A Gyro-Free Quaternion-Based Attitude Determination System Suitable for Implementation Using Low Cost Sensors

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Abstract

Attitude determination systems that use inexpensive sensors and are based on computationally efficient and robust algorithms are indispensable for real-time vehicle navigation, guidance and control applications. This paper describes an attitude determination system that is based on two vector measurements of non-zero, non-colinear vectors. The algorithm is based on a quaternion formulation of Wahba's problem, whereby the error quaternion (q_e) becomes the observed state and can be cast into a standard linear measurement equation. Using the earth's magnetic field and gravity as the two measured quantities, a low-cost attitude determination system is proposed. An iterated least-squares solution to the attitude determination problem is tested on simulated static cases, and shown to be globally convergent. A time-varying Kalman filter implementation of the same formulation is tested on simulated data and experimental data from a maneuvering aircraft.

The time varying Kalman Filter implementation of this algorithm is exercised on simulated and real data collected from an inexpensive triad of accelerometers and magnetometers. The accelerometers in conjucnction with the derivative of GPS velocity provided a measure of the gravitation field vector and the magnetometers measured the earth's magnetic field vector. Tracking errors on experimental data are shown to be less than 1 degree mean and standard deviation of approximately 11 degrees in yaw, and 3 degrees in pitch and roll. Best case performance of the system during maneuvering is shown to improve standard deviations to approximately 3 degrees in yaw, and 1.5 degrees in pitch and roll.

1. Introduction

Attitude determination is a requirement for most navigation and control problems. Classically, this problem has been solved by an Attitude Heading Reference System (AHRS) using gyros that are updated by gravity sensors (pitch and roll) and magnetic field sensors (yaw) with low-pass filters to attenuate errors incurred during turns. Rate gyros are prone to bias- and random-walk errors resulting from the integration of wide-band noise. Successful AHRSs require very expensive sensors that have exceptional long term bias stability. The sensor cost of this kind of attitude determination has limited such AHRS to very expensive applications. In cost-sensitive applications, filtering techniques are applied to bound the attitude error growth. With the recent proliferation of low-cost inertial sensors along with position and attitude sensors like the Global Positioning System (GPS), it has become viable to construct inexpensive attitude determination systems which have attitude errors bounded in time.

Recently, a considerable amount of effort has been directed at developing low-cost systems for attitude determination. For example, [1] discusses an inexpensive attitude determination system for aviation applications. This attitude determination system does not employ any inertial sensors, but instead, relies on a kinematic model of the vehicle along with GPS position and velocity measurements derived from a single GPS antenna to generate what is termed "pseudo-attitude." A different approach is taken by [2, 3] where a triad of inexpensive automotive-grade rate gyros are fused with an ultra-short baseline differential GPS attitude determination system. In this system, the high bandwidth attitude information derived from integrating the output of the rate gyros was blended with the low bandwidth GPS attitude solution using a complementary filter. Even though the rate gyros used exhibited poor long term bias stability, the GPS attitude system was used to update the estimate of the gyro biases continuously. In order to further reduce the cost of this attitude system, an offshoot was created with a single GPS baseline for yaw[4], with roll and pitch obtained by an ad-hoc non-linear acceleration vector matching algorithm[5].

Vector matching in multiple coordinate frames is often referred to as Wahba's problem. First published in 1965[6], Wabha proposed an attitude solution by matching two non-zero, non-colinear vectors that are known in one coordinate frame, and measured in another. Several solutions to this method of attitude determination have been proposed and implemented [7, 8, 9], usually on satellites with star-tracker sensors. In this paper, a novel method of solving Wahba's problem is demonstrated, in which the measurement equation can be cast into a standard form. By using the quaternion representation of attitude, the algorithm does not require the solution of transcendental equations (as would be the case with euler angles), and is suitable for implementation in a small, inexpensive microcomputer. Specifically, by measuring the body-fixed magnetic field, and the body-fixed acceleration, the attitude of a moving aircraft is tracked both in simulation, and experimentally with post-processed data.

