Lower Bounds in Optimal Integrity Monitoring

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ABSTRACT

The goal of integrity monitoring in positioning algorithms consists in finding a test statistic and an estimator that meets both the integrity requirements and the alert requirements under a set of conditions. The search for such test statistics can be cast as an optimization problem where the goal is to minimize the integrity risk while maintaining the alert requirements. In this work, we provide results that extend previous results in two ways. First, we provide a lower bound on the integrity risk for linear unbiased estimators (but not necessarily optimal). Second we provide a lower on the integrity risk in the case of fault detection and exclusion. The results developed in this work here are general. In particular, they are applicable to both snapshot solutions and Kalman filter solutions, and to any combination of sensors.

INTRODUCTION

Until recently, integrity monitoring in radio-navigation was mostly limited to aircraft navigation. It is now being expanded to automotive, rail, and maritime applications [1], [2], [3]. Given the increased awareness of GNSS threats (like spoofing), it is likely that integrity monitoring will pervade most navigation systems. There are many different types of integrity monitoring algorithms, each responding to a different design constraint. In all cases, it can be useful to know what the minimum achievable integrity risk is, for at least three reasons. First, these bounds tell us whether it makes sense to continue improving the algorithm; second, the search itself shows us which class of algorithms will likely perform well, third, if the lower limit is too high, then we know that we should be looking somewhere else to achieve the desired performances (like more measurements, additional structure, or more constraints on the fault modes).

Finding the optimal integrity monitoring algorithm is in general a very difficult problem. It is however possible to define tractable problems that can be proven to provide a lower bound on the achievable integrity risk in the original problems. Optimality results in the range domain pre-date the development of integrity in GNSS ([4], [5]). These results have been adapted to GNSS in at least [6]. In [7], we proved that in the case of one threat, even multi-dimensional, the optimal detection statistic is the solution separation statistic. This was achieved by casting the search of the optimal detection region as a mini-max problem, and using the Neyman-Pearson lemma to limit the search of the detection regions to a class of regions parameterized by a bias. These results allowed us to establish a lower bound on the minimum integrity risk. However, these results were only proven for least squares estimators and for the detection problem only.

In this work, we expand and generalize the theoretical results from [7] in two directions. First, we provide lower bounds on the lowest possible achievable integrity risk given a set of measurements and a threat space in the case of linear estimators (but not necessarily least squares); second, we will consider the case of fault detection and exclusion with non-linear estimators. For this case, we will show that, to obtain optimality results, it is useful to generalize the fault detection and exclusion process. After introducing notations, definitions, and previous results, we provide two inequalities that place a lower bound on the optimal integrity given an alert probability.

ERROR MODEL AND DEFINITIONS

Fault free error model

In this paper, we will assume that the linear approximation holds. The fault free error model is given by the state equation:

$$y = Gx + \varepsilon_y \tag{1}$$

where:

G is the geometry matrix (n by p) (p is 3 plus the number of clock states)

y is the set of measurements (n by 1)

ε_y is the nominal noise (n x 1)

x is the position and clock unknowns.

The nominal noise follows a zero mean Gaussian distribution with covariance W^{-1} :

$$\varepsilon_y \sim N\left(0, W^{-1}\right) \tag{2}$$

Fault error model

The fault error model is the one adopted in [7] which generalizes the fault modes used in RAIM. In this model, the measurements are determined by one error model, and one only, out of N +1 possible error models. Each of these error models, or hypothesis, has a known probability of occurrence p_{Hi} and corresponds to the addition of an unknown state in the measurement equation:

$$y = Gx + A^{(i)} b^{(i)} + \varepsilon_y \tag{3}$$

$A^{(i)}$ and $b^{(i)}$ are an n by m_i matrix and a m_i by 1 vector respectively. A_i is known, and $b^{(i)}$ is arbitrary. In the rest of the paper, we will assume that the matrix $[G\ A_i]$ is full rank and that $n>p+m_i-1$. If there is no change of variable on the nuisance parameters that can make the matrix $[G\ A_i]$ full rank, then the fault cannot be monitored (this would happen for example if $A^{(i)} = G$). Similarly, if the system of equations (3) is underdetermined, which will happen if $n<p+m_i$, then the fault cannot be monitored.

The fault free case corresponds to $i = 0$.

