Numerical Studies of Instabilities in a Magnetized Hall Discharge

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An analysis is presented of low to moderate-frequency wave propagation in Hall thruster channels. Starting with extensive experimental data on the time-averaged, spatially varying plasma properties within a Hall discharge plasma, we subject this plasma to small (linear) perturbations in its properties. For this perturbation analysis, we assume a two-dimensional fluid description that includes a simplified equation for the electron energy. The azimuthal wavenumber is selected to be resonant with the channel circumference, and the growth rate and frequencies of propagating plasma disturbances are obtained by numerical solution of the resulting eigenvalue problem under a quasi-uniform plasma condition, along the entire discharge channel. The results predict the possible emergence of low (10-100 kHz), moderate (100 kHz - 10 MHz), and high (>10^11 MHz) frequency instabilities concentrated largely in the vicinity of the exit plane. The low frequency instability, which is commonly seen in experiments, is found to be associated with the ionization process. While it is intrinsic to the discharge as an m = 1 azimuthal disturbance, the analysis performed here shows that it is also expected to be excited as an m = 4 mode, perhaps due to the four-fold symmetry of the magnetic circuit, and an m = 12 mode, due to the separation of the probes, consistent with recent experimental observations.

I. INTRODUCTION

Crossed-field co-axial Hall discharge thrusters are $E \times B$ plasma devices that generate intense ion beams at high velocities (>10 km/s), suitable for use as plasma thrusters for various space propulsion applications. Their potential as high specific impulse thrusters for space plasma devices that generate intense ion beams at E X B thrusters for various space propulsion applications. Their operating principle for a Hall discharge is seemingly straightforward - electrons emitted by an external cathode are magnetized by the magnetic field and drift in the $E \times B$ direction. Ions, generated through volume ionization by the trapped electrons, having a Hall parameter much less than unity, are not magnetized, and are accelerated by the electrostatic fields established by the retarded electron flow. In a co-axial geometry, the electrons are constrained to move in the azimuthal direction of the closed $E \times B$ drift (hence the name "closed electron drift thruster"). The cross-field drift of electrons towards the anode is known to be anomalously high (greater than that determined by the plasma collisionality) and there is growing evidence that the primary mechanism for this anomalous transport is turbulence in the plasma properties (e.g., plasma density, potential) due to the development of a large number of instabilities in the presence of strong gradients in the plasma density and magnetic field [3-7]. In a recent excellent review of the fluctuations seen in these devices [3, 8], Choueiri classifies the modes according to frequency range and qualitative behavior, suggesting that there are instabilities associated with the ionization process at low frequencies, some of which interact strongly with the external discharge circuit, instabilities associated with axial ion transit at mid frequencies, and azimuthal "drift" waves due to the axial gradients at high frequencies. Regardless of the physical origins, all of these disturbances seem to be concentrated near the exit of the discharge channel, where the plasma density and magnetic field is highest, or where the strongest gradients in these plasma properties exist.

In this paper, we present continued numerical studies aimed at developing an understanding of the fluctuations that often persist in closed electron drift Hall discharges. In prior theoretical research on these instabilities, most notably in the earlier Russian literature [5,6], simplifications were often made to the governing equations so as to derive closed-form analytical expressions for the dispersion characteristics of these waves. While these analyses provided physical insight as to the origins of these waves, ionization was usually neglected in these prior theoretical treatments. It is now well known that the disturbances at lower frequencies, which greatly impact the interaction of the device with the external driving circuit, depend critically on the ionization processes.

One motivation for this work is to test the ability of a multi-fluid treatment of a Hall discharge to capture the characteristics of these plasma fluctuations seen in the laboratory. The success of such a test would lead to added confidence in the direct numerical solution of the fluid equations, or perhaps of a hybrid (fluid/particle) treatment in two or three dimensions [9-11]. In this report, we make use of an extensive and comprehensive experimental mapping of the background (time-averaged) plasma properties in a Hall discharge, which provides us with the necessary local background conditions for the linear perturbation analysis [12,13]. We believe that at present, this is a better approach to understanding fluctuation phenomenon in Hall discharges, since
satisfactory theories for the "stationary" plasma state do not yet exist.

