, 34]. However, to date, such below-threshold multimode quadratures have not been measured directly, and it was not clear how these combs are related to quadrature lattices and the non-Hermitian lattice phenomena that govern them. Additionally the major development of dissipative Kerr microcombs, which has become the focal point of vast research efforts, offers new possibilities in shaping quantum light, but also pose
new limitations. Thus, studying the multimode quantum squeezing and anti-squeezing generated by them is essential for both utilizing their quantum properties and gaining a better understanding of the quantum origin of their formation and their quantum noise limitations.

In this work, we experimentally study non-Hermitian lattice physics in photonic quadratures for the first time. Our quadrature lattice is formed by a tunable dissipative Kerr comb that couples different quadratures along the frequency dimension. We show theoretically and experimentally the relationship between dispersion symmetries and the squeezing properties of the multi-frequency lattice states of the below-threshold comb. Our work explores Kerr combs exhibiting multimode squeezed states with robust single peak frequency structure and double peak structure, thereby revealing fundamental properties related to the quantum nature of Kerr comb formation and quantum noise. We then observe such multimode states experimentally by measuring the anti-squeezing and photon statistics of the below-threshold Kerr comb. Our work demonstrates the relationship between non-Hermitian lattice physics and squeezed Kerr combs. To this end, we use the framework of non-Hermitian lattice physics and draw parallels with seminal classical non-Hermitian observations such as power oscillations caused by parity-time symmetry (PT symmetry) [15], and PT symmetric lattices [35–37]. This advancement will enable the generation of novel quantum optical resources on-chip, manipulation of quantum noise, and enhance the understanding of the quantum origin of Kerr comb formation.

2 Results

We consider a 2-FSR Kerr comb where every other cavity mode (every even-numbered cavity mode) is predominantly populated by coherent light, and light is pair-generated by the Kerr comb on all the other cavity modes (Fig. 1(a)). The loss in the odd cavity modes is greater than the parametric gain, such that these modes are populated by below-threshold light, i.e., a squeezed vacuum state. Each odd cavity mode is also directly coupled to neighboring odd modes (but not to the even modes) by Bragg scattering which originates predominantly from Kerr comb teeth separated in frequency by 2-FSR, creating a 1D lattice of frequency sites populated by vacuum states of light.

Fig. 1  Squeezed quadrature lattices induced by Kerr combs. a. A 2-FSR Kerr comb (red) and a generated sub-threshold multimode state with squeezing frequency structure (green). b. General scheme of the system: A Kerr ring resonator is pumped by CW light, generating a 2-FSR bright comb (red circles). This comb generates a sub-threshold comb (green circles), which has the connectivity of a 1D lattice by pair generation and Bragg scattering. c. A unit cell of the 1D lattice in the bosonic creation and annihilation operator basis, longer range couplings decay with frequency distance.
Such a scenario can be evoked either by injecting a comb, as depicted in Fig. 1a into a Kerr resonator, or by pumping the resonator with continuous-wave (CW) light, causing the other comb teeth to emerge spontaneously via the symmetry-breaking phenomenon of Turing rolls \cite{38} (Fig. 1(b)). The dynamics of the below-threshold light at the odd modes (centered around the pump at \( \mu = 0 \)) is given by the following Hamiltonian\cite{7}

\[
\hat{H} = \sum_{\mu} -\Delta \omega_\mu \hat{a}^\dagger_\mu \hat{a}_\mu - \frac{g_0}{2} \sum_{\mu,\nu,j,k} \delta [\mu + \nu - j - k] \left( A_\mu A_\nu \hat{a}^\dagger_\mu \hat{a}^\dagger_\nu \hat{a}_j \hat{a}_k + 2 A^\dagger_\nu A_\mu \hat{a}^\dagger_j \hat{a}_\mu + \text{h.c} \right)
\]

where \( A_\mu \) are above-threshold amplitudes (the Kerr comb) in even cavity modes \( \mu \), \( \hat{a}^\dagger_\mu \) (\( \hat{a}_\mu \)) are the creation (annihilation) operators of quantum light in odd cavity mode \( \mu \), which are described in the rotating frame of reference of the Kerr comb \( U = \exp(i \hat{R} t) \), \( \hat{R} = \sum_\mu (\omega_\mu + \Delta \Omega_\mu) \hat{a}^\dagger_\mu \hat{a}_\mu \) where \( \Delta \Omega \) is the frequency spacing of the Kerr-comb, \( \Delta \omega_\mu \) is the frequency detuning of the cavity-mode \( \mu \) from the Kerr comb frame of reference, \( g_0 \) is the nonlinear coefficient, the Kronecker \( \delta \) reflects photon azimuthal wave number conservation in the system, and \( \text{h.c} \) stands for Hermitian conjugate.

To illustrate the quadrature lattice geometry, we plot the frequency axis in a “folded” form in which the pump is located at the edge (Fig. 1(b-c)). Thus a unit-cell in this lattice consists of four bosonic field operators \( a_\mu, a^\dagger_\mu, a_{-\mu}, a^\dagger_{-\mu} \), which map to four field quadratures (see Methods 4.1 for the derivation of the quadrature lattice model). The simplified quadratic bosonic lattice model can be used both to understand the fundamental sub-threshold comb behavior and to engineer it. The sub-threshold quadrature dynamics of the Kerr comb constitutes a non-Hermitian 1D lattice \cite{35–37, 39}, which can be either APT symmetric (pseudo Anti-PT symmetric \cite{16, 40}) or non-symmetric around the central point \( \mu = 0 \). In the APT symmetric case all the eigenvalues will be purely real or purely imaginary, while in the non-symmetric case they are complex. In a realistic Kerr micro-comb the APT symmetry is mainly eliminated by a mismatch between the Kerr comb frequency spacing (\( \Delta \Omega \)) and the FSR of the cavity. To understand the impact of the APT symmetry on the quadrature lattice we reduce the number of parameters by writing the Hamiltonian for the odd cavity modes in a unitless form. Only two parameters need to be considered for the minimal model: \( \alpha \) which defines simplified comb amplitudes by:

\[
A_\mu / A_0 = 10^{-R |\mu|/10} \equiv \alpha, R > 0, \text{ and } \Delta \tilde{\omega}_\mu \text{ which is the unitless frequency detuning of the cavity modes from a rotating frame of reference aligned with the Kerr comb.}
\]