In Section 2 of this paper, the attitude determination algorithm is derived in detail. In Section 3, the stability and convergence of the iterated least-squares implementation is verified in a simulated static case. In Section 4, the Kalman filter implementation of the solution is derived and tested on simulated data. In Section 5, the experimental setup is described and the results of the solution on post-processed data are discussed. Finally, conclusions and a summary are presented in Section 6.

2. Algorithm Development

Wahba's problem, attitude determination from vector matching, requires that two or more vectors be known or measured in the respective coordinate frames. Vector matching obviates gyro bias estimation. In addition, the random-walk errors associated with wide-band noise integration are not present. However, the vectors must be of non-zero length, and be non-colinear. All vector matching attitude determination algorithms solve for the rotation that aligns the two vectors into the base coordinate frame. The usage of only one vector, or colinear vectors, results in an ambiguity of rotation about that vector. This work presents a quaternion-based vector-matching attitude determination method, implemented using magnetometer and accelerometer measurements. This method results in attitude errors that are bounded with respect to time. The algorithm developed is general for any n vectors, but the specific implementation demonstrated use the magnetic field vector and the acceleration vector. Again, given any two non-colinear vectors, a unique plane containing the two vectors can be defined, and if the components of these two vectors can be measured in two non-aligned coordinate frames, then the rotation needed to align the two coordinate frames can be determined.

Let the two coordinate frames in which the measurements of vector components are made be designated the "body frame" (denoted by superscript b and fixed to the vehicle's body) and the "navigation frame" (denoted by superscript n at attached to the locally level plane). Furthermore, let the two vectors defining the plane which will be used for the attitude determination be designated $\vec{u}$ and $\vec{v}$. First, the relations involving the vector $\vec{u}$ will be derived. The relations involving $\vec{v}$ will be a repeat of those derived for $\vec{u}$. The transformation between the vector $\vec{u}$ as expressed in the body frame and the navigation frame is:

$$\vec{u}^b = \overset{n\to b}{C}(q)\vec{u}^n. \tag{1}$$

The Direction Cosine Matrix (DCM), $\overset{n\to b}{C}(q)$, for the transformation from navigation to body frame is a function of the attitude quatrenion q. In terms of the attitude quaternion, the DCM is expressed as:

$$\overset{n\to b}{C}(q) = \begin{bmatrix} 1-2({q_2}^2+{q_3}^2) & 2(q_1q_2+q_3q_0) & 2(q_1q_3-q_2q_0) \\ 2(q_1q_2-q_3q_0) & 1-2({q_1}^2+{q_3}^2) & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_2q_0) & 2(q_2q_3-q_1q_0) & 1-2({q_1}^2+{q_2}^2) \end{bmatrix} \tag{2}$$

where the quaternion q is defined as

$$q = [q_0\ \vec{q}]^T \tag{3}$$

composed of a scalar component q_0 and a vector component given by:

$$\vec{q} = [q_1\ q_2\ q_3]^T. \tag{4}$$

Equation 1 can be written in terms of quaternions as follows:

$$\vec{u}^b = q^* \otimes \vec{u}^n \otimes q, \tag{5}$$

where $\otimes$ represents quaternion multiplication and q^* is the complementary rotation of the quaternion q and is defined as:

$$q^* = [q_0\ -\vec{q}]^T. \tag{6}$$

Let $\hat{q}$ be an estimate of the true attitude quaternion q. The small rotation from the estimated attitude, $\hat{q}$ to the true attitude is defined as q_e. The error quaternion is small but non-zero, due to errors in the various sensors. The relationship is expressed in terms of quaternion multiplication as follows:

$$q = \hat{q} \otimes q_e, \tag{7}$$

The error quaternion, q_e, is assumed to represent a small rotation, thus q_e can be approximated as [10]:

$$q_e = [1\ \vec{q}_e]^T \tag{8}$$

The error quaternion q_e is a perturbation to the direction cosine matrix $\overset{n\rightarrow b}{C}(q)$. That is,

$$\delta \overset{n\rightarrow b}{C}(q) \triangleq \overset{n\rightarrow b}{C}(q_e). \tag{9}$$

Noting that the vector components of q_e are small, the perturbation to the DCM in Equation 2 can be written as:

$$\overset{n\rightarrow b}{C}(q_e) = \begin{bmatrix} 1 & 2q_3 & -2q_2 \\ -2q_3 & 1 & 2q_1 \\ 2q_2 & -2q_1 & 1 \end{bmatrix}. \tag{10}$$

