Results on the Optimal Detection Statistic for Integrity Monitoring

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ABSTRACT

One of the most stringent requirements of positioning systems for aviation is integrity. The provision of integrity is the main function of Satellite-based Augmentation systems (like the Wide Area Augmentation System and EGNOS), Ground Based Augmentation Systems, and Receiver Autonomous Integrity Monitoring (RAIM). All these systems have different threat models, different nominal error models, and different constraints, but they all attempt to do the same thing: maximize the availability while maintaining integrity. Despite this common problem structure, there is a wide variety of integrity algorithms, even within the case of RAIM alone.

In this paper, we focus on the choice of the detection statistics, and attempt to determine the optimal statistic given a threat model. There are three contributions in this paper. First, we cast the search of the optimal detection region as a mini-max problem. Second, we use the Neyman-Pearson lemma to limit the search of the detection regions to a class of regions parameterized by a bias. Finally, we have shown that in the case of one threat, even multi-dimensional, the optimal detection statistic is the solution separation, that is, the difference between the all-in-view solution and the solution obtained by assuming that the threat might be present.

INTRODUCTION

The main task of systems like Augmentation Systems (SBAS, GBAS) or Receiver Autonomous Integrity Monitoring (RAIM) is to provide guaranteed error bounds to the satellite navigation position solution, that is, error bounds with integrity. The error bounds must be small enough so that the service they aim to provide is available. The monitoring algorithms at the core of these systems must therefore attempt to maximize availability while maintaining integrity, or at least meet the required availability while maintaining integrity. For this reason, the design of the monitoring algorithms is in essence an optimization problem (of integrity for example) subject to constraints (the availability). The goal of this paper is to make this optimization with constraints problem explicit and provide some results on its solution.

The solution in the case of a fault parameterized by one bias is described in [1], [2], in the context of geodetic networks. Again for the one bias case, this result is presented in [3] as a direct consequence of the Neyman-Pearson lemma. However, the conditions for the Neyman-Pearson lemma are not met, as the H_1 hypothesis (fault) is not well defined. In these papers, the problem is cast in terms of a choice between H_0 or H_1, which is not the correct question in the integrity context. Although these are not the terms employed in these three papers, the suggested statistic is the square of the solution separation statistic (which coincides with the residual test). Because this result is presented as showing that the optimal statistic is a chi-square statistic, it has sometimes been misinterpreted as meaning that the optimal statistic is the chi-square of the residuals –which is a good detection statistics, but not the optimal as will be seen.

In this paper, we formulate the problem as the search of the detection region that minimizes the worst case integrity risk. As a consequence, we will generalize the results included in [1],[2],[3] in two directions. First, we consider threats that can be multidimensional, like simultaneous satellite faults. Second, we include the possibility of having heterogeneous prior probabilities for each fault mode, and attempt to define the detection region given the whole threat model. A possible starting point is given by the Neyman-Pearson lemma, which provides the best test statistic given a known faulted distribution of errors. However, as mentioned earlier, the integrity monitoring problem does not in general fulfill the conditions of this lemma. In particular, in the integrity context, the faulted distribution is typically not completely determined. Instead, it is parameterized by an unknown bias, or vector of biases. The contribution of the fault to the integrity budget is obtained by maximizing over all possible biases. This means that the Neyman Pearson lemma cannot be applied directly.

After introducing our notations, the search of the detection region is formulated as a mini-max problem. After some manipulations we apply the Neyman-Pearson lemma, so that the search is reduced to a class of regions for a very wide range of problems. Then, the exact solution is given in the case where the threat model is reduced to one (possibly multidimensional) threat, which is the solution separation statistic. Finally, we show that while the set of solution separation statistics corresponding to each threat is not the optimal statistic in the general case, it might be very close to it.

ERROR MODEL AND DEFINITIONS

Fault free error model

The fault free error model is given by the state equation:

$$y = Gx + Hx_{other} + \varepsilon \tag{1}$$

G is the geometry matrix, y is the set of measurements, and ε is the nominal noise. The vector x is the target of the integrity bounds, and x_{other} is not. For example, x_{other} could be the clock component in the case of RAIM. The vector ε is a zero mean multivariate normal random variable.

Fault error model

In this paper we are interested in the faults that can be described by adding an additional unknown in the state equation:

$$y = Gx + Hx_{other} + A_i b + \varepsilon \tag{2}$$

A_i and b are a matrix and a vector respectively. A_i is known and b is unknown (it is an additional state).

It is assumed that the measurements follow this fault error model with probability p_i. The set of fault error models and the fault free error model form a partition, that is, there is only one state at a given time.