To the first order of \( \alpha \) we obtain the following form for the lattice Hamiltonian:

\[
\hat{H}_{\text{lat}} = -g_0 A_0^2 \sum_{\mu \in \text{odd, } \mu > 0} \left[ \frac{1}{2} (\Delta \tilde{\omega}_\mu + 2) \hat{c}^\dagger_\mu \hat{c}_\mu + 2 \alpha \hat{d}^\dagger_\mu \hat{d}_\mu \right] + \left[ \frac{1}{2} (\Delta \tilde{\omega}_{-\mu} + 2) \hat{d}^\dagger_{-\mu} \hat{d}_{-\mu} + 2 \alpha \hat{d}^\dagger_\mu \left( \hat{d}^\dagger_{-\mu-2} + \hat{d}^\dagger_{2+\mu} \right) \right] + \text{h.c} + \mathcal{O} (\alpha^2)
\]
where \( c_\mu \equiv a_\mu, d_\mu \equiv a_{-\mu} \) for \( \mu > 0 \), and \( d_{-1} \equiv c_1, c_{-1} \equiv d_1 \). We note that the higher order terms \( O(\alpha^n) \) add Bragg coupling and pair generation terms to \( n \)-nearest neighbours thereby not breaking the lattice structure. In this “folded” representation it is possible to show that if the detuning \( \Delta \tilde{\omega}_\mu \) is symmetric around \( \mu = 0 \), the bosonic quadratic lattice in Eq. 2 becomes a degenerate pseudo APT symmetric lattice which results in a eigenspectrum with eigenvalues that are purely real or purely imaginary (Methods 4.1).

To describe realistic squeezing behavior, the bosonic quadrature lattice of Eq. 2 must be finite, which naturally occurs due to group velocity dispersion (GVD) reducing the Bragg coupling coefficient for modes away from the center. If we define \( \Delta \tilde{\omega}_\mu = \tilde{D}'_1 \mu + \tilde{D}_2 \mu^2 / 2 + O(\mu^3) \), then \( \Delta \tilde{\omega}_\mu \) is symmetric if all the odd terms are zero. Note that \( \tilde{D}'_1 (\tilde{D}'_1) \) is the (unitless) frequency difference between the cavity FSR and the Kerr comb’s frequency spacing. To illustrate the implication of different \( \tilde{D}'_1 \) values, we calculate the quadrature eigenspectrum of the sub-threshold light generated by a 2-FSR Kerr comb with fixed \( A_\mu \) profile and varying \( \tilde{D}'_1 \). For concreteness, we use \( \mathcal{R} \) and \( D_2 \) taken from our experimental system (see Methods 4.2). Figure 2(a) shows the symmetric case in which \( \tilde{D}'_1 = 0 \), while Figs. 2b(c) are non-symmetric with \( \tilde{D}'_1/2\pi = 3(8) \text{MHz} \). We adjust the overall comb power in the plots in Fig. 2 to be close to threshold for detuning \( \Delta \omega_0 \) = \( 2g \sum |A_\mu|^2 \). To compute the quadrature eigenspectrum we write Eq. 2 in the quadrature basis \((p_1, q_1, ... p_N, q_N)\) where \( p_i \) and \( q_i \) are the quadratures of mode \( i \), and \( N \) is the number of participating cavity modes. Since the transformation to quadratures is not a unitary operation, we obtain a non-Hermitian Hamiltonian \( \mathcal{M} \). To study threshold behavior we introduce \( \Gamma = \kappa / 2I_{2N} \times 2N \), where \( \kappa \) is the uniform intensity loss rate.

In Fig. 2 we analyze the eigenspectrum of \( \mathcal{M} - \Gamma \) while varying the parameter \( \tilde{D}'_1 \). We observe three distinct regimes. For \( \tilde{D}'_1 = 0 \) (Fig. 2(a)), the eigenvalues are either purely real or have a real part that equals \( -\kappa / 2 \) resulting from their initial purely real eigenvalue shifted uniformly by \( \kappa / 2 \). Interestingly, due to the multimode nature of the system, even when the symmetry is removed (i.e. when \( \tilde{D}'_1 \neq 0 \)) some states close to threshold remain with imaginary values of 0 (i.e the partially symmetric case, see Fig. 2(b)). This gradual decrease in the number of purely real eigenvalues or robustness is analyzed in Methods 4.2. For the highest \( \tilde{D}'_1 \) in Fig. 2(c), the modes close to threshold all have a double peak structure (i.e., imaginary part is non zero).

Each eigenvalue in Fig. 2(a-c) represents a multimode squeezed state, residing on the odd modes. As depicted in green in Fig. 1(a) the squeezing spectrum of each supermode uniformly evolves around each cavity mode [33]. The spectral structure around each cavity mode is frequency dependent squeezing, a concept which recently started to be utilized in metrology application through noise spectrum engineering [41].

To compute the multimode frequency-dependent squeezing structure, we calculate the supermodes of \( \mathcal{M} - \Gamma \), using the Bloch-Messiah decomposition (details in Methods 4.2). The structure of the supermodes with the highest real part (gain) in Fig. 2(a-c) is presented in Fig. 2(d-g). Supermodes with the highest gain will dominate in total photon number over other supermodes and have the narrowest peaks,
being closest to the optical parametric oscillator (OPO) threshold. The OPO threshold occurs when $\Re(\lambda) = 0$, where $\lambda$ denotes the eigenvalue of the supermode with the highest real part. The frequency spacing in the double peak structure is determined by the imaginary value of the corresponding supermode. Therefore, supermodes with imaginary value zero, exhibit a single peak structure (“un-detuned” from the frame of reference of the Kerr comb), while supermodes with imaginary value that is not zero exhibit double-peak spectrum (“detuned” from the frame of reference of the Kerr comb).

The partially symmetric case exhibits both double-peak and single peak evolution for different supermodes with high gain simultaneously, with frequency spacing between the peaks which is generally smaller than in the completely non-symmetric case. The different plotted lines in each spectral shape in Fig. 2(d-g) represent the squeezing for different distances from threshold (by changing the overall power of the Kerr comb). Together with an intra-modal frequency-dependent structure, each supermode has a spectral structure in the cavity mode basis, shown for the partially symmetric case in Fig. 2(h-i). The bandwidth of these supermodes is predominantly determined by the values of $D'_1$ and $D_2$. 

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**Fig. 2** Dispersion symmetry and the properties of supermode squeezed states. **a.-c.** Eigenvalues of the multimode states of a 2-FSR comb with different values of $D'_1$: 0, 3, and 8 MHz respectively. For larger $D'_1$ the purely real eigenvalues gradually become complex. Highlighted quadrature supermodes are those close to threshold. **d.-g.** The frequency($\omega$)-dependent squeezing spectrum of the supermodes close to threshold (highlighted). Each line represents the supermode spectrum for a linearly swept intra cavity power, and the supermodes can be seen to narrow as they approach threshold. The supermode in f. does not narrow asymptotically because c. reaches threshold first. **h.,i.** The supermode composition as a function of cavity modes of the multimode squeezed states, for the supermodes in e. and f. respectively.
Next, we aim to experimentally observe the frequency-dependent quadrature variance directly for the different cases described above using integrated microring resonators [42]. In this work, we utilize 4H-silicon carbide-on-insulator [43], which offers low losses and high third-order nonlinearity, enabling low-power operation. We excite a localized 2-FSR Kerr comb by injecting CW light near an avoided mode crossing, generating strong local anomalous dispersion. The comb is naturally phase-locked via cascaded parametric processes and centered within the telecommunications c-band (1545 nm). The characterization of the device parameters, including dispersion, is presented in Methods 4.3 and Methods 4.5).