In a more compact form Equation 10 can be written as

$$\overset{n\rightarrow b}{C}(q_e) = I_{3\times 3} - 2[\vec{q}_e]^\times \tag{11}$$

where $[\vec{q}_e]^\times$ is a skew symmetric matrix whose entries are the components of $\vec{q}_e$. Equation 7 relating $\hat{q}$ and q can be written in terms of DCM's as

$$\overset{n\rightarrow b}{C} = \delta \overset{n\rightarrow b}{C}\ \overset{\widehat{n\rightarrow b}}{C}. \tag{12}$$

$\overset{\widehat{n\rightarrow b}}{C}$ is the estimate of the direction cosine matrix or the equivalent of $\hat{q}$. Transposing both sides of the above equation to recast the equation in terms of body to navigation frame direction cosine matrices results in:

$$\overset{b\rightarrow n}{C} = \overset{\widehat{b\rightarrow n}}{C}\left[I_{3\times 3} + 2[\vec{q}_e]^\times\right]. \tag{13}$$

If this is substituted into the Equation 1 the following relation is obtained:

$$\begin{aligned} \vec{u}^n &= \overset{\widehat{b\rightarrow n}}{C}\left[I_{3\times 3} + 2[\vec{q}_e]^\times\right]\vec{u}^b && (14)\\ &= \hat{\vec{u}}^b + 2[\vec{q}_e]^\times \vec{u}^b && (15)\\ &= \hat{\vec{u}}^b - 2[\vec{u}^b]^\times \vec{q}_e && (16)\end{aligned}$$

Let the following definition be made:

$$\delta\vec{u}^n \triangleq \vec{u}^n - \hat{\vec{u}}^n. \tag{17}$$

If a similar argument is carried out for the second vector $\vec{v}$, then the following linear measurement equation (standard form) is obtained for the vector portion of quaternion error:

$$\begin{bmatrix} \delta\vec{u}^n \\ \delta\vec{v}^n \end{bmatrix} = \begin{bmatrix} -2[\vec{u}^n]^\times \\ -2[\vec{v}^n]^\times \end{bmatrix} \vec{q}_e. \tag{18}$$

The highly non-linear attitude determination problem has been recast into a standard form of a linear measurement equation:

$$\vec{z} = H\vec{q}_e \tag{19}$$

Where the measurement vector $\vec{z}$ is defined as:

$$\vec{z} = \begin{bmatrix} \delta\vec{u}^n \\ \delta\vec{v}^n \end{bmatrix}. \tag{20}$$

and the observation matrix, H, is defined as:

$$H = \begin{bmatrix} -2[\vec{u}^n]^\times \\ -2[\vec{v}^n]^\times \end{bmatrix}. \tag{21}$$

Note that the observation matrix, H, is unique to each measurement, but this can still be used in linear time-varying solutions. The next section explores the simulated performance of the iterated least-squares solution based on the above equations.

3. Iterated Least Squares

The physical implementation of this low cost system relies on a 3-axis magnetometer (Honeywell HMR2300) to resolve the earth's magnetic field into body coordinates, and three MEMS accelerometers to resolve the earth's gravitational field into body coordinates. In the case where the vehicle is not accelerating, then the only acceleration measured on the accelerometers is due to gravity. The magnitude of the earth's magnetic field and gravitational field are both well known, and well modelled. Thus two non-colinear vectors (of non-zero length) are available to determine attitude in the static case. An iterated least-squares solution of the measurement equation is formulated as follows:

Define $\vec{m}^b$ as the magnetic field vector, as measured in the body frame and $\vec{m}^n$ as the magnetic field vector in the local level frame (navigation frame). Likewise, define $\vec{a}^b$ as the acceleration measured in the body fixed frame, and $\vec{a}^n$ as the acceleration in the local level frame (including gravity).