For notation purposes, we set:

$$A_0 = 0 \tag{3}$$

$$p_0 = 1 - \sum_{i=1}^{N_{fault}} p_i$$

Integrity Risk: Probability of exceeding the error bounds and passing the test

The objective of the test is to make sure that the probability of exceeding the error bounds (Alert Limits in the case of RAIM) is below the allowable integrity risk (IR). We label Ω_{AL} the region within which the estimation error must lie with a probability of at least 1 – IR. The region of measurements for which there is no alert is labeled **Ω**. The test region **Ω** must be such that:

$$P\left(x - \hat{x} \notin \Omega_{AL}, y \in \Omega\right) \le IR \tag{4}$$

This expression can be developed as follows:

$$P\left(x - \hat{x} \notin \Omega_{AL}, y \in \Omega\right) = \sum_{i=0}^{N_{\text{fault modes}}} p_i P\left(x - \hat{x} \notin \Omega_{AL}, y \in \Omega \mid i\right) \tag{5}$$

Now we develop each term in this sum:

$$P\left(x - \hat{x} \notin \Omega_{AL}, y \in \Omega \mid i\right) = \max_{b^{(i)}} P\left(x - \hat{x} \notin \Omega_{AL}, y \in \Omega \mid i\right) \tag{6}$$

False Alert

In addition, there is a false alert requirement: under fault free conditions, the probability that the measurements are outside of Ω must not exceed the false alert budget P_{fa}:

$$P\left(y \notin \Omega \mid i = 0\right) \le P_{fa} \tag{7}$$

SEARCH OF THE DETECTION REGION AS AN OPTIMIZATION PROGRAM

The goal is to obtain a region Ω such that:

$$P\left(x-\hat{x}\notin\Omega_{AL}, y\in\Omega\right)\le IR \quad (8)$$

$$P\left(y\notin\Omega \mid i=0\right)\le P_{fa}$$

The search for a solution can be cast as the following minimization program:

$$\text{Minimize } P\left(x-\hat{x}\notin\Omega_{AL}, y\in\Omega\right) \quad (9)$$

$$\text{s.t. } P\left(y\notin\Omega \mid i=0\right)\le P_{fa}$$

If we develop the above expression, the problem appears as a mini-max problem:

$$\text{Minimize } \sum_{i=0}^{N_{\text{fault modes}}} p_i \max_{b^{(i)}} P\left(x-\hat{x}\notin\Omega_{AL}, y\in\Omega \mid i\right) \quad (10)$$

$$\text{s.t. } P\left(y\notin\Omega_{meas} \mid i=0\right)\le P_{fa}$$

The mini-max nature of the problem is more apparent if we take the maximum out of the summation:

$$\text{Minimize } \max_{b^{(i)}} \sum_{i=0}^{N_{\text{fault modes}}} p_i P\left(x-\hat{x}\notin\Omega_{AL}, y\in\Omega \mid i\right) \quad (11)$$

$$\text{s.t. } P\left(y\notin\Omega \mid i=0\right)\le P_{fa}$$

The unknown in this problem is the region Ω. From this point, the paper will examine the case where x is a scalar and:

$$\Omega_{AL}=\left[-L,L\right] \quad (12)$$

CHANGE OF VARIABLES – PROJECTION ON PARITY SPACE

Filtering out the effect of the actual position and projecting onto the parity space

The first step consists on filtering out the effect of the position on the measurements. This is done by projecting the measurements on the parity space. Let P be a matrix whose rows form a basis of the kernel of *[G H]*. After the appropriate normalization and orthogonalization, the random component can be assumed to be a vector of independent zero mean unit Gaussian distributions:

$$Py=P\left(Gx+Hx_{other}+A_i b^{(i)}+\varepsilon\right)=PA_i b^{(i)}+P\varepsilon=PA_i b_i+z \quad (13)$$

We have:

$$z\sim N\left(0,I\right) \quad (14)$$

Similarly, we have:

$$\begin{aligned}\hat{x}-x&=s^T y-x=s^T\left(Gx+Hz+A_i b^{(i)}+\varepsilon\right)-x\\&=s^T A_i b_i+s^T\varepsilon\end{aligned} \quad (15)$$

We have:

$$s^T\varepsilon\sim N\left(0, std\left(s^T\varepsilon\right)\right) \quad (16)$$

We define:

$$\eta=\frac{s^T\varepsilon}{std\left(s^T\varepsilon\right)} \quad (17)$$

Without loss of generality, we can assume that A_i is full rank and:

$$\frac{s^T A_i b^{(i)}}{std\left(s^T\varepsilon\right)}=b_1^{(i)} \quad (18)$$

This is done by performing the appropriate change of variables and parameterization of the bias b.

To lighten the notations we redefine A_i and L as follows:

$$A_i\to PA_i$$

$$L\to\frac{L}{std\left(s^T\varepsilon\right)}$$

$$y\to Py \quad (19)$$

$$\hat{x} - x \to x$$

At the end of this step, and after updating the notations, we have reduced the problem to the form:

$$\text{Minimize } \max_{b^{(i)}} \sum_{i=0}^{N_{\text{fault modes}}} p_i P\left(\left|\eta + b_1^{(i)}\right| > L, y = z + A_i b^{(i)} \in \Omega\right) \tag{20}$$

$$\text{s.t. } P\left(y \notin \Omega\right) \le P_{fa}$$

Notice that η and z are not necessarily independent. In the above expression, the no fault case has always a zero bias. We will therefore assume in the rest of the paper that :

$$b^{(0)} = 0$$

(The max is to be understood concerning i>0 only).

