Errata sheet for OPTIMAL POSITIONING FOR ARAIM

Juan Blanch, May 3, 2017

Although the final formula for the Hessian is correct, Equation B8 is incorrect. The term that is written in this equation only appears after summing over all contributions to the Hessian.

Hessian Computation

We have:

$$\nabla L(x) = \sum_i \alpha_i \frac{(x - a_i)}{|x - a_i|} h_i(x)$$

Where we have defined:

$$h_i(x) = \frac{\frac{p_i}{\sigma_i} p\left(\frac{L - \alpha_i |x - a_i|}{\sigma_i}\right)}{\sum_j \frac{p_j}{\sigma_j} p\left(\frac{L - \alpha_i |x - a_j|}{\sigma_j}\right)} = \frac{f_i(x)}{\sum_j f_j(x)}$$

The Hessian is obtained by taking the gradient of each component of $\nabla L(x)$:

$$\nabla\nabla L(x) = \nabla\left(\sum_i \alpha_i \frac{x - a_i}{|x - a_i|} h_i(x)\right) = \sum_i \alpha_i \frac{x - a_i}{|x - a_i|} \nabla h_i(x)^T - \sum_i \alpha_i \frac{(x - a_i)(x - a_i)^T}{|x - a_i|^3} h_i(x) + \delta_{kl} \sum_i \alpha_i \frac{h_i(x)}{|x - a_i|}$$

The second and third term do not need to be developed further. For the first, we write:

$$\nabla h_i(x) = \frac{\nabla f_i(x)}{\sum_j f_j(x)} - f_i(x) \frac{\sum_j \nabla f_j(x)}{\left(\sum_j f_j(x)\right)^2}$$

So that:

$$\sum_i \alpha_i \frac{x-a_i}{|x-a_i|}\nabla h_i(x)^T = \sum_i \alpha_i \frac{x-a_i}{|x-a_i|}\left(\frac{\nabla f_i(x)}{\sum_j f_j(x)} - f_i(x)\frac{\sum_j \nabla f_j(x)}{\left(\sum_j f_j(x)\right)^2}\right)^T$$

$$= \sum_i \alpha_i \frac{x-a_i}{|x-a_i|}\frac{\nabla f_i(x)}{\sum_j f_j(x)} - \sum_i \alpha_i \frac{x-a_i}{|x-a_i|}\left(f_i(x)\frac{\sum_j \nabla f_j(x)}{\left(\sum_j f_j(x)\right)^2}\right)^T$$

$$= \sum_i \alpha_i \frac{x-a_i}{|x-a_i|}\frac{\nabla f_i(x)^T}{\sum_j f_j(x)} - \nabla L(x)\frac{\sum_j \nabla f_j(x)^T}{\left(\sum_j f_j(x)\right)}$$

$$= \sum_i \left(\alpha_i \frac{x-a_i}{|x-a_i|} - \nabla L(x)\right)\frac{\nabla f_i(x)^T}{\sum_j f_j(x)}$$

We also have:

$$\nabla f_i(x) = \left(\nabla L(x) - \alpha_i \frac{x-a_i}{|x-a_i|}\right)\frac{\alpha_i|x-a_i| - L(x)}{\sigma_i^2} f_i(x)$$

We therefore have:

$$\nabla\nabla L(x) = \sum_i \left(\nabla L(x) - \alpha_i \frac{x-a_i}{|x-a_i|}\right)\left(\nabla L(x) - \alpha_i \frac{x-a_i}{|x-a_i|}\right)^T \frac{L(x)-\alpha_i|x-a_i|}{\sigma_i^2} h_i(x)$$

$$-\sum_i \alpha_i \frac{(x-a_i)(x-a_i)^T}{|x-a_i|^3} h_i(x) + \delta_{kl}\sum_i \alpha_i \frac{h_i(x)}{|x-a_i|}$$

$$= \sum_i \left(\nabla L(x) - \alpha_i \frac{x-a_i}{|x-a_i|}\right)\left(\nabla L(x) - \alpha_i \frac{x-a_i}{|x-a_i|}\right)^T \frac{L(x)-\alpha_i|x-a_i|}{\sigma_i^2} h_i(x)$$

$$+\sum_i \alpha_i \left(I - \frac{(x-a_i)(x-a_i)^T}{|x-a_i|^2}\right)\frac{h_i(x)}{|x-a_i|}$$

The convexity of the *L* is apparent in this formula, as it is a weighted sum of projections, all the weights being positive.