1. $\hat{q} = [1\,0\,0\,0]^T$, and $q_e = [1\,0\,0\,0]^T$
2. $\hat{\vec{m}}^n = \hat{q} \otimes \vec{m}^b \otimes \hat{q}^*$ and likewise for $\hat{\vec{a}}^n$
3. $\delta\hat{\vec{m}}^n = \vec{m}^n - \hat{\vec{m}}^n$ and likewise for $\delta\hat{\vec{a}}^n$
4. Form H (Equation 21), take the pseudo-inverse: $[H^T H]^{-1} H^T$
5. $q_e(+) = \alpha[H^T H]^{-1} H^T$
6. $\hat{q}(+) = \hat{q}(-) \otimes q_e$.
7. Return to step (2), repeat until converged.

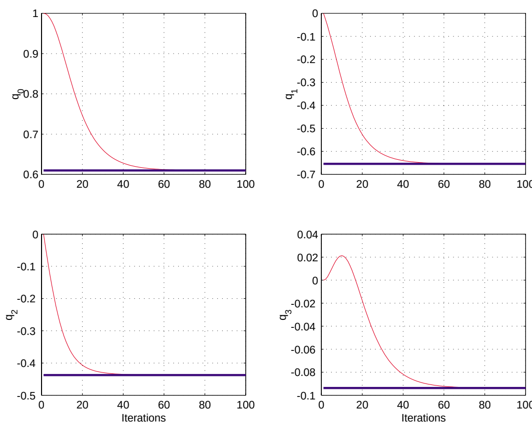

Figure 1: Quaternion Convergence History.

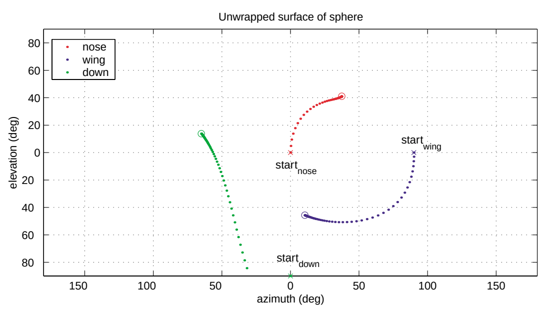

Figure 2: Attitude Convergence Map.

To validate the algorithm, a Monte-Carlo simulation was performed where a random starting attitude was given to the algorithm. The body-fixed measurements were corrupted with appropriate levels of sensor noise, and the algorithm was allowed 100 iterations to converge with a tuning parameter (α) of 1/10. Figure 1 shows a time history for the attitude quaternion components during a single run in these series of Monte-Carlo simulations. As can be seen the convergnce to the correct attitude is rapid and assured. Figure 2 shows a mercator projection of the traces made by the tips of the body-fixed unit vector triad during the above mentioned Monte-Carlo run. The algorithm was found to be globally convergent, in the static case. The range of the tuning parameter (α) was found to be robust, ensuring convergence as long as α was within [$1/n$ to 2] where n is the number of iterations. Though the formulation of this solution methodology assumed a small q_e, the algorithm performs very well even when the initial estimate of the attitude, $\hat{q}$ is not close to the final attitude.

4. Kalman Filter Implementation

The previous section dealt with a "snap-shot" solution, that used the vector measurements as the only available information. The solution method worked well for the static case with no dynamics. More information is available, however, in the form of past attitude. Typically, a Kalman filter is used to smoothly join the measurements with a dynamic model of the plant.

In order to implement a Kalman filter, equations accounting for the dynamics must be included in the fomulation. If angular rate information is available (from rate gyros), a dynamic model for $\vec{q}_e$ based on the kinematics of the attitude problem can be included[11]. If a dynamic model for the rate of change of the quaternion error $\vec{q}_e$ is not included, a lag in the attitude solution will be present. However, if the dynamics are assumed to have a relatively low frequency content, the lag will be inconsequential. In this case, the quaternion errors may be modeled as an exponentially correlated (Gauss-Markov) process. The dynamics for the state variable of interest, $\vec{q}_e$, can be written as:

$$\dot{\hat{\vec{q}}}_e(t) = F\hat{\vec{q}}_e(t) + G\vec{w}(t) \tag{22}$$

where

$$F = -\frac{1}{\tau}I_{3\times 3} \tag{23}$$

and

$$G = -\tau F. \tag{24}$$

If the problem is formulated as a time-varying Kalman Filter, then the measurement update equations will be stated as follows:

$$\vec{q}_e^{\,(+)} = \vec{q}_e^{\,(-)} + L(\vec{z} - H\vec{q}_e^{\,(-)}) \tag{25}$$

where the $(-)$ and $(+)$ represent values before and after the measurement update respectively. Both the measurement vector, $\vec{z}$, and the observation matrix, H, were defined previously in Equations 20 and 21 respectively. L is the time-varying Kalman gain computed as in[12]:

$$L = P(-)H^T(H^TP(-)H + R_v)^{-1} \tag{26}$$

where P is the state error covariance matrix and R_v is the *a priori* measurement noise matrix. The time update of P is carried out as:

$$P(+) = \Phi P(-)\Phi^T + C_d \tag{27}$$

where Φ is the discrete equivalent of F and C_d is the discrete process noise matrix. Finally, the measurement update of P is accomplished using the standard

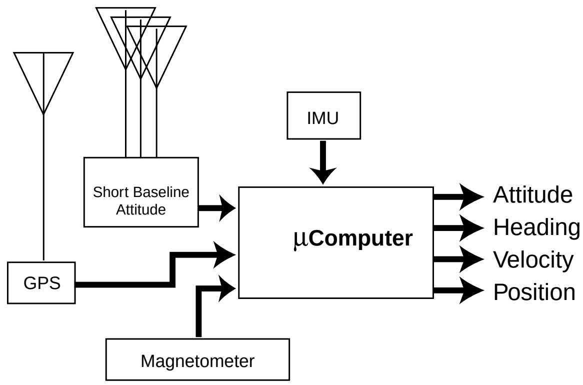

Figure 4: Block Diagram for Hardware in the Experimental Setup.

Kalman filter covariance update equation:

$$P(+) = (I - LH)P(-). \tag{28}$$

To confirm tracking performance, a simulation was performed. The simulation includes both attitude and gross motion dynamics. In this case, the acceleration measured in the body frame is no longer simply gravity, but rather, the aircraft apparent gravity ($\vec{g}-\vec{a}$). The local level acceleration, $\vec{a}$, is obtained by differentiating GPS velocity measurements. The simulation included the correction for aircraft apparent gravity as well as modeled sensor noise. The results of this simulation are shown in Figure 3. Even in the presence of high dynamics, and sensor noise, the solution tracks attitude very little lag.

There is a tradeoff in the tuning of the Gauss-Markov parameter τ; a larger τ corresponds to longer correlation times and slows down the tracking response. This, however, also smoothes the solution and effectively filters out noise. Decreasing τ improves tracking performance at the expense of making the solution more sensitive to noise. Inspection of figure 3 will show the initial transients as the filter locks onto the correct attitude. The initial transients could be improved by first using the iterated least-squares "snapshot" solution to start from a good initial guess of attitude.

5. Experimental Setup and Results

A block diagram of an experimental setup that was used validate this algorithm is shown in Figure 4. Experimental validation of the two vector matching algorithm was performed on a Beechcraft/Raytheon QueenAir test aircraft. This aircraft has been used extensively in the WAAS/LAAS experimentation done at Stanford University, and is equipped with several high quality sensors. For attitude reference, a Honeywell YG1851 IRU navigation-grade (1 nm/hr drift) inertial reference unit output position and attitude at 50 Hz. The data from the IRU was recorded on a separate computer and later aligned with the GPS-time-tagged data. A second, less expensive inertial measurement unit (CrossBow DMU-FOG) output data to the main computer at a rate of 125 Hz. These measurements are aligned to GPS time internally. The raw output of the Crossbow DMU accelerometers were used as the body-fixed accelerometer measurements ($\vec{g}^b$). The local level acceleration, $\vec{a}$, was computed by differencing velocities derived from GPS augmented by the Stanford University Wide Area Augmentation System (WAAS) prototype. This measurement was used to correct the aircraft apparent gravity (accelerometer specific force) measurements during turns. A low cost magnetometer triad (Honeywell HMR 2300) was used to measure the earth's magnetic field vector in the body frame $\vec{B}^b$. The output of the magnetometer was recorded serially on the main computer and aligned with GPS- time at 1 Hz.