NEYMAN-PEARSON LEMMA

In this section, we consider the following problem:

$$\begin{aligned} &\text{minimize} \int_{\Omega} f\left(y\right) dz \\ &s.t. \int_{\Omega} g\left(y\right) dy \ge 1 - P_{fa} \end{aligned} \tag{21}$$

The solution to this problem is provided by the Neyman-Pearson lemma [4]. The optimal region **Ω** is given by:

$$\Omega = \left\{ y \mid \frac{f\left(y\right)}{g\left(y\right)} \le T \right\} \tag{22}$$

The threshold T is chosen to meet the false alert requirement:

$$\int_{\frac{f(y)}{g(y)} \le T} g\left(y\right) dy = 1 - P_{fa} \tag{23}$$

Problems (20) and (21) are very similar. There is however a complication in the first one: there is a maximum taken across the biases b. The idea to solve (20) consists on switching the min and max, so that we can use (22).

SYMMETRIZATION

The min and the max in (20) cannot be switched without manipulation, because there is no set of biases b that would produce an optimal region. In this section we show that we can modify the problem to an equivalent problem in which we can switch the min and max. Specifically, we show that under some conditions the following problem is equivalent to the original one:

$$\text{Minimize}$$
$$\max_{b^{(i)}} \frac{1}{2} \sum_{i=0}^{N_{\text{fault modes}}} p_i \begin{pmatrix} P\left(\left|\eta + b_1^{(i)}\right| > L, y = z + A_i b^{(i)} \in \Omega\right) \\ +P\left(\left|\eta - b_1^{(i)}\right| > L, y = z - A_i b^{(i)} \in \Omega\right) \end{pmatrix} \tag{24}$$

$$\text{s.t. } P\left(y \notin \Omega \mid \text{no fault}\right) \le P_{fa} \tag{25}$$

We lighten the notations by noting b the vectors $b^{(i)}$ and F the function:

$$F\left(b, \Omega\right) = \sum_{i=0}^{N_{\text{fault modes}}} p_i P\left(\left|\eta + b_1^{(i)}\right| > L, z + A_i b^{(i)} \in \Omega\right) \tag{26}$$

Problem (24) is then written:

$$\text{Minimize } \max_{b} \frac{1}{2}\left(F\left(b, \Omega\right) + F\left(-b, \Omega\right)\right) \tag{27}$$

$$\text{s.t. } P\left(z \notin \Omega\right) \le P_{fa}$$

Be Ω^* the solution to this problem, we will show that the solution to this problem is symmetric, such that:

$$F\left(b, \Omega^*\right) = F\left(-b, \Omega^*\right) \tag{28}$$

By the definition of Ω^*, for any other region Ω, there exists b such that:

$$\begin{aligned} &\max_{b'} \frac{1}{2}\left(F\left(b', \Omega^*\right) + F\left(-b', \Omega^*\right)\right) \\ &= \max_{b'} F\left(b', \Omega^*\right) \le \frac{1}{2}\left(F\left(b, \Omega\right) + F\left(-b, \Omega\right)\right) \end{aligned} \tag{29}$$

This means that there exits b such that:

$$\max_{b'} F\left(b', \Omega^*\right) \le F\left(b, \Omega\right) \tag{30}$$

As a consequence, we have:

$$\max_{b'} F\left(b',\Omega^*\right) \le \max_{b} F\left(b,\Omega\right) \quad (31)$$

Since this is true for any Ω, we have:

$$\max_{b'} F\left(b',\Omega^*\right) \le \min_{\Omega} \max_{b} F\left(b,\Omega\right) \quad (32)$$

This shows that if Ω^* is symmetric with respect to zero, it is a solution of (20) .

SWITCHING MIN AND MAX

In this section, we treat the problem:

$$\max_{b} \min_{\substack{\Omega \\ s.t. P(z\notin\Omega)\le P_{fa}}} \frac{1}{2}\left(F\left(b,\Omega\right)+F\left(-b,\Omega\right)\right) \quad (33)$$

That is, for each set of biases b, we find the optimal detection region Ω_b, solution of the problem:

$$\min_{\substack{\Omega \\ s.t. P(z\notin\Omega)\le P_{fa}}} \frac{1}{2}\left(F\left(b,\Omega\right)+F\left(-b,\Omega\right)\right) \quad (34)$$

Then, we maximize over the bias b. We then will consider the argument b_{max} that maximizes the function:

$$b \mapsto \frac{1}{2}\left(F\left(b,\Omega_b\right)+F\left(-b,\Omega_b\right)\right) \quad (35)$$

We will show that $\Omega_{b_{\max}}$ is a solution of (20). For now, we go back to (34) and cast it such that we can use (22). We have:

$$\begin{aligned} P\left(\left|\eta+b_1^{(i)}\right| > L, z+A_i b^{(i)} \in \Omega\right) &= \int_{\substack{\left|\eta+b_1^{(i)}\right|>L \\ z+A_i b^{(i)}\in\Omega}} \phi(\eta,z)\, d\eta dz \\ &= \int_{\substack{|\eta|>L \\ y\in\Omega}} \phi\left(\eta-b_1^{(i)}, y-A_i b^{(i)}\right) d\eta dy \\ &= \int_{y\in\Omega}\left(\int_{|\eta|>L} \phi\left(\eta-b_1^{(i)}, y-A_i b^{(i)}\right) d\eta\right) dy \end{aligned} \quad (36)$$

In these formulas $\phi(\eta,z)$ is the joint density of the random variables η and z.

We note:

$$g_{A_i}\left(y,b^{(i)}\right) = p_i \int_{|\eta|>L} \phi\left(\eta-b_1^{(i)}, y-A_i b^{(i)}\right) d\eta \quad (37)$$

And:

$$P\left(y\in\Omega\right) = \int_{\Omega} \varphi(y)\, dy \quad (38)$$

The function φ is the density of a zero mean unit multivariate Gaussian (*N(0,I)*).