We begin by characterizing the modal structure of the generated 2-FSR comb in its native basis. Using a custom-built single-photon optical spectrum analyzer (SPOSA)[7], which provides > 100 dB of dynamic range with single-photon sensitivity, we simultaneously capture the photon populations above- and below-threshold, as shown in Fig. 3(a). Mapping out the two-photon correlation matrix of the below-threshold state can reveal the inter-modal connectivity in the Hamiltonian, and provide initial indications of the formation of the supermodes introduced in Fig. 1(b,c). We perform pairwise second-order photon correlation measurements, $g^{(2)}_{ij}(\tau) = \langle \hat{a}_i^\dagger(0)\hat{a}_j^\dagger(\tau)\hat{a}_i(\tau)\hat{a}_j(0) \rangle / \langle \hat{a}_i^\dagger(0)\hat{a}_i(0) \rangle \langle \hat{a}_j^\dagger(\tau)\hat{a}_j(\tau) \rangle$ on each pair of below-threshold modes in the comb, and the resulting matrix is shown in Fig. 3(b). The signature of supermodes is seen as pair-wise correlations between modes close in their distance from the center pump mode (consistent with the folded picture in Fig. 1(b)).

We proceed to demonstrate the transition between the non-symmetric regime and the symmetric regime through continuous tuning of the resonator dispersion properties. Gradually heating the microring changes the mode splitting, and therefore the threshold point, allowing us to vary $D_1'$ (see Methods 4.3). Tuning our below-threshold comb above threshold, we observe through two-tone homodyne detection that adjusting the voltage of the heater changes the secondary comb frequency structure above threshold discontinuously (Fig. 3(c) and Methods 4.4). Figure 3(d-g) shows the above-threshold peaks for different applied voltages. We observe three distinct voltage regimes: In the first, the threshold presents a double peak structure with the peak separation of approximately 300 MHz (Fig. 3(d)), significantly larger than the cavity linewidth of 100 MHz, thus corresponding to the strongly non-symmetric regime of Fig. 2(c,g). Further tuning the voltage, we transition discontinuously to a regime in which the detuning is small, around 30 MHz, which corresponds to the partially symmetric regime of Fig. 2(b,e,f) followed by a transition to a single peak (corresponding to the symmetric case threshold crossing of Fig. 2(a,d)). The discontinuous change of the frequency spectrum with the voltage occurs because each regime in Fig. 2 corresponds to a different supermode crossing threshold, which is not a continuous process. Afterward, the system abruptly transitions to the partially-symmetric state again (Fig. 3(g)). This matches completely to the regimes described in Fig. 2.

The above-threshold light shown in Fig. 3(d-g) presents only a partial picture of the eigenspectrum evolution, since it shows the single mode that reaches threshold first, without the below-threshold quadrature variance structure. To observe the competing multimode processes in the quantum regime, we also measure the below-threshold
quadrature vacuum fluctuations (anti-squeezing), whose evolution precedes the formation of the above-threshold light shown in Fig. 3(d-g). To observe the below-threshold light, we reduce shot noise using on-chip and off-chip filters (Methods 4.3). Furthermore, we tune the power ratio of our two-tone LO to a ratio of $LO_1 = 10 \cdot LO_{-1}$, which we found to maximize the overlap with the amplified quadrature supermode (Fig. 4(a)). Figure 4(b) shows our 2-FSR Kerr comb at the soliton crystal stage, with symmetry corresponding to the partially symmetric case (Fig. 2(b,e,f)). Below threshold, we observe both the single peak and double peak modes of Fig. 2(b,e,f) simultaneously. By further red-tuning the input pump frequency, we observe that while the detuned supermode is amplified and narrows further until it crosses threshold, the non-detuned supermode does not (Fig. 4(b)).

Focusing on the non-detuned state (Fig. 4(d,e)), we observe the oscillations (induced by the phase sweep) dip to shot noise. Squeezing is not observed due to out-coupling losses and incomplete overlap of the double tone LO with the supermode, but anti-squeezing is visible. The phase-dependent quadrature variance is consistent with the picture presented in Fig. 2(b,e,f), of the partially symmetric state, visible only through below-threshold quadrature measurements. In principle, the fine structure of the anti-squeezing is equivalent to that of the squeezing but more robust to losses making it useful for exploration. Using these states as a squeezed quantum resource would require overcoupling to the resonator and utilizing a LO with higher bandwidth, along with very low out-coupling losses.

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**Fig. 3** 2-FSR Kerr comb and observation of the secondary threshold at different peak separation. 

- **a.** Single-photon optical spectrum analyzer (SPOSA) measurement of photon populations in above threshold (red) and below threshold (green) modes of the 2-FSR state. 
- **b.** The two-photon correlation ($g^{(2)}$) matrix of the below threshold light in the odd modes, showing the correlations generated by Bragg scattering and pair-generation in the quadrature lattice. 
- **c.** Illustration of the homodyne measurement of the quadrature lattice, the LO is generated by an electro-optic modulator (EOM) and does not have the same spacing as the Kerr comb. 
- **d-g.** Homodyne measurements for four different applied heater voltages, showing the different regimes from symmetric to non-symmetric. The double peak is also observed with a single tone LO and with a LO located at other modes as well.
Tuning more toward the symmetric state, we are able to measure the sub-threshold spectrum of the non-detuned supermode. We perform the same measurement as in Fig. 4(a) but for every two adjacent odd cavity modes simultaneously using a separate laser (Fig. 4(f)). The separate laser is not phased-locked to the pump; therefore, it does not show the quadrature phase dependency but does provide the quadrature mean variance value and spectrum. With this approach, we can resolve 3 separate peaks in each measurement (Fig. 4(g)). Two broad peaks represent the quadrature variance signals of two odd cavity modes, with comb leakage in between, which is narrow (since it is classical above-threshold light). Thus, retrieving each pair of neighbouring odd modes allows us to measure the general non-detuned quadrature supermode structure, as shown in Fig. 4(h). The structure is similar in width to the observed correlations in Fig. 3(b). Similar analysis was performed on the detuned supermode giving similar multi-mode bandwidth since it is dictated by $D_2$ (Methods 4.6).

The co-existence of quadrature supermodes and the robustness of single-peaked supermodes have significant implications for the process of comb formation and evolution: The two-peaked sub-threshold gain profile dictates the secondary comb frequency structure. It is revealing to understand how the different types of anti-squeezing structures affect the threshold from primary 2-FSR comb to a 1-FSR secondary comb. Before presenting the experimental results, we wish to gain further insights from the

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**Fig. 4** Below-threshold quadrature measurements: partially symmetric regime  

**a.** Scheme of the below-threshold homodyne measurement. Unlike the measurement in Fig. 3(d-g) here the ratio of LO tones is optimized and shot noise is filtered.  