The basis for our analysis, which has been described in detail in a previous paper [14], is a 2-D multi-fluid description of the Hall discharge plasma. Below, we review the basic features of this model, and the approach to solving the dispersion relations for the temporal waves that are rendered unstable over the operating conditions for which experimental data is available for the stationary state. Finally, results are presented for operating conditions over which plasma density fluctuations have been obtained within the discharge channel [15].

II. MODEL

A. Multi-Fluid Model

The geometry for the analysis is selected so that the x-axis is defined as being along the thruster axis (along the direction of the applied electric field, and the direction of ion flow), the y-axis as along the azimuthal coordinate, and the z-axis is along the thruster radius corresponding to the principle direction of the applied radial magnetic field. With this geometry, the azimuthal electron drift velocity is in the negative y direction. The plasma is assumed to be uniform in the z-direction, and we restrict our attention to motion in the x-y plane, rendering the problem to two-dimensions.

The plasma response to the electric and magnetic field is governed by a set of thirteen multi-fluid magnetohydrodynamic equations that determine the electron, ion, and neutral xenon number densities, corresponding velocities, the electron temperature, magnetic field (assumed to vary only along x), and the electric field components along x and y.

The species conservation equations explicitly account for finite-rate ionization and wall losses of electrons and ions:

\[
\frac{\partial n_e}{\partial t} + \frac{\partial n_e u_x}{\partial x} + \frac{\partial n_e u_y}{\partial y} = \alpha_i n_i n_e - n_e v_{wall} \tag{1}
\]

\[
\frac{\partial n_i}{\partial t} + \frac{\partial n_i v_x}{\partial x} + \frac{\partial n_i v_y}{\partial y} = \alpha_i n_e n_i - n_i v_{wall} \tag{2}
\]

Here, \( n_e, n_i \) are the electron and ion number densities, \( u_x, u_y, v_x, v_y \) the corresponding x and y-component velocities, \( \alpha_i \) is the volumetric rate constant for ionization (which depends on electron temperature), and \( v_{wall} \) the rate at which electrons and ions are lost at the channel wall. In expressing the species conservation equations in this way, we assume that there is a net loss of electrons/ions at the wall, controlled largely by the electron diffusion. While the net wall loss rate is expected to depend in a complex way on the secondary electron emission process, we determine its value experimentally, from the measured axial variations in the background plasma properties, including the electron current density, \( J_e \), and from the computed volumetric ionization rate constant (assuming a Maxwellian electron energy distribution at the measured electron temperature):

\[
v_{wall} = \alpha_i n_e = \frac{1}{en_e} \frac{dJ_e}{dx} \tag{3}
\]

It is noteworthy that ionization rate and collision frequencies are taken to be time-independent, allowing a linearization of the equations in plasma and neutral density, and so the ionization instabilities captured here are driven to first order by fluctuations only in the plasma density (i.e., not associated with fluctuations in the electron temperature).

B. Stability Analysis

In general, for the case of an axially non-uniform plasma as the background stationary condition, a stability analysis would require the solution of a set (13 in this case) of homogeneous ordinary differential equations for the perturbation amplitudes, obtained by assuming a perturbation of the form

\[
f = f_0(x) + f_1(x, y)e^{-i\omega t} \tag{4}
\]

where, \( f \) is the plasma property of interest, \( f_0 \) is the steady-state background property value, dependent on position for the non-uniform plasma case, and \( f_1 \) is its small perturbation amplitude (which can also depend on position, and the mode frequency). In this study, we shall simplify the analysis by assuming that the plasma is initially uniform in the azimuthal direction (y), and that the characteristic length of the axial perturbations (~ \( \lambda_x \), the wavelength component along x) are smaller than the characteristic length scale over which the background properties change, i.e.

\[
\lambda_x < \left( \frac{1}{f_0 \frac{\partial f_0}{\partial x}} \right)^{-1} \tag{5}
\]