OPTIMAL DETECTION REGION

The design of the integrity algorithm is therefore equivalent to the determination of a detection region *Ω* such that:

$$P\left(\left|x_k - \hat{x}_k\right| \geq L, y \in \Omega\right) \leq P_{HMI} \tag{4}$$

Where:

$\hat{x}$ is the estimate of x obtained from the measurements y

L is the Alert Limit

P_{HMI} is the required integrity risk

In addition, there is a false alert requirement: under fault free conditions, the probability that the measurements are outside of *Ω* must not exceed the false alert budget P_{fa}:

$$P\left(y \notin \Omega \mid H_0\right) \leq P_{fa} \tag{5}$$

The optimal detection region can be defined as the region that minimizes the integrity risk given a false alarm rate, that is, it is the solution to the optimization problem:

$$\text{Minimize } P\left(\left|x_k - \hat{x}_k\right| \geq L, y \in \Omega\right) \tag{6}$$

$$\text{s.t. } P\left(y \notin \Omega \mid i = 0\right) \leq P_{fa}$$

PREVIOUS RESULT

From the results shown in [7], the most useful one concerned the case with one multidimensional fault mode and where the all-in-view solution is the optimal one under fault free conditions. This result allowed us to compute a lower bound on the achievable integrity in the case with multiple faults:

Optimal detection region for one multi-dimensional threat with a least squares all-in-view solution

For a fixed false alarm probability, a detection region that minimizes the integrity risk when only one threat is considered *(N=1)* is given by:

$$\Omega^* = \left\{ y \mid \left| \left(s_k^T - s_k^{(i)T} \right) y \right| \leq T \right\} \tag{7}$$

where $s_k^{(i)T}$ is the k^th^-row of the least squares estimator of x_k assuming the measurement model fault $S^{(i)}$(3):

$$S^{(i)} = \left(\begin{bmatrix} G^T \\ A^{(i)T} \end{bmatrix} W \begin{bmatrix} G & A^{(i)} \end{bmatrix} \right)^{-1} \begin{bmatrix} G^T \\ A^{(i)T} \end{bmatrix} W \tag{8}$$

The threshold T is set to meet the false alarm requirement (P_{fa}):

$$T = \sigma_{ss}^{(i)} Q^{-1}\left(\frac{P_{fa}}{2}\right) \tag{9}$$

$$\sigma_{ss}^{(i)} = \sqrt{\left(s_k^T - s_k^{(i)T}\right) W^{-1} \left(s_k^T - s_k^{(i)T}\right)}$$

This result means that the optimal detection statistics is the solution separation between the all-in-view solution and the least squares solution that is immune to the fault mode. When $A^{(i)}$ corresponds to the addition of independent biases to a set of satellites, the least squares solution immune to the fault mode is the least squares solution that excludes the satellites affected by the fault mode.

One of the goals of this paper is to extend this result when the all-in-view estimator is not necessarily the optimal one for accuracy (but still a linear one). This is important because in some cases, it is useful to offset the all-in-view position solution from the most accurate solution to improve integrity ([8],[9],[10],[11],[12],[13]).

LOWER BOUND ON OPTIMAL INTEGRITY FOR LINEAR ESTIMATORS

In this section, we provide a lower bound on the optimal integrity when the all-in-view estimator is a linear unbiased estimator (which covers [8],[9],[10],[11],[12],[13]), that is:

$$\hat{x} = Sy \tag{10}$$

where S produces an unbiased estimate [8]:

$$E(Sy) = x \tag{11}$$

which implies that:

$$SG = I \tag{12}$$

There are two steps in this process: first we develop a lower bound as a function of the integrity achieved by the solution separation statistic.

We consider the detection region defined as in (7), but with a non-least squares estimator s_{NLS}:

$$\Omega^{SS}\left(P_{fa}\right) = \left\{ y \mid \left| \left(s_{NLS,k}^T - s_k^{(i)T} \right) y \right| \le T_{fa} \right\} \tag{13}$$

where the threshold T_{fa} is set to meet a false alert of T_{fa}. Note that this detection region is not necessarily optimal.