**b.** Below-threshold quadrature oscillations induced by the phase shifter, showing the co-existence of two quadrature supermodes - non-detuned and detuned simultaneously, showing the partially symmetric regime directly.  

**c.** Tuning closer to resonance for the state in (b), we observe that the detuned state is increasing in gain while the non-detuned state remains weak, proving that there are two separate supermodes competing.  

**d.-e.** Zooming in on the frequency axis on the quadrature anti-squeezing oscillations of the un-detuned below-threshold state.  

**f.-g.** Scheme and a measurement of mapping the non-detuned quadrature supermode by measuring 3 neighbouring modes with an additional laser. Each measurement produces two below-threshold quadrature variance together with a weak filtered comb line.  

**h.** Quadrature variance of the non-detuned peaks, showing the profile of the non-detuned variance of the quadrature supermode.
Lugiato–Lefever equations (LLE) [44] and simulate the classical non-linear evolution at the threshold (see LLE simulation details in Methods 4.5). By tuning the quality factors and dispersion in the simulation to impose a small $D'_1$, we can simulate a non-detuned threshold to a secondary comb (Fig. 5(a-b)). In this type of threshold, the 2-FSR comb turns into a 1-FSR comb but does not form an rf beatnote as is generally the case [45]. Therefore, only after the first threshold to an un-detuned 1-FSR comb, a second threshold of a detuned comb occurs between the 1-FSR comb and an additional 1-FSR comb, which in-turn leads to an rf beatnote and transmission fluctuations Fig. 5(b).

Returning to our experiment, we follow the sub-threshold light with the same homodyne measurement as in Fig. 4(a). We take multiple snapshots of the quadrature structure during the process of threshold crossing. In Fig. 5(c), we observe how the non-detuned sub-threshold anti-squeezed light crosses the threshold. The first threshold crossing results in a 1-FSR comb, and no rf beatnote. This is followed by a second quadrature supermode, this time detuned, crossing the threshold and giving rise to sub-combs (Fig. 5(d)). One can clearly see that after the threshold crossing of the un-detuned mode, the detuned mode still exists and keeps growing until it also crosses the threshold. Thus, our measurements reveal the mechanism for creating a 1-FSR comb from a 2-FSR primary comb without formation of an rf beatnote. The case of the un-detuned threshold contrasts with the case in which the detuned quadrature supermode reaches threshold first. In that case, as we show in two different measurements of the same quadrature supermode in Fig. 5(e,f), a comb with an rf beatnote can appear immediately after the first threshold. The two measurements are single tone homodyne measurements at different modes ($\mu = 1, \mu = 3$). Additionally, we present five more measurements in Methods 4.6 to show the uniformity of the quadrature variance structure along different cavity modes. This demonstrates how both types of quadrature supermodes result in different types of threshold behavior - one threshold vs two consecutive thresholds.

Fig. 5 Primary to secondary threshold behavior of the quadrature supermodes for different regimes. a-b. LLE simulation of a non-detuned supermode that cross threshold and becomes a secondary 1-FSR comb. The simulation shows that it is followed by a second threshold in which a detuned state crosses threshold, which is then followed by the generation of light that causes an rf beatnote and transmission oscillations. c-d. The experimentally observed evolution of a non-detuned supermode into a secondary comb in the symmetric regime; c. shows the first threshold of a, d. is the second threshold that follows c and is detuned. e-f. A direct detuned threshold corresponding to the completely non-symmetric case. Unlike a-d there is only a single threshold before transmission oscillations. Plots e and f show the same threshold at different modes (+1 and +3).
3 Discussion

In this work we demonstrated non-Hermitian lattice effects in photonic multimode quantum fluctuations described as photonic quadrature lattices. We used Kerr combs that generate photonic quadrature lattices to experimentally explore their non-Hermitian properties for the first time. By controlling the resonator dispersion via an integrated microheater, we were able to observe the three theoretically predicted regimes corresponding to different non-Hermitian lattice behavior. Namely, we observed the symmetric case, partially symmetric, and non-symmetric. Kerr combs allow scalability and rich complexity on a compact on-chip platform. By measuring the anti-squeezing of 2-mode projection of the quadrature supermode, we revealed the existence of detuned and robust non-detuned squeezed supermodes and explained how these form in the context of lattice symmetries. Exploring such fundamental properties paves the way for designing more complex lattice phenomena by injecting or probing different above-threshold states. Furthermore, understanding comb formation and the quantum noise of Kerr combs with our approach can lead to improved comb technology, particularly for combs that are governed by quantum noise which dictates the ultimate limit on comb stability. The concepts studied here form a bridge that will enable the exploitation of the rich non-Hermitian lattice physics in quadrature bosonic lattices to engineer complex quantum resources on-chip. The complex transport of quadratures offered by the Kerr effect and demonstrated in this work will lead to lattice phenomena such as non-Hermitian skin effect, Anderson localization, edge-states and topological phases. These phenomena would manifest as quadrature modes offering new opportunities to control quantum light.

4 Methods

4.1 Band model for quadrature lattice induced by a Kerr comb

As shown in Eq. 2 in the main text, 2-FSR Kerr combs generate 1D quadrature lattices on the frequency axis. The lattice geometry allows reducing the complexity of modeling the physics from a general network of quadratures to a 1D lattice geometry. The goal of this section is to explain the physics of the quadrature lattice by deriving an analytical expression for the band model of the quadrature lattice and its Brillouin zone. First, we transform the Hamiltonian in Eq. 2 into the quadrature basis \( \vec{R} = (p_{-N}, \ldots, p_N, q_{-N}, \ldots, q_N)^T \) where \( p_\mu = \frac{1}{\sqrt{2}} (a_\mu^\dagger + a_\mu) \), \( q_\mu = \frac{i}{\sqrt{2}} (a_\mu^\dagger - a_\mu) \) and write the quadrature non-Hermitian Hamiltonian \( \mathcal{M} \) [46], which satisfies the Heisenberg equation:

\[
\frac{\partial \vec{R}}{\partial t} = \mathcal{M} \vec{R} \tag{3}
\]