The known portion of the vector measurements, the earth's gravitational field and magnetic field vector in the navigation frame (i.e., $\vec{g}^n$ and $\vec{B}^n$), are well known as readily modeled. The earth's magnetic field was modeled using Schmidt-normalized coefficients from the 1995 International Geomagnetic Reference Field [13] as a function of GPS-position. The gravitational field of the earth was assumed to remain down in the local level coordinate frame (navigation frame), and to have a magnitude of 9.81 m/s^2. The accelerometers of the DMU are MEMS accelerometers similar to what will be used for the final system.

The test flight lasted approximately 45 minutes, from takeoff to landing, and included many steep turns. Pitch and roll doublets were also performed in order to deviate from straight and level flight. Following the flight, the data was aligned to GPS-time in post-processing, and the magnetometer and accelerometer data were aligned to each other, in order to simulate running the system in real time. The Honeywell IRU was interpolated to match the sample points of the magnetometer and accelerometers. Finally, the GPS velocity measurements were differenced to create the measurement of $\vec{a}^n$.

The accelerometer bias and scale factor calibration was already performed by the Crossbow DMU, thus no further processing of the accelerometer data was required. The magnetometer required calibration for misalignment errors, hard and soft iron errors (bias), and scale factors errors. The method for calibrat-

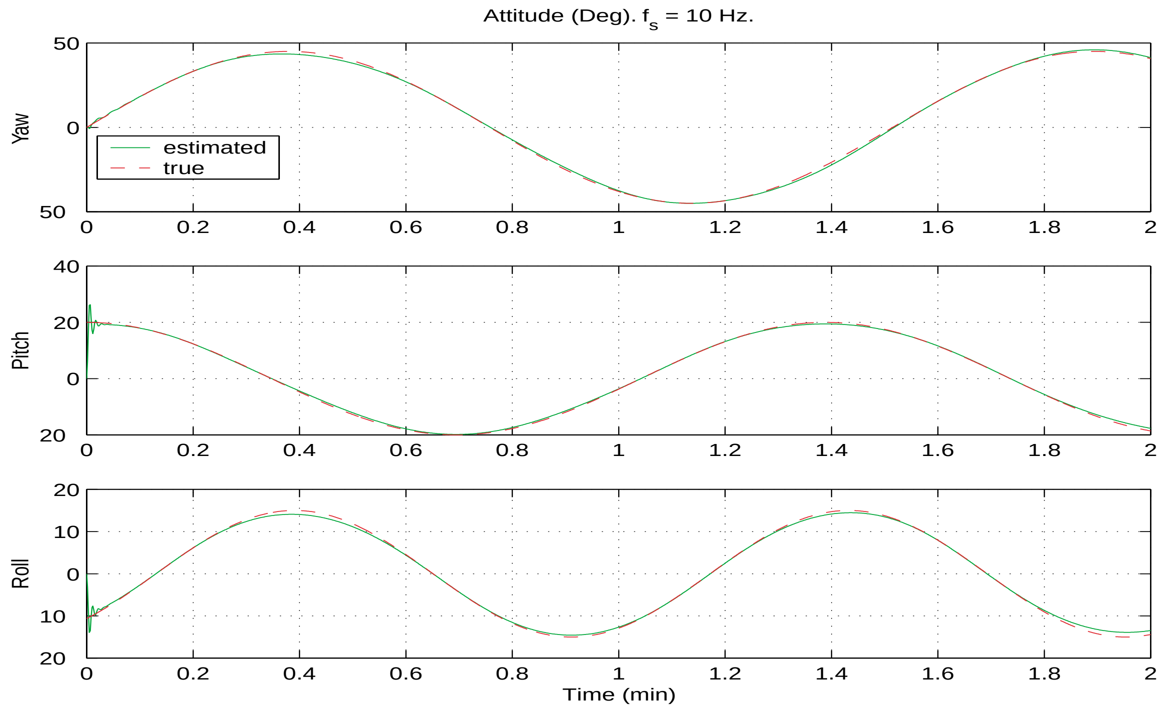

Figure 3: Attitude Time History (Simulation)

ing the magnetometer was based on a two-step, non-linear least-squares estimator similar to the ones discussed in [14]. The full treatment of "swinging" the magnetometer is the subject of another paper by the authors.