With these notations, (34) is written:

$$\begin{aligned} &\text{minimize} \int_{z\in\Omega} \frac{1}{2}\left(\sum_{i=0}^{N_{\text{fault modes}}} g_{A_i}\left(y,b^{(i)}\right)+g_{A_i}\left(y,-b^{(i)}\right)\right) dz \\ &s.t. \int_{\Omega} \varphi(y)\, dy \ge 1-P_{fa} \end{aligned} \quad (39)$$

Under this form, we can use (22). The optimal region Ω_b is given by:

$$\Omega_b = \left\{ y \mid \frac{\frac{1}{2}\left(\sum_{i=0}^{N_{\text{fault modes}}} g_{A_i}\left(y,b^{(i)}\right)+g_{A_i}\left(y,-b^{(i)}\right)\right)}{\varphi(y)} \le T \right\} \quad (40)$$

Again, the threshold T is chosen to meet the false alert requirement.

Symmetry of Ω_b

The functions $g_{A_i}\left(y,b^{(i)}\right)$ and *φ(y)* are symmetric with respect to zero:

$$\begin{aligned} g_{A_i}\left(-z,b^{(i)}\right) &= p_i \int_{|\eta|>L} \phi\left(\eta-b_1^{(i)}, -z-A_i b^{(i)}\right) d\eta \\ &= p_i \int_{|\eta|>L} \phi\left(-\eta+b_1^{(i)}, z+A_i b^{(i)}\right) d\eta = p_i \int_{|\eta|>L} \phi\left(\eta+b_1^{(i)}, z+A_i b^{(i)}\right) d\eta \\ &= g_{A_i}\left(z,-b^{(i)}\right) \end{aligned} \quad (41)$$

This means that Ω_b is symmetric with respect to zero.

Convexity of Ω_b

For a fixed η and b, the following function is convex:

$$y \mapsto \frac{\phi\left(\eta - b_1^{(i)}, y - A_i b^{(i)}\right)}{\varphi(y)} \quad (42)$$

(It is the exponential of a linear function of z). As a consequence, the integral over η is also convex:

$$y \mapsto \frac{\int_{|\eta|>L} \phi\left(\eta - b_1^{(i)}, y - A_i b^{(i)}\right)}{\varphi(y)} = \frac{g_{A_i}\left(y, b^{(i)}\right)}{\varphi(y)} \quad (43)$$

For the same reason, the function:

$$y \mapsto \frac{\frac{1}{2}\left(\sum_{i=0}^{N_{\text{fault modes}}} g_{A_i}\left(y, b^{(i)}\right) + g_{A_i}\left(y, -b^{(i)}\right)\right)}{\varphi(y)} \quad (44)$$

is convex. The region Ω_b is a sublevel set of this function, so it is convex [5].

Unimodality as a function of b

We now consider $\Omega_{b_{\max}}$ as defined in (35) and show that the following function is unimodal in a region that will be defined below:

$$\Gamma : b \mapsto \frac{1}{2}\left(F\left(b, \Omega_{b_{\max}}\right) + F\left(-b, \Omega_{b_{\max}}\right)\right) \quad (45)$$

Since $\Omega_{b_{\max}}$ is symmetric, we have:

$$\frac{1}{2}\left(F\left(b, \Omega_{b_{\max}}\right) + F\left(-b, \Omega_{b_{\max}}\right)\right) = F\left(b, \Omega_{b_{\max}}\right) \quad (46)$$

We write:

$$F\left(b, \Omega_{b_{\max}}\right) = \sum_{i=0}^{N_{\text{fault modes}}} p_i P\left(\left|\eta + b_1^{(i)}\right| > L, y = z + A_i b^{(i)} \in \Omega_{b_{\max}}\right)$$
$$= \sum_{i=0}^{N_{\text{fault modes}}} p_i \int_{\substack{|\eta|>L \\ y\in\Omega}} \phi\left(\eta - b_1^{(i)}, y - A_i b^{(i)}\right) d\eta dz \quad (47)$$

We will consider this function for the biases b such that:

$$b_1^{(i)} \geq 0 \quad (48)$$

At this point, we make an approximation which simplifies the derivation (but should not change the conclusions). We write that:

$$F\left(b, \Omega_{b_{\max}}\right) = p_0 \int_{\substack{|\eta|>L \\ z\in\Omega}} \phi(\eta, z) d\eta dz +$$
$$\sum_{i=1}^{N_{\text{fault modes}}} p_i \int_{\substack{|\eta|>L \\ y\in\Omega}} \phi\left(\eta - b_1^{(i)}, y - A_i b^{(i)}\right) d\eta dy$$
$$\simeq \int_{\substack{|\eta|>L \\ y\in\Omega}} \phi(\eta, y) d\eta dy + \sum_{i=1}^{N_{\text{fault modes}}} p_i \int_{\substack{\eta>L \\ y\in\Omega}} \phi\left(\eta - b_1^{(i)}, y - A_i b^{(i)}\right) d\eta dy \quad (49)$$

The idea is here is that the left hand side contribution in the integral will not have any effect on the unimodality of the function in the domain we are interested in, so that we only need to consider the right hand side. A formal proof of this fact is not included here, so we rely on (49). A justification for (49) can be found in [6].