This scale length is approximately the length of the discharge channel (~8 cm), but, in some cases, is better characterized by the length of the ionization zone (~2 cm). Under these conditions, the problem reduces to a homogeneous set of linear, algebraic equations, with

\[
f_1(x, y, t) = \hat{f}_1 e^{i(k_x x + k_y y - \omega t)} \tag{6}
\]

where \( \hat{f}_1(\omega) \) is the mode frequency-dependent amplitude of the perturbation. Although simplified, this first approach leads to a direct comparison between the predicted and observed behavior at various locations within the discharge channel. The analysis results in the prediction of linear perturbation frequencies that depend on position. In the analysis, gradients in these background properties are evaluated explicitly, based on their local values, determined experimentally [14].
C. Solution Method

Our approach is to solve the so-called “temporal formulation” of the problem, where the wave vectors \( k_x \) and \( k_y \) are taken as real quantities, and solve the system of equations for the complex \( \omega \), searching through the roots for positive frequencies (real component of \( \omega \)) and positive growth rates (imaginary component of \( \omega \)). For purely azimuthal modes, we constrain \( k_y = \frac{m}{r} \) (negative sign since we are looking for disturbances propagating in the negative Hall current direction). Here, \( m \) represents the mode number and \( r = 4 \) cm, the radius of the midpoint of the annular discharge channel. In our experimental study of fluctuations and wave propagation within the discharge channel [15], we have seen the persistence of two intrinsic azimuthal modes; \( m = 1 \) and \( m = 4 \), the later believed to be anchored by the four-fold symmetry of the pole pieces in the magnetic circuit. Also, we have seen weak modes of varying integer mode numbers depending on the placement of the two probes used to characterize the azimuthal wave propagation, for example, the \( m = 12 \) mode, when the probes are located 30° apart azimuthally. Here we examine the propagation of all of these azimuthal modes for the discharge conditions studied.

For predominantly azimuthal waves with small propagation angles \( \alpha \) out of the azimuthal plane, we use \( k_x = k_y = \tan (\alpha) \) for the out of plane component of the wave vector. For our particular discharge channel, we find that we satisfy the weak axial non-uniformity condition above provided that \( 20° \geq \alpha \geq 70° \). For other propagation angles, the axial component of the wavelength is comparable to and slightly greater than the length of the channel, and so, the results for these intermediate propagation angles must be interpreted cautiously.

To find the mode frequencies, we solve a generalized eigenvalue problem of the form

\[
\begin{bmatrix}
1 & A - \omega I
\end{bmatrix}
\begin{bmatrix}
f
\end{bmatrix} = 0,
\]

where, the vector describing the perturbation amplitudes is:

\[
\tilde{f} = \begin{bmatrix}
\hat{n}, \hat{n}_u, \hat{n}_v, \hat{u}, \hat{v}, \hat{w}, \hat{E}, \hat{E}_r, \hat{B}_r
\end{bmatrix}
\]

and all of the elements of the coefficient matrix \( A \) are functions of the axial position, known from experiments. Here, \( I \) is the identity matrix. For each eigenmode representing a non-damped physical instability, we can determine the corresponding set of perturbation amplitudes (and relative phases) as complex components of that specific eigenvector. It is noteworthy that to cast the equation set in this eigenvalue form, we had to neglect the local acceleration (inertia terms) in the electron momentum equation. We find that for the low to
moderate frequency modes of interest here, these terms do not affect the overall results of the calculation.

III. RESULTS

The eigenvalue solution was applied to the conditions for Hall discharge operation at low voltage (100V), intermediate (161 V), and high voltages (200 V), where a complete set of time-averaged data is available within the discharge channel of our thruster [12, 13]. As described in Refs. [12,13], all of the background plasma properties needed to determine the coefficient matrix, \( A \), are measured for the length of the entire discharge channel. Apart from the trivial solution \( (\omega = 0) \), we generally find four or five unstable roots, two of which describe a mode that propagates in opposite directions, and very close to the electron cyclotron frequency, \( \omega_{ce} \approx 10^{11} \) Hz. These are most likely electromagnetic in origin (e.g., the ordinary mode). We often see a root in the range of \( 10^7 \) Hz that propagates in the Hall drift direction and possibly related to the high frequency azimuthal drift waves seen in the PIC simulations of Harakawa [16]. We occasionally see a mode at slightly lower frequency, \( 10^6 \) Hz, propagating opposite to the Hall direction. And we most frequently capture one (and sometimes two) unstable modes between 10 and 100 kHz, which we have discussed in greater detail in Ref. [14], that are strongly coupled to the ionization process. Of these low frequency roots, that which has the largest growth rate is selected for display. The frequencies of these roots are often found to depend strongly on both the axial position and on the propagation angle.