We obtain translation symmetry of the lattice by assuming periodic boundary conditions and flat dispersion \( \Delta \tilde{\omega}_\mu \) in the folded lattice, meaning that \( \Delta \tilde{\omega}_\mu = \delta_A \Theta(\omega - \mu) + \delta_B \Theta(\omega + \mu) \) where \( \Theta \) is the step function and \( \delta_A \) and \( \delta_B \) are real numbers. We again use the operators \( \hat{c}_\mu, \hat{d}_\mu \) and their appropriate quadratures: \( \hat{p}_{c,\mu}, \hat{q}_{c,\mu}, \hat{p}_{d,\mu}, \hat{q}_{d,\mu} \). The difference \( \delta_A - \delta_B \) creates an asymmetry in the dispersion \( \Delta \tilde{\omega}_\mu \), which removes the anti-parity symmetry, thus it functions as a simplified representation of the mismatch.
By using the translation symmetry, we transition to the reciprocal space defined by the good quantum number $k$, obtaining a $4 \times 4$ Hamiltonian:

$$
\mathcal{M}_k = \begin{pmatrix}
0 & g_{A,k} & 0 & f_k \\
-g_{A,k} & 0 & f_k & 0 \\
f_k & 0 & 0 & g_{B,k} \\
f_k & 0 & -g_{B,k} & 0
\end{pmatrix}
$$

(4)

where the variables $g_{A(B),k} = (\tilde{\delta}_{A(B)} + 2) + 2\alpha e^{ik} + 2\alpha e^{-ik}$ originate from the dispersion, Bragg scattering, and self phase modulation, while $f_k = -1 - 2\alpha e^{ik} - 2\alpha e^{-ik}$ are pair generation terms, and $\mathcal{M} = \sum_k \mathcal{M}_k + \mathcal{O}(\alpha^2)$. The Hamiltonian $\mathcal{M}_k$ has pseudo anti-parity symmetry (implying that the eigenvalues of $\mathcal{M}_k$ are either purely real or purely imaginary) if $\tilde{\delta}_A = \tilde{\delta}_B$, which shows why there exist a detuned threshold when $\Delta \tilde{\omega}_\mu$ is not symmetric around $\mu = 0$.

![Fig. 6](image)

**Fig. 6** 2-band model of a quadrature lattice induced by a 2-FSR Kerr comb. a. Illustration of the above threshold comb (red) and below threshold comb (green) in the model, and of the different terms - pair generation and Bragg scattering that couple the different modes. b. The 1D lattice unit cell, in the generalized quadrature basis $|\tilde{p}, \tilde{q}\rangle$. Different band topologies (different number of exceptional points) of the Brillouin zone of the lattice in b. plot c corresponds to $A_{-2}/A_0 = 0.25, \Delta \tilde{\omega} = 2$, plot d corresponds to $A_2/A_0 = 0.25, A_{-2}/A_0 = 0.2, \Delta \tilde{\omega} = 2$ and plot e to $A_{+2}/A_0 = 0.25, \Delta \tilde{\omega} = 0$. f-i. The number of exceptional points for different 2-FSR Kerr combs on different slices of the 3D parameter space. The parameter space is spanned by the two ratios of the sidebands and the overall detuning

Therefore, by considering the case in which $\tilde{\delta}_A = \tilde{\delta}_B \equiv \tilde{\delta}$ the quadrature Hamiltonian $\mathcal{M}_k$ becomes degenerate. We simplify the treatment further by transitioning to a new quadrature basis in the unit cell according to $\tilde{R}' = U \tilde{R}$ where $\tilde{R}' = (\tilde{r}'_{1,k}, \tilde{q}'_{1,k}, \tilde{r}'_{2,k}, \tilde{q}'_{2,k})$ is a generalized quadrature basis, and the unitary is:

$$
U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 \end{pmatrix}
$$

(5)
Applying the change of coordinates with $U$ leads to the following 2 by 2 Hamiltonian:

$$
\mathcal{M}'_k = \begin{pmatrix}
1 + 2\alpha e^{ik} + 2\alpha e^{-ik} & \left(\tilde{\delta} + 2\right) + 2\alpha e^{ik} + 2\alpha e^{-ik} \\
-\left(\tilde{\delta} + 2\right) - 2\alpha e^{ik} - 2\alpha e^{-ik} & -1 - 2\alpha e^{ik} - 2\alpha e^{-ik}
\end{pmatrix}
$$

which is related to $\mathcal{M}_k$ of Eq. 4 by $\mathcal{M}_k = U^\dagger \mathbf{1}_{2,2} \otimes \mathcal{M}'_k U$. The 2-band model of Eq. 6 represents the most basic lattice quadrature physics and is pseudo Hermitian with APT symmetry. This means that its eigenvalues are either purely real or purely imaginary and it has exceptional points for appropriate lattice parameters. To illustrate the APT symmetry consider the operators $\mathcal{P} = \sigma_x$ and $\mathcal{T} = K$, where $\sigma_x$ is the x Pauli matrix, and $K$ is the conjugation operator. These operators $\mathcal{PT}$ and $\mathcal{M}'_k$ are anti-commuting: $\{\mathcal{PT}, \mathcal{M}'_k\} = 0$ which shows the APT symmetry.

Due to the degeneracy, we use only two variables: $(p'_k, q'_k)$, allowing us to explore some of the fundamental physics of our lattices. Figure 6 analyzes the 2-band model of Eq. 6. Figure 6(a) shows different non-linear terms and the structure of the comb (red) and sub-comb (green), while Fig. 6(b) presents the folded representation where translation symmetry exists.

Next, we analyze the parameter space of the comb shown in Fig. 6(a) where $\alpha < 0.5$. In this parameter sub-space there can exist 0, 2 or 4 exceptional points in the Brillouin zone (Fig. 6(c-d)). We map the 3D parameter space for combs (also for combs with $A_2 \neq A_{-2}$) presenting the different band topologies (Fig. 6(f-i)). We note that having 4 exceptional points is not an inherent maximum; beyond the presented parameter space, more exceptional points can exist. Additionally, the relative phases between the comb tones can influence the number of exceptional points when more than two pumping amplitudes are considered. Therefore, while we provide the basic mechanisms governing such lattices, they can become significantly more complex. The analysis of the 2-band or 4-band models sheds light on the eigen-vector and eigen-value structures. However, the infinite lattice cannot reach threshold by definition and is therefore not suitable for studying multi-mode squeezing phenomena on its own, and must be truncated for that purpose. More details on the truncation and its implication on the results of the 2-band model will be discussed in the next section.

4.2 The finite model and analysis of the robustness of non-detuned quadrature lattice modes

In this section, we continue with the theoretical analysis akin to Methods 4.1 focusing on finite lattices. Similar to structures in physical space, a realistic lattice in the frequency domain is inherently finite. Naturally, dispersion is never completely flat (i.e. group velocity dispersion or higher-order terms are never zero), implying that as the lattice extends towards higher and lower frequencies away from the pump, the detuning between the cold cavity dispersion and the quadrature lattice modes increases. This effect reduces the coupling efficiency of Bragg scattering and possibly pair generation, leading to "soft edges" and resulting in a gradual decrease in transport between frequencies. In contrast, edges can also be made abrupt through engineering of mode crossing or other type of frequency dependent defects [20]. These sharp edges
can be desirable as they may host confined edge-states in the synthetic frequency dimension [47]. While Kerr resonators can accommodate both options, our focus here is on naturally occurring edges due to non-zero group velocity dispersion.