The function:

$$\eta, z, b^{(i)} \mapsto \phi\left(\eta - b_1^{(i)}, y - A_i b^{(i)}\right) I\left(\eta > L, y \in \Omega_{b_{\max}}\right) \quad (50)$$

is log-concave (it is the product of two log-concave functions [5]). The indicator function is log-concave because its support is convex. According to [5], the marginal distribution obtained by integrating over η and y is also log-concave. That is, the following function is log-concave:

$$b^{(i)} \mapsto \int \phi\left(\eta - b_1^{(i)}, y - A_i b^{(i)}\right) I\left(\eta > L, y \in \Omega_{b_{\max}}\right) d\eta dy$$
$$= \int_{\substack{\eta>L \\ y\in\Omega}} \phi\left(\eta - b_1^{(i)}, y - A_i b^{(i)}\right) d\eta dy \quad (51)$$

As a consequence, it is unimodal: there is only one extremum and it is the global maximum. This means that the function:

$$\Gamma : b \mapsto F\left(b, \Omega_{b_{\max}}\right) \simeq \int_{\substack{|\eta|>L \\ y\in\Omega_{b_{\max}}}} \phi(\eta, y) d\eta dy +$$
$$\sum_{i=1}^{N_{\text{fault modes}}} p_i \int_{\substack{\eta>L \\ y\in\Omega_{b_{\max}}}} \phi\left(\eta - b_1^{(i)}, y - A_i b^{(i)}\right) d\eta dy \quad (52)$$

is also unimodal and has only one extremum (Γ is separable in each $b^{(i)}$).

Proof that b_{max} maximizes Γ

We have shown that Γ has only one extremum, and it is the global maximum. In this paragraph we show that the maximum is reached at b_{max}. Let us consider the function:

$$\theta : b \mapsto F\left(b, \Omega_b\right) \tag{53}$$

At b_{max} we have:

$$\frac{\partial \theta}{\partial b}\left(b_{\max}\right) = \frac{\partial F}{\partial b}\left(b_{\max}, \Omega_{b_{\max}}\right) + \frac{\partial \Omega}{\partial b}\frac{\partial F}{\partial \Omega}\left(b_{\max}, \Omega_{b_{\max}}\right) = 0 \tag{54}$$

The definition of $\mathbf{\Omega}_b$ guarantees that:

$$\frac{\partial F}{\partial \Omega}\left(b_{\max}, \Omega_{b_{\max}}\right) = 0 \tag{55}$$

As a consequence, we have:

$$\frac{\partial F}{\partial b}\left(b_{\max}, \Omega_{b_{\max}}\right) = 0 \tag{56}$$

Or equivalently:

$$\frac{\partial \Gamma}{\partial b}\left(b_{\max}\right) = 0 \tag{57}$$

This means that b_{max} is the only extremum in the domain of interest and that therefore, b_{max} is the global maximum.

Proof that $\Omega_{b_{\max}}$ is a solution to the original problem

Be Ω^* the solution of (24). By the definition of $\Omega_{b_{\max}}$ we have:

$$F\left(b_{\max}, \Omega_{b_{\max}}\right) \le \frac{1}{2}\left(F\left(b_{\max}, \Omega^*\right) + F\left(-b_{\max}, \Omega^*\right)\right) \tag{58}$$

As a consequence we have:

$$\begin{aligned} F\left(b_{\max}, \Omega_{meas, b_{\max}}\right) &\le \max_b \frac{1}{2}\left(F\left(b, \Omega^*\right) + F\left(-b, \Omega^*\right)\right) \\ &= \min_\Omega \max_b \frac{1}{2}\left(F\left(b, \Omega\right) + F\left(-b, \Omega\right)\right) \end{aligned} \tag{59}$$

We have shown in the previous paragraph that for any bias b:

$$F\left(b, \Omega_{b_{\max}}\right) \le F\left(b_{\max}, \Omega_{b_{\max}}\right) \tag{60}$$

As a consequence we have:

$$\max_b F\left(b, \Omega_{b_{\max}}\right) \le F\left(b_{\max}, \Omega_{b_{\max}}\right) \tag{61}$$

This implies that:

$$\min_\Omega \max_b F\left(b, \Omega\right) \le \max_b F\left(b, \Omega_{b_{\max}}\right) \le F\left(b_{\max}, \Omega_{b_{\max}}\right) \tag{62}$$

And therefore:

$$\min_\Omega \max_b F\left(b, \Omega\right) = \max_b F\left(b, \Omega_{b_{\max}}\right) \tag{63}$$

As a consequence, the region $\Omega_{b_{\max}}$ is a solution of problem (24). Since $\Omega_{b_{\max}}$ is symmetric with respect to zero, it is also a solution of the original problem (20).

Summary

We have shown that the solution $\mathbf{\Omega}_{opt}$ of problem (20) is such that:

$$\Omega_{opt} = \left\{ y \mid \frac{\frac{1}{2}\left(\sum_{i=0}^{N_{\text{fault modes}}} g_{A_i}\left(y, b_{\max}^{(i)}\right) + g_{A_i}\left(y, -b_{\max}^{(i)}\right)\right)}{\varphi(y)} \le T \right\} \tag{64}$$

for a set of biases b_{max}. Although this result does not say what the biases b_{max} are, it does show the general shape of the optimal detection region. Figure 1 shows the contours of the function defining the region for a two dimensional y and assuming that η and z are decorrelated. Appendix B gives more information on the biases b_{max} in the case where η and z are decorrelated.