We have the following result:

$$\min_{P(y\notin\Omega|i=0)\le P_{fa}} \max_{\theta} P\left(\left|x_k-\hat{x}_{NLS,k}\right|\ge L, y\in\Omega\,|\,\theta\right)\ge$$
$$\max_{\theta} P\left(\left|x_k-\hat{x}_{NLS,k}\right|\ge L, y\in\Omega^{SS}\left(P_{fa,1}\right)|\,\theta\right)-2Q\left(\frac{L-T_{fa,1}}{\sigma_i}\right)\frac{P_{fa}}{P_{fa,1}} \quad (14)$$

This lower bound is weak, because it is negative in some cases. However, for small P_{fa} values, it is possible to find values of $P_{fa,1}$ that place a strictly positive lower bound on the achievable integrity.

The next bound is only a conjecture, since we did not manage to prove it, (but believe it to be true):

$$\min_{P(y\notin\Omega|i=0)\le P_{fa}} \max_{\theta} P\left(\left|x_k-\hat{x}_{NLS,k}\right|\ge L, y\in\Omega\,|\,\theta\right)\ge\left(1-\frac{2P_{fa}}{P_{fa,1}}\right)P\left(\left|x_k-\hat{x}_{NLS,k}\right|\ge L, y\in\Omega^{SS}\left(P_{fa,1}\right)\right) \quad (15)$$

This is a much tighter bound, and is always strictly positive.

Elements for the proof

The integrity risk can be written as follows:

$$P\left(|\varepsilon|>L,\left(\hat{\varepsilon}(y),z\right)\in\Omega\,|\,\theta\right)=\int_z P\left(|\varepsilon|>L,\hat{\varepsilon}(y)\in\Omega_z\,|\,z,\theta\right)p(z)\,dz \quad (16)$$

Where:

ε is the position error

$\hat{\varepsilon}$ is the estimate of the errors given the measurements (which was proven to be the solution separation statistic)

z completes $\hat{\varepsilon}$ into a basis of the parity space

θ is the fault bias state (noted $b^{(i)}$ above)

We start by considering a parameter θ^* that realizes the maximum of the integrity risk for the optimal integrity region Ω^*. We have:

$$\min_{P(y\notin\Omega|i=0)\le P_{fa}} \max_{\theta} P\left(|\varepsilon|\ge L, y\in\Omega\,|\,\theta\right)=P\left(|\varepsilon|\ge L, y\in\Omega^*\,|\,\theta^*\right) \quad (17)$$

We now consider the region that is optimal for the parameter θ^* only, which we label $\Omega(\theta^*)$. We have by definition:

$$P\left(|\varepsilon|\ge L, y\in\Omega\left(\theta^*\right)|\,\theta^*\right)\le P\left(|\varepsilon|\ge L, y\in\Omega^*\,|\,\theta^*\right) \quad (18)$$

In what follows, we develop a lower bound of the left hand side term.

First, it can be shown that for the optimal detection region, there is a lower threshold dependent on z, $T^-(z)$ and an upper threshold $T^+(z)$, such that:

$$\Omega_z = \left[-T^-(z), T^+(z)\right] \tag{19}$$

where Ω_z is the projection of $\Omega(\theta^*)$ onto a given z.

So that:

$$P\left(|\varepsilon| > L, \left(\hat{\varepsilon}(y), z\right) \in \Omega \mid \theta^*\right) = \int_z P\left(|\varepsilon| > L, -T^-(z) \le \hat{\varepsilon}(y) \le T^+(z) \mid z, \theta^*\right) p(z)\, dz \tag{20}$$

In the next step, we decompose the second term according to the size of the threshold:

$$\begin{aligned}
&\int_z P\left(|\varepsilon| > L, -T(z) \le \hat{\varepsilon}(y) \le T^-(z) \mid z, \theta^*\right) p(z)\, dz \\
&= \int_{\substack{T^+(z)>T_1 \\ T^-(z)>T_1}} P\left(|\varepsilon| > L, -T^-(z) \le \hat{\varepsilon}(y) \le T^+(z) \mid z, \theta^*\right) p(z)\, dz + \\
&\int_{\substack{T^+(z)<T_1 \\ or \\ T^-(z)<T_1}} P\left(|\varepsilon| > L, -T(z) \le \hat{\varepsilon}(y) \le T(z) \mid z, \theta^*\right) p(z)\, dz
\end{aligned} \tag{21}$$