Figure 1 plots the wave dispersion and growth rates for the low-frequency mode, for 100V, 161V, and 200V discharge conditions [see Ref. 15], and for three axial locations very near the discharge exit plane, \( x = 0 \) mm. At low voltages (100 V, say) it is apparent that the predicted growth rates are highest for the pure azimuthal (\( \alpha = 0 \) propagation) direction, and the modes are no longer unstable beyond \( \alpha = 50^\circ \). The computed frequencies tend to be higher closer to the exit plane for large propagation angles, although at the point of highest growth rate, the frequencies are in the range of 10 kHz - 20 kHz, remarkably consistent with experimental observations [15]. It is noteworthy that for a distance of 2 cm upstream of the exit plane, the predicted growth rate diminishes greatly, a finding that is also consistent with experiments.

At higher voltages, the predicted growth rates become less sensitive to propagation direction, and the waves are seen to be unstable beyond \( 50^\circ \) for the 161V (intermediate voltage) case, and beyond \( 60^\circ \) for the 200V
Fig. 3. Computed frequencies (top) and growth rates (bottom) for the low frequency mode propagating azimuthally over a range of axial positions.

The case (with the exception perhaps, of the x = -10 mm case). At the 200 V discharge condition, the model predicts relatively strong, azimuthal propagation behavior ($\alpha<45^\circ$) to frequencies as high as 40 – 100 kHz, although the frequencies are predicted to be the highest, 1 cm upstream of the exit plane. The predictions also suggest that at 200 V, quasi-longitudinal waves should be prevalent, in the range of 10 – 40 kHz. The 161 V case is qualitatively similar to that of the higher voltage case, although the predicted frequencies do not extend well beyond 60 kHz in the vicinity of the exit plane. The 200 V case shows the persistence of a weak 0 - 10 kHz mode that has the propensity to propagate longitudinally, which may be weak evidence of an ionization wave frequently referred to as the “breathing” mode, in the Hall thruster literature.

Figure 2 depicts the predicted frequencies and growth rates for the same cases as in Fig. 1, but without the gradients in the plasma properties considered. It is noteworthy that there are substantial differences in these two cases. For example, at low voltages (100 V), the preference for propagation at low angles is no longer apparent, and the predicted frequency range for the instability is much lower that indicated in Fig. 1. At higher voltages, the differences are more apparent for the upstream cases (x = -10 mm and -20 mm), where the predicted growth rates and frequencies are seen to increase moderately.

Figure 3 plots the axial variation in the predicted frequency and growth rates for the low-frequency mode, propagating in the pure azimuthal direction ($\alpha = 0$). With the exception perhaps of the low-voltage (100 V) case, the waves are seen to have the highest growth rate 5-10 mm upstream of the exit plane, with the peak frequencies varying between 10 and 50 kHz. The concentration of instabilities in plasma properties very near, and upstream of the exit plane was noted in Ref. 15. It is noteworthy that the gradient in the magnetic field is positive in this region (not negative), however, in this region, the gradient in the inhomogeneity scale parameter $\ln(n_e/B)$, is indeed negative because of the spatial variation in the plasma density, $n_e$.