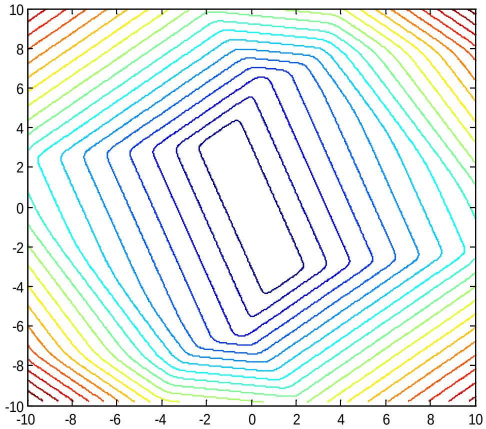

Figure 1. Contours of the function defining the region Ω_b in a two dimensional case

CASE OF ONE THREAT

In this section, we analyze the case where there is only one threat. The problem can then be written:

$$\text{Minimize } \max_b P\left(\left|\eta + b_1\right| > L, y = z + Ab \in \Omega\right) \tag{65}$$

$$\text{s.t. } P\left(y \notin \Omega \mid \text{no fault}\right) \le P_{fa}$$

The shape of the solution is given by (64). There exists $b_{\max}$ such that:

$$\Omega_{opt} = \left\{ y \mid \frac{\frac{1}{2}\left(g_A\left(y, b_{\max}\right) + g_{A_i}\left(y, -b_{\max}\right)\right)}{\varphi\left(y\right)} \le T \right\} \tag{66}$$

Let us assume that η and z are decorrelated (which is the most common case in RAIM). We have:

$$\begin{aligned} g_A\left(y,b\right) &= p \int_{|\eta|>L} \phi\left(\eta - b_1, y - Ab\right) d\eta \\ &= p\text{Prob}\left(\left|\eta - b_1\right| \ge L\right)\varphi\left(y - Ab\right) \end{aligned} \tag{67}$$

Now we perform a change of variables as follows. We define:

$$h_1^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}\left(A^T A\right)^{-1} A^T \tag{68}$$

We perform the following change of variables:

$$w_1 = \frac{h_1^T y}{\sigma_{b_1}} \tag{69}$$

$$\sigma_{b_1} = sqrt\left(A^T A\right)^{-1}_{1,1} \tag{70}$$

The purpose of this change of variable is to project the measurement y on the best estimate of b_1. The normalization is there to keep a unit Gaussian. We now complete h_1 into an orthonormal basis by h_2 etc and define for k>1:

$$w_k = h_k^T y \tag{71}$$

We also redefine b_k for k>1 (not for k=1) as:

$$b_k' = h_k^T b_k \tag{72}$$

After this change of variables (and dropping the prime to lighten the notations) we have (φ designates a zero mean unit Gaussian of the appropriate dimension):

$$g_A\left(w,b\right) = p\text{Prob}\left(\left|\eta - b_1\right| \ge L\right)\varphi\left(w_1 - \frac{1}{\sigma_{b_1}} b_1\right)\varphi\left(w_2 - b_2\right)\cdots\varphi\left(w_n - b_n\right) \tag{73}$$

The vector w is also a unit Gaussian with mean:

$$q = E\left(w\right) = \begin{bmatrix} \frac{1}{\sigma_{b_1}} b_{\max,1} & b_{\max,2} & \cdots & b_{\max,n} \end{bmatrix}^T \tag{74}$$

We have:

$$\Omega_{opt} = \left\{ z \mid \frac{\frac{1}{2}\left(p\text{Prob}\left(\left|\eta - b_{\max,1}\right| \ge L\right)\varphi\left(w - q\right) + p\text{Prob}\left(\left|\eta + b_{\max,1}\right| \ge L\right)\varphi\left(w + q\right)\right)}{\varphi\left(w\right)} \le T \right\} \tag{75}$$

It turns out that this region has a much simpler expression. We have:

$$\text{Prob}\left(\left|\eta - b_1\right| \geq L\right) = \text{Prob}\left(\left|\eta + b_1\right| \geq L\right) \quad (76)$$

We can redefine the threshold so that:

$$\Omega_{opt} = \left\{ w \mid \frac{\varphi(w-q)+\varphi(w+q)}{\varphi(w)} \leq T \right\} \quad (77)$$

We have:

$$\frac{\varphi(w-q)+\varphi(w+q)}{\varphi(w)} = f\left(q^T w\right) \quad (78)$$

where *f* is convex and symmetric, so that the sublevel sets are symmetric intervals. There exists a threshold τ such that:

$$\Omega_{opt} = \left\{ w \mid \left|q^T w\right| \leq \tau \right\} \quad (79)$$

Without loss of generality, we can assume that the norm of q is one. The threshold τ is then only a function of the false alert requirement. In what follows, we show that we necessarily have:

$$q_k = 0 \text{ for } k \geq 2$$

We note:

$$\beta = \begin{bmatrix} \frac{1}{\sigma_{b_1}} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$$

$$F\left(b, \Omega_{opt}\right) = p\text{Prob}\left(\left|\eta - b_1\right| \geq L\right) \int_{\left|q^T w\right| \leq \tau} \varphi(w-\beta)\, dw \quad (80)$$