The Kalman filter implementation of the two vector attitude algorithm was performed on the post-processed data as described above. An initial guess of the quaternion estimate, $\hat{q}$, and the initial error quaternion, q_e, was used and the Kalman filter run sequentially through the post-processed data. It is important to note that the Kalman filter is providing an estimate of q_e, not $\hat{q}$. The estimated attitude is computed from q_e and continously updated, and is the output of interest. Figure 5 compares the time history of the attitude as recorded by the navigation grade INS and the time history of the attitude solution as generated by the Kalman filter. In general, agreement with the INS is excellent, with virtually no lag in the solution whatsoever. For the entire flight, the means and standard deviations for the vector matching Kalman filter (truth provided by IMU), are summarized in Table 1. Several times in the flight, the pitch measurement appears to diverge from the INS, but then recovers. Close inpection of figure 5 will show that the yaw oscillates from the true attitude at the same time. These divergences appear periodically, at 8,13,17,22 minutes into the flight, and

Error	Mean (μ)	Standard Deviation (σ)
Yaw	-0.831	10.507
Pitch	0.0959	3.694
Roll	-0.757	2.062

Table 1: Attitude Error Statistics (Nominal Performance)

during the time from 29-33 minutes and 36-37 minutes. The reason for which the algorithm performs poorly during these transients is that the maneuvering aircraft aligns the body z-axis during a coordinated turn with the magnetic field of the earth. This condition violates the basic requirement that the two vectors be non-colinear. During this transient moment, while the aircraft nose swings through due east or due west, the magnetic field vector is aligned with the aircraft apparent gravity, $\vec{g}-\vec{a}$. During this transient, there is insufficient information to determine aircraft attitude. The Kalman filter, however, continues to perform, albeit at a degraded performance level, even with the lack of gyros. Rate gyros would allow the attitude determination algorithm to bridge these outages without the large transients, but with increased cost and complexity.