Furthermore, we will numerically demonstrate that the non-detuned multimode states persist even when dispersion is non-flat and are robust against the breaking of dispersion symmetry around the pump frequency $\mu = 0$. Starting from Eq. 2, we assume $\Delta \tilde{\omega} = \omega_p + D_1 \mu + D_2 \mu^2$, where $D_1 = \Delta \Omega - D_1$ and $D_1$ represents the FSR. When $D_2 \neq 0$ and/or $D_1 \neq 0$ the multimode states localize, allowing us to analyze them using a finite Hamiltonian $\mathcal{M}$. For this analysis, we assume equal quality factors for all resonances $\Gamma = \kappa / 2 \mathcal{I}_{N} \times 2^N$. Since $\Gamma$ is real and proportional to the unit matrix, it only translates the real part of the eigenspectrum of $\mathcal{M} - \Gamma$, and does not influence the imaginary part of the eigenvalues or the peak separation of the squeezed multimode states. Following the conventions in [46] the input-output relation of the system and its Fourier transform are given by:

$$\frac{dR_{out}}{dt} = (-\Gamma + \mathcal{M}) R(t) + \sqrt{2}\Gamma R_{in}(t)$$

(7)

$$R_{out}(\omega) = S(\omega) R_{in}(\omega)$$

(8)

The linear response matrix $S(\omega)$ describes in our case how the system amplifies and squeezes the quantum fluctuations in the quadrature lattice and is given by:

$$S(\omega) = \sqrt{2\Gamma} (i\omega I + \Gamma - \mathcal{M})^{-1} \sqrt{2\Gamma} - I$$

(9)

The matrix $S(\omega)$ can be decomposed by the Bloch-Messiah decomposition which leads to:

$$S(\omega) = U(\omega) D(\omega) V^\dagger(\omega)$$

(10)

where the matrices $U$ and $V$ are unitary and $D(\omega) = \text{diag}\{d_1(\omega), ..., d_N(\omega), d_1^{-1}(\omega), ..., d_N^{-1}(\omega)\}$ corresponds to squeezing and anti-squeezing values of the different quadrature supermodes. Next we aim to show more broadly the impact of group velocity dispersion and asymmetry in dispersion relative to the comb on the multi-mode states. In Fig. 7(a-b) we present a typical complex eigenvalue band structure for a quadrature lattice with $D_2 \neq 0$, setting $D_2/2\pi = 1.2 MHz, \mathcal{R} = 0.225$, and $Q_i = 1.74 \cdot 10^6$, which are experimental parameters also used in Fig. 2 of the main text. Clearly, dispersion compresses the spectrum bandwidth of the quadrature supermode and the number of participating supermodes. By focusing on the most relevant supermodes, we observe that APT symmetry persists: Supermodes are either purely real (un-detuned) or purely imaginary (detuned), with the purely real ones exhibiting higher gain.

Introducing a mismatch between the cavity FSR and the Kerr-comb frequency spacing eliminates lattice symmetry, yielding more general complex eigenvalue solutions for the $\mathcal{M}$ matrix. When the mismatch is sufficiently small, as depicted in Fig. 7(c-d), we observe that some supermodes remain with purely real eigenvalues while others do not. Figure 7(c,f) shows the existence of un-detuned (imaginary eigenvalue is 0) supermodes as a function of $D_2$ and constant pump detuning. Figure 7(e)
Fig. 7 Existence of non-detuned state in the presence of non-zero $D_2$ and $D'_1$. a. Eigenvalue solution of the quadrature Hamiltonian $\mathcal{M}$ for a 2-FSR Kerr comb with side band drop ratio of $R = 0.225$, and $D_2/2\pi = 1.2 \text{ MHz}$. b. Zoom-in on the quadrature supermodes with higher-gain showing that all of them are undetuned (imaginary value is 0.) c-d. Same as a. and b. only with $D'_1/2\pi = 2.8 \text{ MHz}$. e-f. Numerically counting the un-detuned states as a function of underlying GVD ($D_2$) and detuning of the comb for the symmetric case of $D'_1 = 0$ and the non-symmetric case $D'_1/2\pi = 2.8 \text{ MHz}$.

4.3 Additional data on fabricated device and experimental setup

Our devices are etched from a 550 nm thick 4H-SiC-on-Insulator \cite{43} and capped with SiO$_2$. Microheaters are patterned via liftoff of 100 nm platinum with a 5 nm titanium adhesion layer. Coupling to and from the chip is achieved using lensed fibers, with collection efficiency of up to 50%. The ring resonators exhibit intrinsic quality factors of up to 8 million, with typical values around 1.5 million. The resonators exhibit a $D_2/2\pi$ of 1.2MHz, and an FSR of 153 GHz. Additionally, the ring is multimode, causing mode splitting through modal interactions (Fig. 8(b), which allows a broad range of narrow-bandwidth states to be observed in the devices, such as 2-FSR primary combs (Fig. 9(a-b)), as well as the transition to 1-FSR combs and soliton crystals. The heaters modify the mode-splitting properties, thereby altering the detuning at threshold and consequently the Kerr comb’s frequency spacing $\Delta\Omega$. Figure 9 presents the experimental evolution of the comb for the symmetric case. The comb starts as a 2-FSR primary comb (Fig. 9(a-b)) and undergoes an undetuned (single peak) threshold
to a 1-FSR phase locked comb (Fig. 9(c)). Following the threshold, a second detuned (double peak) threshold occurs followed by additional thresholds and an rf beatnote in the comb (Fig. 9(d,e)). Finally, the comb abruptly turns to a 2-FSR soliton crystal (Fig. 9(f)).

Fig. 9 Kerr comb for different pump detuning - a single evolution. a.-b. A 2-FSR primary comb, where b. is closer to the secondary threshold (the pump is at a longer wavelength) than a. c. Tuning the pump to longer wavelengths, the 2-FSR comb evolves to a single phase locked 1-FSR comb corresponding with Fig. 5(c-d). d.-e. Additional thresholds creating additional 1-FSR combs with subcombs corresponding with Fig. 5(d). f. Red tuning the pump further results in a 2-FSR soliton crystal (corresponding with the measurements in Fig. 4)

Next, we provide a more detailed description of the experimental setup (Fig. 10(a-b)). We utilize two tunable CW telecom lasers named CW laser 1 (Velocity, 1520-1570 nm, TLB-6728) and CW laser 2 (Toptica CTL1500: 1460-1570nm). The lasers are split into two channels. The first channel (upper channel in Fig. 10(b)) receives light from laser 1 and after amplification through an EDFA excites the Kerr resonator. The Kerr resonator outputs two channels: the first carries the quantum signal with the filtered Kerr comb directly to the homodyne unit, while the second channel contains the Kerr comb passing thorough the filter ring to a photodiode.