We have:

$$\int_{\left|q^T w\right| \leq \tau} \varphi(w-\beta)\, dw = Q\left(\tau + q^T\beta\right) - Q\left(-\tau + q^T\beta\right) \quad (81)$$

It can be verified that if we had q_k or k>1 different than zero we would have:

$$\max_b F\left(b, \Omega_{opt}\right) = Q(\tau) - Q(-\tau) \quad (82)$$

(b_1 can be made to go to infinity while keeping $q^T \beta = 0$). On the other hand, if $q_k = 0$ for $k \geq 2$, we have:

$$F\left(b, \Omega_{opt}\right) = p\text{Prob}\left(\left|\eta - b_1\right| \geq L\right) \left(Q\left(\tau + \frac{1}{\sigma_{b_1}} b_1\right) - Q\left(-\tau + \frac{1}{\sigma_{b_1}} b_1\right) \right) < Q(\tau) - Q(-\tau) \quad (83)$$

We therefore have:

$$\Omega_{opt} = \left\{ w \mid \left|w_1\right| \leq \tau \right\} \quad (84)$$

The above equation means that we need to perform a threshold test on w_1, the best estimate of the error (normalized by its standard deviation). Appendix B shows an alternate proof.

Summary

We have shown that the best statistic in the case of one threat (even multidimensional) is a threshold test on the best estimate of the user error given the measurements. This result holds when the nominal error is decorrelated from the nominal noise on the measurements. Appendix A suggests that the correlated case appears to be more complex.

BEST ESTIMATE OF ERROR GIVEN THE MEASUREMENTS: SOLUTION SEPARATION STATISTIC

In order to complete the previous result, we show that the best estimate of the error given the measurements is the solution separation statistic. The next derivation also works in the case where the user error is correlated with the measurements. The notations in this section are different than in the previous ones.

$$y = Gx + Ab + \varepsilon \quad (85)$$

$$z = s^T\left(\varepsilon + Ab\right)$$

We would like to test *y* to test whether *z* exceeds a certain limit. As shown in the previous section, the optimal choice for this is the best estimate of *z* given *y*.

We use a Minimum Mean Square [7] approach with:

$$x \sim N\left(0, C_X\right) \tag{86}$$

$$b \sim N\left(0, C_B\right)$$

$$\varepsilon \sim N\left(0, C_\varepsilon\right)$$

We note:

$$C_Y = \operatorname{cov}(y) \tag{87}$$

$$C_{ZY} = \operatorname{cov}(z, y)$$

Taking into account that the means are all zero, we have:

$$\hat{z} = C_{ZY} C_Y^{-1} y \tag{88}$$

We now replace the above matrices by their expression as a function of the given parameters. We have:

$$C_Y = \operatorname{cov}(y) = G C_X G^T + A C_B A^T + C_\varepsilon \tag{89}$$

$$C_{ZY} = \operatorname{cov}(z, y) = s^T \left(A C_B A^T + C_\varepsilon\right) \tag{90}$$

Therefore:

$$C_{ZY} C_Y^{-1} = s^T \left(A C_B A^T + C_\varepsilon\right)\left(G C_X G^T + A C_B A^T + C_\varepsilon\right)^{-1} \tag{91}$$

At this point, we would like to make C_B and C_X go to infinity, because we have no prior information on b and x. To do this, we change the form of the above formula:

$$\begin{aligned} & s^T \left(A C_B A^T + C_\varepsilon\right)\left(G C_X G^T + A C_B A^T + C_\varepsilon\right)^{-1} = \\ & s^T - s^T G C_X G^T \left(G C_X G^T + A C_B A^T + C_\varepsilon\right)^{-1} \end{aligned} \tag{92}$$

We now use the following general formula:

$$\Sigma R^T \left(R \Sigma R^T + \Gamma\right)^{-1} = \left(R^T \Gamma^{-1} R + \Sigma^{-1}\right)^{-1} R^T \Gamma^{-1} \tag{93}$$

With:

$$R = \begin{bmatrix} G & A \end{bmatrix} \tag{94}$$

$$\Sigma = \begin{bmatrix} C_X & 0 \\ 0 & C_B \end{bmatrix}$$

$$\Gamma = C_\varepsilon$$

We get:

$$\begin{aligned} & \begin{bmatrix} C_X G^T \\ C_B A^T \end{bmatrix} \left(G C_X G^T + A C_B A^T + C_\varepsilon\right)^{-1} = \\ & \left(\begin{bmatrix} G^T \\ A^T \end{bmatrix} C_\varepsilon^{-1} \begin{bmatrix} G & A \end{bmatrix} + \begin{bmatrix} C_X^{-1} & 0 \\ 0 & C_B^{-1} \end{bmatrix} \right)^{-1} \begin{bmatrix} G^T \\ A^T \end{bmatrix} C_\varepsilon^{-1} \end{aligned} \tag{95}$$

$$\begin{aligned} & \begin{bmatrix} C_X G^T \left(G C_X G^T + A C_B A^T + C_\varepsilon\right)^{-1} \\ C_B A^T \left(G C_X G^T + A C_B A^T + C_\varepsilon\right)^{-1} \end{bmatrix} = \\ & \left(\begin{bmatrix} G^T C_\varepsilon^{-1} G & G^T C_\varepsilon^{-1} A \\ A^T C_\varepsilon^{-1} G & A^T C_\varepsilon^{-1} A \end{bmatrix} + \begin{bmatrix} C_X^{-1} & 0 \\ 0 & C_B^{-1} \end{bmatrix} \right)^{-1} \begin{bmatrix} G^T C_\varepsilon^{-1} \\ A^T C_\varepsilon^{-1} \end{bmatrix} \end{aligned}$$