Figure 4 plots the axial variation in the predicted frequency and growth rates for the low-frequency mode, propagating in the near axial (quasi-longitudinal) direction, with $\alpha = 75^\circ$. The growth rates are predicted to be the highest for the higher voltage conditions, and it is apparent in Fig. 3 that the activity is predicted to shift towards upstream positions with increasing voltage. The model predicts that the frequencies for these quasi-longitudinal modes should be generally higher than the azimuthal modes, a finding that is also supported by experiments [15].
As mentioned above, the model predicts the existence of a moderately high frequency \((10^7 - 10^8) \text{ kHz}\) disturbance, the origin of which is still under investigation. A plot of the predicted frequency verses axial position for the azimuthally propagating version of this mode is given in Fig. 5. The mode is unstable at low voltages over regions where the magnetic field is strongest, peaking somewhat outside of the discharge. At higher voltage, this mode seems to be unstable upstream of the exit plane. Few experimental details are available for the propagation behavior of this mode, although it was seen to exist in Hall discharges studied by the Russians back in the 1970s. It is noteworthy that Hirakawa predicted the existence of a high frequency azimuthal mode through PIC calculations [16].

While all of what is presented thus far is for the case of an \(m = 1\) principle azimuthal mode, experiments [15] have also suggested that the four-fold symmetry of the magnetic coils may excite an \(m = 4\) mode, that propagates over a range of directions. This mode persists only at the lowest voltage conditions studied in our discharge [15], where there is an observed four-fold symmetry in the time-averaged plasma density.

Figure 5 plots the predicted frequencies and growth rates for low frequency \(m = 4\) modes, verses propagation direction, for the 100V case. It is seen that the growth rate is predicted to be higher very near the exit plane,
where the frequencies are higher (20 − 100 kHz). Upstream of the exit plane, this \( m = 4 \) mode is predicted to be weaker, and of lower frequency (< 30 kHz). Both of these observations are in good qualitative agreement with experiments [15]. It is noteworthy that the weak dependence in the growth rate with propagation angle suggests that there is no strong favoring in the propagation angle, also apparent in the experimental results by the presence of a scatter of points, bounded by the line defining the \( \alpha = 0 \) case [15].

Finally, in Fig. 6., we compare the propagation behavior of the low frequency, \( m = 1, m = 4, \) and \( m = 12 \) modes, for the \( x = 0 \) mm, 100 V case. Recall that previous experiments provide strong evidence for the excitation of higher order modes corresponding to wavelength components equal to the separation of the probes along the azimuthal direction. All three modes show an increasing frequency dependence on propagation angle, and a corresponding decrease in growth rate. The \( m = 1 \) and \( m = 4 \) modes seem to be excited over a larger range of angles (< 40°), whereas the \( m = 12 \) mode has a positive growth rate over angles of less than 10°. These results suggest that the \( m = 12 \) mode should be excited at lower frequencies than its low order counterparts, and should have well defined angles of propagation.

V. CONCLUSIONS

The availability of experimental data on the background (time-averaged) plasma properties within a Hall discharge has permitted a study of the response of this plasma to linear perturbations in its properties. A two-dimensional multi-fluid description, accounting for electron heating is used as the basis for the unsteady analysis. The disturbances are constrained to have azimuthal component wavelengths to be resonant with the channel circumference, and either excited intrinsically (e.g., \( m = 1 \), as seen experimentally), or by the four-fold perturbing influence of the plasma probes themselves (for example, separated by 30°, i.e., \( m = 12 \)). The growth rate and frequencies of the plasma disturbances are obtained by numerical solution of the resulting eigenvalue problem under a quasi-uniform plasma condition, along the entire discharge channel. The results identify the persistence of low (10 − 100 kHz) and moderate (10^6 − 10^7 kHz) frequency instabilities that are concentrated largely near the exit plane of the discharge. The low frequency instability is intimately tied to the ionization process, and has been shown previously to be damped when the ionization rate is set to zero. The qualitative behavior of this low frequency mode, and that at moderate frequency is in general agreement with recent experimental studies, providing further support for the validity of a multi-fluid model in describing qualitative (and in some cases, quantitative) unsteady Hall discharge behavior.

While this analysis does not provide an analytical solution for the dispersion of these ionization instabilities, it provides a test bed for determining the important plasma processes that are responsible for the growth and propagation of these instabilities (various terms in the model can be turned on and off). Future papers will report on such an exercise, so that a theoretical analysis can follow on the development of a simple analytical understanding of these ionization instabilities.

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