The second channel (bottom in Fig. 10(b)), receives light either from CW laser 1 or CW laser 2 or both. CW laser 1 is used for measuring modes -1 and 1, while CW laser 2 allows measuring other modes but is not phase-locked to CW laser 1. To span a 3-FSR range with our LO, we employ an electro-optic modulator (EOSpace LiNbO3 20GHz phase modulator). Subsequently, a programmable WaveShaper (4000A multiport optical processor) transmits specific sidebands that overlap with the modes.
of the Kerr comb, filtering out all other lines. The frequency tones that overlap with
the classical comb are mixed with the teeth that pass through the filter ring in order
to track the frequency difference between the Kerr comb and the LO (center of the
diagram in Fig. 10(b)). The frequency tones of the LO that overlap with the quantum
signal are passed through a piezo fiber stretcher that stretches the fiber to change
the phase of the LO. This continuous phase ramp is imprinted on the homodyne
signal, allowing to distinguish it from other non-phase dependent fluctuations. The
signal is then amplified with a low input power EDFA (OptiLab-MSA Pre-Amp EDFA
Module, C-band) and then mixed with the quantum signal from the Kerr comb and
fed into a balanced photodetector pair (WL-BPD1GA 1 GHz Dual-Balanced InGaAs
Low Noise), to perform the homodyne measurement.

4.4 Multimode Homodyne detection

To measure the maximal squeezing and anti-squeezing of the multimode below-
threshold light, balanced homodyne detection must be performed with a multi-tone
local oscillator (LO). The intensity and relative phase of each line of the LO should
match the maximally-squeezed supermode of the squeezed vacuum \[10\]. In this work,
the multi-tone LO is generated from a 20 GHz EOM comb \[48\] driven by the same
pump laser. The low bandwidth of a single phase modulator (or a few cascaded phase
modulators) requires either working with relatively large micro-resonators or only par-
tial overlap with the maximally squeezed supermode. Large micro-resonators pose
additional challenges, including higher operation powers (for the same intrinsic quality
factor) and a lower \(D_2\) for the same resonator cross-section. The theory and exper-
imental validation we present in this work does not overlap optimally with the squeezed
supermode, but is sufficient to capture the frequency-dependence of the most squeezed
states. Therefore we are able to use a two-tone local oscillator despite the fact that
the connectivity expands beyond two modes. We found that the two-tone local oscillat-
or measures significantly stronger quadrature variance signal than a single-tone local
oscillator by improving the overlap with the squeezed mode. Additionally, the two-
tone symmetric LO may measure a time-independent phase relative to the signal in
the absence of active phase locking of the EOM comb to the repetition rate of the
Kerr comb \[49\].

We now provide further mathematical details on the multi-tone homodyne
measurement.

To calculate the noise variance spectrum for a multi-tone local oscillator at the
repetition rate of the comb \[33\], we can define a supermode decay operator

\[
\hat{L}(t) = \sum_j \alpha_j \hat{a}_j(t)
\]

whose Hermitian conjugate creates a photon in a superposition of azimuthal modes
\(\hat{a}_j(t)\) across the frequency comb. \(\alpha_j = |\alpha_j|e^{i\phi_j}\) defines the amplitude and phase
composition of the supermode. Its associated supermode output operator is

$$\hat{L}_{\text{out}}(t) = \sum_j \alpha_j \hat{b}_{\text{out},j}(t)$$

(12)

where $\hat{b}_{\text{out},j}(t)$ is the output operator of mode $\hat{a}_j(t)$. The corresponding Hermitian quadrature operators of $\hat{L}_{\text{out}}$ are:

$$\hat{L}^{(+)}_{\text{out}}(t) = \frac{1}{\sqrt{2}} (\hat{L}^{+}_{\text{out}}(t) + \hat{L}_{\text{out}}(t))$$

(13)

$$\hat{L}^{(-)}_{\text{out}}(t) = \frac{i}{\sqrt{2}} (\hat{L}^{+}_{\text{out}}(t) - \hat{L}_{\text{out}}(t))$$

(14)

The noise variance is then defined as

$$V^{(\pm)}(\omega) = \int_{-\infty}^{\infty} d\tau \langle \hat{L}^{(\pm)}_{\text{out}}(t) \hat{L}^{(\pm)}_{\text{out}}(t + \tau) \rangle e^{i\omega\tau}.$$  

(15)

The maximally-squeezed supermodes can be obtained from the columns of $U(\omega)$ (Eq. 10), which defines the linear combination of quadratures $(q_1(t), ..., q_n(t)|p_1(t), ..., p_n(t))^T$ for a given supermode. When $U(\omega)$ is real, then the composition for the $k$th maximally-(anti)squeezed supermode at $\omega$ may be written:

$$|\alpha_j| = \sqrt{U_{j,k}(\omega)^2 + U_{j+n,k}(\omega)^2}$$

(16)

$$\phi_j = \text{atan2} \left( U_{j+n,k}(\omega), U_{j,k}(\omega) \right)$$

(17)

The quantum (anti)-squeezing is therefore significant when the local oscillator composition overlaps significantly with a multimode state that has large value $d_i(\omega)$. In this approach maximum quadrature variance can be obtained through this method for the undetuned squeezing. Since the undetuned squeezing occurs at $\omega = 0$, the value of its eigenvector is real $U(0) \in \mathbb{R}$ and a time-independent local oscillator (in the frame rotating at the repetition rate of the comb) will measure maximal (anti)-squeezing. In the case of detuned squeezing, $U(\omega)$ is complex and a stationary local oscillator will not necessarily measure maximal squeezing.

However, when measuring a symmetric double peak configuration one may obtain maximal squeezing [49]: when a local oscillator is equidistant between a nondegenerate squeezed state, measurement at the sideband frequency probes both modes. If the double peak is symmetric, we are summing two peaks with opposite Fourier components. Of course, as shown in Methods 4.6 the double peak quadrature variance spectrum can be asymmetric.

4.5 Additional data on LLE simulation

In order to support the theoretical conclusions and experimental observations, we preformed LLE simulations of a 2-FSR Kerr comb. The LLE simulates the classical
comb dynamics [44], and allows us to focus on specific aspects, some not previously addressed in the experiment. Particularly, the squeezing structures, both detuned and non-detuned, significantly influence the formation of secondary combs. Hence, studying the primary comb to secondary comb transitions in a classical framework is valuable in the context of the findings presented in this work.