We can now make C_B and C_X go to infinity, so that we get:

$$\begin{aligned} & \begin{bmatrix} C_X G^T \left(G C_X G^T + A C_B A^T + C_\varepsilon\right)^{-1} \\ C_B A^T \left(G C_X G^T + A C_B A^T + C_\varepsilon\right)^{-1} \end{bmatrix} = \\ & \left(\begin{bmatrix} G^T C_\varepsilon^{-1} G & G^T C_\varepsilon^{-1} A \\ A^T C_\varepsilon^{-1} G & A^T C_\varepsilon^{-1} A \end{bmatrix} \right)^{-1} \begin{bmatrix} G^T C_\varepsilon^{-1} \\ A^T C_\varepsilon^{-1} \end{bmatrix} \end{aligned} \tag{96}$$

The right hand term is the matrix giving the least squares estimate of x (in the presence of the fault Ab) and b as a function of y. That is, we have:

$$\begin{bmatrix} \hat{x}_A \\ \hat{b} \end{bmatrix} = H_A y \tag{97}$$

$$H_A = \left(\begin{bmatrix} G^T C_\varepsilon^{-1} G & G^T C_\varepsilon^{-1} A \\ A^T C_\varepsilon^{-1} G & A^T C_\varepsilon^{-1} A \end{bmatrix} \right)^{-1} \begin{bmatrix} G^T C_\varepsilon^{-1} \\ A^T C_\varepsilon^{-1} \end{bmatrix} \tag{98}$$

This means that the best estimate of z is given by:

$$\hat{z} = C_{ZY} C_Y^{-1} y = s^T y - s^T G \hat{x}_A = s^T y - \hat{x}_A^v \tag{99}$$

This last equation shows that is coincides with the solution separation of the all-in-view solution and the solution assuming the presence of the fault.

COMMENTS ON THE GENERAL SOLUTION

The general solution as illustrated in Figure 1 appears to be very close to the intersection of half-spaces defined by the solution separation statistics. Appendix B shows that in the case where the user error and the nominal measurement error are decorrelated the optimal detection region is actually a function of the solution separation statistic for each threat. Each term in the function defining the detection region is akin to the term in the single threat case.

CONCLUSION

There are three contributions in this paper. First, we cast the search of the optimal detection region as a mini-max problem. Second, we have used the Neyman-Pearson lemma to limit the search of the detection regions to a class of regions parameterized by a bias. Finally, we have shown that in the case of one threat, even multi-dimensional, the optimal detection statistic is the solution separation, that is, the difference between the all-in-view solution and the solution obtained by assuming that the threat might be present. The one threat case shows that the multiple threat case might be very well approximated by the solution separation statistics corresponding to each possible threat.

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APPENDIX A

Be c the correlation between η and w. We have:

$$g_A(w,b) = \text{constant}$$
$$\left(Q\left(L - b_1 + c^T(w-b)\right) + Q\left(L + b_1 - c^T(w-b)\right)\right)$$
$$\exp\left(-\frac{1}{2}\left((w-b)^T(w-b) - \left(c^T w\right)^2\right)\right)$$

Taking Equation (66) we can see that the detection region in the correlated case will depend $b^T w$ but also on the projection $c^T w$.

APPENDIX B

In the decorrelated case and after the appropriate change of variables (F_i being the contribution of each term in the sum (39)), we have:

$$F_i\left(b^{(i)},\Omega\right) = \left(Q\left(L - b_1^{(i)}\right) + Q\left(L + b_1^{(i)}\right)\right) \int_\Omega \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(w - b^{(i)}\right)^T\left(w - b^{(i)}\right)\right) dw \quad (100)$$

Because of the symmetry of Ω, we have:

$$F_i\left(\begin{bmatrix} b_1^{(i)} & b_2^{(i)} & \cdots & b_n^{(i)} \end{bmatrix},\Omega\right) = F_i\left(\begin{bmatrix} b_1^{(i)} & -b_2^{(i)} & \cdots & -b_n^{(i)} \end{bmatrix},\Omega\right)$$

Using the same argument as above, F_i is log concave in the variable $\begin{bmatrix} b_2^{(i)} & \cdots & b_n^{(i)} \end{bmatrix}$. Since it is also symmetric it means that the maximum is attained at

$$\begin{bmatrix} b_2^{(i)} & \cdots & b_n^{(i)} \end{bmatrix} = 0$$

According to (56) this means that we have

$$\begin{bmatrix} b_{\max,2}^{(i)} & \cdots & b_{\max,n}^{(i)} \end{bmatrix} = 0$$

(In the coordinate frame where (100) can be written). This means that each term in the definition of the detection region is a function of the solution separation statistic.

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