It is important to clarify that the large transients are

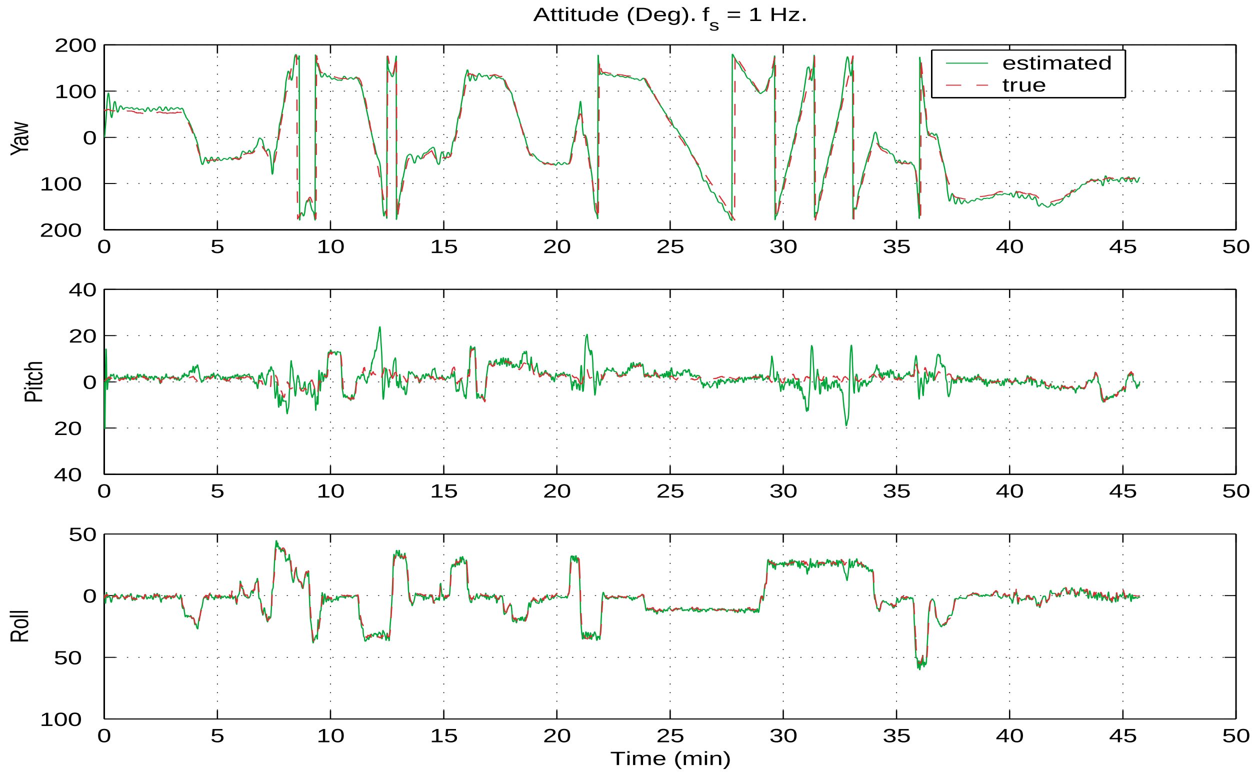

Figure 5: Attitude History from a Flight Test.

Error	Mean (μ)	Standard Deviation (σ)
Yaw	-0.823	3.286
Pitch	0.952	1.380
Roll	-0.956	1.259

Table 2: Attitude Error Statistics (Best Performance)

not a failure of the algorithm itself, but rather an operation outside the assumptions of the basic Wahba's problem. The roll angle is approximately equal to the magnetic declination, and the yaw angle lines up the aircraft apparent gravity vector with the earth's magnetic field. This only happens when the nose of the aircraft is turning through a yaw angle of ± 90 degrees. Close inspection of the results verifies that this is indeed the case.

Without these transients, where the basic assumptions of the algorithm are not violated, the tracking performance is much better. During the time from 22 to 26 minutes of the flight, which includes a steep banked turn, the tracking performance is summarized in Table 2. This should be considered as the best-case performance of the gyro-free quaternion-based attitude determination algorithm. Finally, the aircraft environment is very poor for the performance of the magnetometer, with large electromagnetic transients that cannot be calibrated out of the measurements. Also, the interpolation and alignment of the INS data introduces some errors that are difficult to quantify. Future work includes the creating of a realtime attitude system that can be flight tested directly. The authors expect that a real time solution will improve the attitude tracking performance.

6. Conclusion

An attitude determination system that is based on two vector measurements of non-zero, non-colinear vectors is demonstrated. The algorithm is based on a quaternion formulation of Wahba's problem, whereby the error quaternion (q_e) becomes the observed state and can be cast into a standard linear measurement equation. Using the earth's magnetic field and gravity as the two measured quantities, a low-cost attitude determination system is proposed. An iterated least-squares solution to the attitude determination problem is tested on simulated static cases and shown to be globally convergent. A time-varying Kalman filter implementation of the same formulation is tested on simulated data, as well as on experimental data from a maneuvering aircraft.

The overall performance of the system is shown to have a mean of less than one degree on each axis, and standard deviations of approximately 11 degrees on yaw, and 3 degrees on both pitch and roll. The algorithm is shown to track through periods where the non-colinear assumption is violated due to the aircraft apparent gravity and the earth's magnetic field lining up, though with degraded performance. Outside of those operating conditions, the standard deviations drop to roughly 3 degrees in yaw and 1 degrees for pitch and roll. Rate gyros would improve the response, allowing the true dynamics to be tracked through the outages, but at the expense of more expensive hardware. Future research will implement the algorithm in a low-cost, real-time system.

7. Acknowledgments

We wish to acknowledge the FAA Satellite Navigation Product Team and The Office of Technology and Licensing at Stanford University for sponsoring the research reported in this paper. We also like to thank flight test services of Sky Research Inc. and the test pilot Ben Hovelman. The support from Cross-Bow Inc. regarding modification to inertial sensor is gratefully acknowledged. Finally, the authors wish to thank all the friends and colleagues at Stanford that have helped in this effort; in particular Rich Fuller and Sharon Houck who helped in data collection during flight tests and Roger Hayward for developing systems and software for data collection that facilitated data collection.

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