We have:

$$\begin{aligned}
&\int_{\substack{T^+(z)>T_1 \\ T^-(z)>T_1}} P\left(|\varepsilon| > L, -T^-(z) \le \hat{\varepsilon}(y) \le T^+(z) \mid z, \theta^*\right) p(z)\, dz \ge \\
&\int_{\substack{T^+(z)>T_1 \\ T^-(z)>T_1}} P\left(|\varepsilon| > L, -T_1 \le \hat{\varepsilon}(y) \le T_1 \mid z, \theta^*\right) p(z)\, dz
\end{aligned} \tag{22}$$

and:

$$\begin{aligned}
&\int_{\substack{T^+(z)<T_1 \\ or \\ T^-(z)<T_1}} P\left(|\varepsilon| > L, -T_1 \le \hat{\varepsilon}(y) \le T_1 \mid z, \theta^*\right) p(z)\, dz \le \\
&\max\left(P\left(|\varepsilon| > L, -T_1 \le \hat{\varepsilon}(y) \le T_1 \mid z, \theta^*\right)\right) \int_{\substack{T^+(z)<T_1 \\ or \\ T^-(z)<T_1}} p(z)\, dz
\end{aligned} \tag{23}$$

We have the upper bound:

$$\max\left(P\left(|\varepsilon| > L, -T_1 \le \hat{\varepsilon}(y) \le T_1 \mid z, \theta^*\right)\right) \le \max_\theta\left(P\left(\left|\varepsilon - \hat{\varepsilon}(y)\right| > L - T_1\right) \mid \theta^*\right) \tag{24}$$

As a consequence:

$$\int_z P\left(|\varepsilon| > L, -T(z) \le \hat{\varepsilon}(y) \le T(z) \mid z, \theta^*\right) p(z)\,dz \ge P\left(|\varepsilon| > L, -T_1 \le \hat{\varepsilon}(y) \le T_1 \mid \theta^*\right)$$
$$-\max\left(P\left(|\varepsilon - \hat{\varepsilon}(y)| > L - T_1\right)\right) \int_{\substack{T^+(z)<T_1 \\ or \\ T^-(z)<T_1}} p(z)\,dz \qquad (25)$$

We define:

$$\int_{\substack{T^+(z)<T_1 \\ or \\ T^-(z)<T_1}} p(z)\,dz = q \qquad (26)$$

We have:

$$P_{fa} = \int_{\substack{T^+(z)>T_1 \\ T^-(z)>T_1}} P\left(\hat{\varepsilon} > T(z) \mid z\right) p(z)\,dz + \int_{\substack{T^+(z)<T_1 \\ or \\ T^-(z)<T_1}} P\left(\hat{\varepsilon} > T(z) \mid z\right) p(z)\,dz$$
$$\ge q \frac{P_{fa,T_1}}{2} \qquad (27)$$

Therefore:

$$\int_{T(z)\le T_1} p(z)\,dz = q \le \frac{2P_{fa}}{P_{fa,T_1}} \qquad (28)$$

which concludes the proof.

LOWER BOUND ON INTEGRITY FOR NON LINEAR ESTIMATORS FOR THE FAULT DETECTION AND EXCLUSION PROBLEM

Most RAIM algorithms are structured so that detection and exclusion in RAIM are two clearly defined steps in the algorithm. At each step, a linear unbiased estimator is used. After exclusion, the linear estimator corresponds to the best linear unbiased estimator corresponding to one of the fault modes. This approach will not necessarily lead to the minimum integrity risk.

In this section we formulate the problem of fault detection and exclusion (FDE) as the search of two objects: an estimator and an acceptability region. For the FDE problem there are two requirements. The integrity requirement is identical to the one described above. We can formulate it as follows:

$$P\left(\left|\hat{x}(y)-x\right|>AL, y\in\Omega\right)\leq PHMI \tag{29}$$

The false alert requirement becomes an Alert requirement. We want to have:

$$P(y\notin\Omega)\leq P_{Alert} \tag{30}$$

In this equation, $\hat{x}(y)$ is the estimator of the position. The optimization problem can then be written:

$$\text{minimize } P\left(\left|\hat{x}(y)-x\right|>AL, y\in\Omega\right)\leq PHMI$$

$$\text{s.t } P(y\notin\Omega)\leq P_{Alert} \tag{31}$$

As mentioned above, most RAIM FDE algorithms are such that:

$$\hat{x}(y)=\hat{x}^{(i)}(y) \text{ if } y\in\Omega_i \tag{32}$$

where $\hat{x}^{(i)}(y)$ is a linear unbiased estimator that is not affected by fault *i* and Ω_i is the region that leads to the choice of the estimator *i*. However, the above requirements suggest that the estimator $\hat{x}(y)$ and Ω that minimizes the PHMI (subject to the PFA) is most likely not given by an estimator with the structure given by (32).