We start by simulating a perturbed dispersion yielding a comb similar to that of our experiment. The dispersion perturbation in our setup changes in response to heating. For the same reason, the perturbation also changes as a function of power in the cavity (i.e. it is changing during the evolution of the comb). However, since the effects we describe do not require this behavior, we can observe the main features under constant dispersion. We use experimentally extracted parameters of \( D_2/2\pi = 1.2 \text{MHz} \) and extract the perturbation parameters from the measured dispersion. For simplicity, we only use the perturbed modes near the pump modes \(-2, -1, 0, 1, 2\). For quality factors, we use a constant value of 1.83 million, which we reduce at modes distant 17 FSR away from the pump to avoid pair generation there. The normalized pump power is \( f = 5.93 \) where \( f^2 = 8g_0\kappa_cP_{in}/\kappa^3\hbar\omega_0 \) and \( P_{in} \) is the input power, \( \kappa_c \) is the outcoupling power loss, and \( \omega_0 \) is the pump angular frequency.

Importantly, the formation of our 2-FSR comb is not a direct result of the anomalous dispersion, but it is formed locally due to the dispersion defect (see Methods 4.3). This means that the initial comb formation may be governed by adjusting the dispersion of sites \(-2, 0, 2\). The detuning value of the threshold to a 1-FSR comb depends on the dispersion of the odd modes and the perturbation of modes \(-1, 1\). By adjusting these parameters, we are able to simulate a comb evolution that is qualitatively similar to what we observe in the experiment.

Figures 5(a-b) and 11(a) show the comb evolution in the simulation, which is similar to our experiment (Fig. 9). The process begins with a 2-FSR primary comb which transitions to a 1-FSR secondary comb. Figure 11 demonstrates the transitions from one rolling Turing pattern to another. Since this transition is un-detuned, the first 1-FSR comb is a single phase locked comb and does not have sub combs.

![Fig. 11 Intracavity field in real space for different regimes in the comb formation - LLE with perturbed dispersion - undetuned threshold. a. Transmission as a function of continuous pump detuning (toward longer wavelengths). This plot is the same as in Fig. 5(a) but the shaded areas mark different regimes plotted in (b-d). b. Intracavity field in real space for the transition between the primary comb (2-FSR) and the secondary comb (1-FSR) corresponding to the shaded red region in a. c. Same as b but for the transition from the secondary comb (1-FSR) to a second detuned 1-FSR comb which oscillates (shaded green region). d. The region of a 2-FSR soliton crystal (shaded blue region).](image)

The next transition involves the threshold crossing of a detuned 1-FSR comb, which produces an rf beatnote with the existing 1-FSR comb. The oscillations in this state
are evident in both the transmission and the intra-cavity field (Fig. 11(c)). Finally the system transitions to a 2-FSR soliton crystal (Fig. 11(d)). To better understand the unique symmetric transition from a 2-FSR comb to a single phase locked 1-FSR comb, we explore the comb state at a pump detuning close to this threshold at 0.115 GHz. Figure 12(a-b) shows the comb from Fig. 11 for the detuning of 0.115 GHz in both the frequency (mode) space and real physical space respectively. Using the obtained comb amplitudes and the hot-cavity dispersion, we calculate the $g^{(2)}$ correlations of each cavity-mode $\mu$ with each cavity-mode $\nu$, which resembles the experimental $g^{(2)}$ in Fig. 3(b). Next, we calculate the quadrature Hamiltonian $M$ associated with the 2-FSR comb at 0.115GHz. The eigenspectrum is plotted in Fig. 12(d), showing that the non-detuned state (purely real eigenvalue) is approaching threshold. Correspondingly, the squeezing structure is a single peak (Fig. 12(e)), localized near the pump (Fig. 12(f)).

In addition to the perturbed dispersion case, we wish to explore in simulation the case of unperturbed dispersion with a $D_2 > 0$ term. Throughout this work, a perturbation in dispersion somewhat detached the general dispersion of the cavity from the frequency spacing of the 2-FSR Kerr comb, allowing for more possible behaviors. However, when a comb is formed without perturbation, the general dispersion and the comb's frequency spacing are more inherently linked, reducing the number of possible outcomes. While this case is conceptually simpler than the perturbed dispersion case, it can be more challenging to realize due to the high anomalous dispersion that is required for the 2-FSR comb to form (together with a relatively small FSR). Additionally, high anomalous dispersion causes the squeezed supermodes to localize in four modes, reducing the relevance of the lattice model description (Fig. 6). Nevertheless, this case does highlight some aspects which are relevant to the main case studied in this work.

To generate a 2-FSR comb, we use high anomalous dispersion of $D_2/2\pi = 86.4 \text{ MHz}$, an intrinsic Q of 3.66 million and coupling Q of 4.25 million. These parameters result
Fig. 13 Simulation of a 2-FSR primary comb transition to a 1-FSR secondary comb in a unperturbed dispersion. a. Transmission vs pump-detuning of the 2-FSR comb without dispersion perturbation. b. Spectrum of the 2-FSR Kerr comb (red) in an unperturbed dispersion. c. Eigen-spectrum showing 40 supermodes. In this case all the modes are detuned (purely imaginary eigenvalues) while the Hamiltonian is symmetric. d. Zoom-in on the spectrum in c. e. The squeezing structure, showing the double peak which correspond with the detuned state that crosses threshold in c,d f. The spectrum of the most squeezed state

in a 2-FSR primary comb, which we pump at 12.55mW with CW light at 1545nm. This comb reaches threshold at a pump detuning of $-0.04 \text{GHz}$ (Fig. 13(a)). At this pump detuning value, even though the dispersion is symmetric and $D_1' = 0$, the threshold is still detuned (double peak). The symmetry implies that the eigenvalues are purely real or purely imaginary. Thus, if all the states are purely imaginary, there are no undetuned states in existence, leading to a double peak threshold. Correspondingly, the 2-FSR comb in Fig. 13b, for the detuning marked by the orange dot in Fig. 13(a), has a fully detuned spectrum (Fig. 13(c,d)). Such a spectrum is associated with a detuned threshold, i.e., a squeezing spectrum that peaks away from zero. The localization that high $D_2$ dispersion causes can be seen in the spectrum of the supermode in Fig. 13(f). The localization occurs for all supermodes, as Bragg scattering is not efficient away from the center of the comb.

4.6 Single supermode dominance

As we demonstrate in the theoretical and experimental measurements of this work, the quantum noise in the different cavity modes is dominated by one or two supermodes. Additionally, the squeezing frequency spectrum is uniform for each supermode, meaning that the below threshold light, has similar anti-squeezing in all cavity modes in our experiment. In this section we show experimentally how a single supermode dominates the spectrum close to threshold. We focus on the non-symmetric case, which shows distinct two peaks, and perform a single-tone homodyne measurement in each cavity mode. The results are presented in Fig. 14, which shows how similar two peaks cross the threshold for each cavity mode.

Note that the two peaks appear both in the below threshold light (lower line plots in each sub-figure) and above threshold light (upper line-plots).

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Fig. 14 Threshold crossing of detuned supermode by individually homodyning modes -5 to 7. a.-g. Extension of the data in Fig. 5(e,f), showing additional 5 homodyne measurements of the threshold crossing of a double peak (detuned) threshold, showing the dominance of a single supermode in all modes

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