Determining an optimal estimator might be feasible, but seems a very complex problem. We can however develop a lower bound on the achievable integrity.

We start by developing the terms using the law of total probability:

$$P\left(\left|\hat{x}(y)-x\right|>AL, y\in\Omega\right)=\sum_i P\left(\left|\hat{x}(y)-x\right|>AL, y\in\Omega \mid H_i\right)p(H_i) \tag{33}$$

We now bound each of the terms as follows. We have by definition:

$$P\left(\left|\hat{x}(y)-x\right|>AL, y\in\Omega \mid H_i\right)=P\left(\left|\hat{x}(y)-x\right|>AL \mid H_i\right)-P\left(\left|\hat{x}(y)-x\right|>AL, y\notin\Omega \mid H_i\right) \tag{34}$$

We have:

$$P\left(\left|\hat{x}(y)-x\right|>AL, y\notin\Omega \mid H_i\right)\leq P(y\notin\Omega \mid H_i) \tag{35}$$

Combining Equations (34) and (35), we get:

$$P\left(\left|\hat{x}(y)-x\right|>AL, y\in\Omega \mid H_i\right)\geq P\left(\left|\hat{x}(y)-x\right|>AL \mid H_i\right)-P(y\notin\Omega \mid H_i) \tag{36}$$

Now we notice that we always have:

$$P\left(\left|\hat{x}(y)-x\right|>AL\right)\geq P\left(\left|\hat{x}^{(i)}(y)-x\right|>AL \mid H_i\right) \tag{37}$$

The above expression results from the fact that, for a given hypothesis H_i, the estimator that minimizes the integrity risk is the fault tolerant estimator. With our assumptions about the nominal errors, we can calculate an explicit formula for the right hand side in Equation (37). We have:

$$P\left(\left|\hat{x}^{(i)}(y)-x\right|>AL \mid H_i\right)=Q\left(\frac{AL}{\sigma_i}\right) \tag{38}$$

Where $\sigma_i = std\left(\hat{x}^{(i)}(y)-x\right)$. We therefore have:

$$P\left(\left|\hat{x}(y)-x\right|>AL, y\in\Omega \mid H_i\right)\geq Q\left(\frac{AL}{\sigma_i}\right)-P\left(y\notin\Omega \mid H_i\right) \tag{39}$$

Now, summing over all modes, we get:

$$P\left(\left|\hat{x}(y)-x\right|>AL, y\in\Omega\right)\geq \sum_i p(H_i)Q\left(\frac{AL}{\sigma_i}\right)-\sum_i p(H_i)P\left(y\notin\Omega \mid H_i\right) \tag{40}$$

In the right hand side of this inequality, we recognize the Alert probability written developed using the law of total probability:

$$\sum_i p(H_i)P\left(y\notin\Omega \mid H_i\right)=P\left(y\notin\Omega\right)=P_{Alert} \tag{41}$$

We finally have:

$$P\left(\left|\hat{x}(y)-x\right|>AL, y\in\Omega\right)\geq \sum_i p(H_i)Q\left(\frac{AL}{\sigma_i}\right)-P_{Alert} \tag{42}$$

This expression is a lower limit on the integrity risk that can be achieved with any estimator.

SUMMARY

Lower bounds on the achievable integrity risk are useful to determine whether it makes sense to improve a given monitoring approach. They also indicate whether we will need more knowledge to attain a certain performance. In this work we provide lower bounds on the achievable integrity risk in two cases. First, in the case of linear unbiased estimators, we formulate a lower bound as a function of the integrity risk achieved using the solution separation test statistic. Second, for any estimator (and therefore covering the Fault Detection and Exclusion case), we formulate an explicit lower bound which is a function of the fault tolerant estimators corresponding to the fault modes.

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